Arithmetic Average Futures Contracts

Aa A Hedge Against Expiration day Effects

Jin Yoo

Hanyang University, College of Economics and Finance

222 Wangsimni-ro, Seongdong-gu,

Seoul, 133-791, Korea

phone: +82-2-2220-1026, fax:+82-2-2296-9587

e-mail: jyoo@hanyang.ac.kr

Arithmetic Average Futures Contracts

Aa A Hedge Against Expiration day Effects

Abstract

In this paper, I define and price arithmetic average (AV) futures contracts, show how they can reduce price manipulation, and explore their features as investment choices. They offer a number of advantages, the fore most important of which is protecting investors against price manipulation by reducing the level of price manipulation and manipulators' profits. The study also shows that the mean and the variance of the price of an AV futures contract are functions of its reference dates and that it can be flexibly designed to meet various hedging needs of investors. With these features, AV futures can serve as a good complement to existing futures.

Keywords:

Expiration day effects; Price manipulation; Futures contracts; Reference dates

JEL classification: G13, G14

In general, stock prices and trading volumes undergo high volatility when financial instruments such as index futures and options approach their expiration dates. This 'expiration day effect' is a phenomenon common in the U.S., Germany, Canada, Spain, India, and Taiwan and is used to account for the flurry of activity such as price reversals or excessive volatility of stock prices as options and futures traders unwind their positions before these contracts expire, wary of the impact this might have on volume and price (Chamberlain et al. (1989), Hsieh (2009), Illueca and Lafuente (2006), Schlag (1996), Stoll and Whaley (1987), and Vipul (2005)). Index arbitrage and price manipulation are seen as the culprits here (Hsieh (2009), Hsieh and Ma (2009), and Stoll and Whaley (1997)). Since price manipulation is difficult to detect and index arbitrage is likely to remain popular, plain vanilla futures will be subject to expiration day effects in the future, too.

Besides, plain vanilla futures might not always meet the various hedging needs of a range of individuals or firms. For instance, global exporters like Apple and Toyota do not receive their payments in Euros or US dollars on some specific day. They sell iPhones and Camrys in foreign countries throughout the year and receive Euros or dollars year round. Therefore, to hedge their exchange rate risks, they need to convert foreign currencies into their own domestic currencies at an *average* exchange rate over a time period such as a quarter. For this reason, not plain vanilla futures but *average price futures (AV futures)* might better serve the needs of those exporters and importers in hedging their exchange rate risks over a given period.

I find that the volatilities of AV futures contracts are in fact a function of their multiple reference dates, so the risk profiles could be flexibly, optimally determined, thereby increasing investors' utility and attracting new investors to the market.[1] In addition, this research on AV futures contracts can shed some light on pricing of currently available index futures such as CAC40 futures in France, Ibex35 futures in Spain, HIS futures in Hong Kong, and MSCI TW futures in Taiwan since each of these is a kind of AV futures contract with an average of multiple prices on its expiration day being its final settlement price. To date, however, their accurate, no-arbitrage pricing has never been proposed yet. This paper, therefore, will also discuss how accurate pricing of such futures contracts can be achieved.

Meanwhile, if average price options offer more benefits to certain investors, I believe that average price futures will do so, too. It is well known that average price options have been not only traded in developed economies as shown in Hull (2012, page 112) but also researched academically as in Kemna and Vorst (1990), Milevsky and Posner (1998), and Turnbull and Wakeman (1991). In fact, arithmetic average *futures* have many advantages over arithmetic average *options* in that the former, with closed form solutions, are easier to understand, easier to price, and easier to use. Therefore, AV futures contracts can be even a better financial instrument for hedgers or speculators than average price options can.

In this paper, I will define and price *AV futures contracts*, show how they can reduce the risk of expiration day effects, and explore their features as investment choices.[2] I will show that an AV futures' potential for reducing price manipulation and its financial

[1] The reference dates of arithmetic average futures are the pre-specified dates that determine the final settlement price of the futures at expiry.

[2] I interchangeably use the theoretical value of a forward and that of a corresponding futures contract. In fact, the two values are the same when interest rates are constant, as we assume in this paper (Hull (2009), pages 126-127). Also, "an AV futures" and "AV futures" are short forms of "an AV futures contract" and "AV futures contracts." In fact, whenever I use *a, an, this*, etc. before the word "futures," I mean one "futures contract."

instrument qualities are functions of its reference dates and that it can, *therefore*, be designed to meet the unique needs of different investors. The remainder of the paper is organized as follows. In Section 1, I will define and price AV futures contracts. In Section 2, I will show how they reduce the risk of price manipulation. In Section 3, I will explore important features of AV futures as financial instruments, and in Section 4, I will conclude the paper.

1. Pricing of Arithmetic Average Futures

The notation used in this paper is as follows.

① n: the number of reference dates to determine the settlement price of an AV futures at expiry (n = 1, 2, 3, …)[3]

② T (= T_1); the expiry date of AV futures

③ $T_1, T_2, \dots, T_n$: reference dates for AV futures, where $T_n < T_{n-1} < \cdots < T_2 < T_1$

④ r: risk-free interest rate per annum (assumed to be constant)

⑤ S_t: the market price at time t of the underlying asset of AV futures

⑥ $F_{X,t,TH}$: the theoretical value of a futures contract X at time t

⑦ $F_{X,t}$: the market price of a futures contract X at time t

First, we define AV futures.

Definition 1 (AV futures with n reference dates): A futures contract with the final

[3] Plain vanilla futures contracts are a special kind of AV futures contracts with n = 1.

settlement price at expiry T_1 being $\frac{1}{n} \cdot \sum_{i=1}^{n} (S_{T_i})$, where $T_n < T_{n-1} < \cdots < T_2 < T_1$.

$T_1, T_2, \ldots, T_k$ can be k different times on m different days, where $m \leq k$. For instance, they will be k different times on an expiration day if $m = 1$. In this sense, T_1 should be appropriately named "expiration *time,*" but we call T_1 by the more familiar term, "expiration date." To price an AV futures contract with n reference dates, we replicate its cash flow at expiry, $\frac{1}{n} \cdot \sum_{i=1}^{n} S_{T_i}$, assuming that $T_{k+1} \leq t < T_k$, where t is the current time and $1 \leq k \leq n-1$, which is case ii) of Proposition 1. First, let us divide $\frac{1}{n} \cdot \sum_{i=1}^{n} (S_{T_i})$ into two parts such that $\mathrm{A} = \frac{1}{n} \cdot \sum_{i=1}^{k} (S_{T_i})$ and $\mathrm{B} = \frac{1}{n} \cdot \sum_{i=k+1}^{n} (S_{T_i})$. As of time t, A is unknown and B is known since $T_{k+1} \leq t < T_k$. Accordingly, to secure B at expiry, one should invest the dollar amount of $\mathrm{B} \cdot e^{-r(T_1 - t)}$ in a risk-free asset and hold it until T_1. To secure A at expiry, investors should buy $\frac{1}{n} \cdot \sum_{i=1}^{k} (e^{-r(T_1 - T_i)})\, shares$ now, and, at each future time T_i, sell $\frac{1}{n} \cdot e^{-r(T_1 - T_i)}$ shares and invest the proceeds, $\frac{1}{n} \cdot e^{-r(T_1 - T_i)} \cdot S_{T_i}$, in a risk-free asset and hold it until T_1. Since this transaction at each T_i will secure them a cash flow of $\left(\frac{1}{n} \cdot e^{-r(T_1 - T_i)} \cdot S_{T_i}\right) \cdot e^{r(T_1 - T_i)} = \frac{1}{n} \cdot S_{T_i}$ at T_1, ultimately, they will get $\frac{1}{n} \cdot \sum_{i=1}^{k} (S_{T_i}) = A$ at T_1. Lastly, the total cost of this replication of the AV futures contract as of time T_1 is:

$$\left[\mathrm{B} \cdot e^{-r(T_1 - t)} + \left(\frac{1}{n} \cdot \sum_{i=1}^{k} \left(e^{-r(T_1 - T_i)}\right)\right) S_t\right] e^{r(T_1 - t)} = \mathrm{B} + \left(\frac{1}{n} \cdot \sum_{i=1}^{k} \left(e^{r(T_i - t)}\right)\right) S_t$$

$= \frac{1}{n} \left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t \cdot e^{r(T_i - t)}\right)$ as in ii) of Proposition 1. Case iii) is simpler and similarly derived, and case $i)$ is trivial.

Proposition 1 (Theoretical price of an AV futures at t with n reference dates):

$$i)\ \frac{1}{n}\cdot\sum_{i=1}^{n} S_{T_i}, \qquad\qquad where\ t = T_1$$

$$ii)\ \frac{1}{n}\left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t \cdot e^{r(T_i-t)}\right), \qquad where\ T_{k+1} \le t < T_k, \qquad 1 \le k \le n-1$$

$$iii)\ \frac{1}{n}\cdot\sum_{i=1}^{n} S_t \cdot e^{r(T_i-t)}, \qquad\qquad where\ 0 \le t < T_n$$

Notice that Proposition 1 generally holds as long as $T_n < T_{n-1} < \cdots < T_2 < T_1$ is satisfied, which implies that AV futures can be created in a very flexible way, depending on the various needs of their users. For instance, n reference dates could be set equally spaced as $T_1 - T_2 = T_2 - T_3 = \ldots = T_{n-1} - T_n = \Delta t$.[4] But even if the intervals are not set as such, Proposition 1 still holds only if $T_n < T_{n-1} < \cdots < T_2 < T_1$ is satisfied. If the market price of an AV futures contract is different from Proposition 1, investors can take advantage of the difference and make an arbitrage profit, the proof of which is provided in the Appendix.

2. Reduction of Price Manipulation

As Kemna and Vorst (1990) claimed, it is intuitively appealing that average value

[4] For instance, Δt could be a day, a week, 10 days, etc.

derivative securities reduce the risk of price manipulation at expiry. In addition to this generally accepted but yet to be proven claim, we show how AV futures contracts reduce the risk. Price manipulation can occur in various ways while it is not clearly documented how it is being actually done. Thus this study on how AV futures can serve as a mechanism for reducing price manipulation has to be based on following assumptions:

First, in the general economy, a representative stock S exists along with its arithmetic average futures contract F_k with k reference dates, where $1 \leq$ k. S_t and $F_{k,t}$ refer to the time-t market prices of S and F_k, respectively;

Second, the stochastic process of S is, $dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot dB$, where μ and σ are the mean and the standard deviation of the continuously compounded returns of S_t, and B_t is a standard Brownian motion;

Third, F_k´s contract size and expiration date are N shares of S and T_1, respectively. Also, F_k´s reference dates are $T_k, T_{k-1}, \ldots, T_2, T_1$, where $T_k < T_{k-1} < \ldots < T_2 < T_1$;

Fourth, there is a (male) risk-neutral, representative price manipulator, M;

Fifth, if M takes a position in F_k at any time before $T_1 - \Delta$t, he can try to manipulate S_{T_1} at time $T_1 - \Delta$t, where Δt is the shortest time to execute one's order in the stock market[5];

Sixth, for convenience, we assume that M takes a long position in one futures contract at the price of $F_{k,\tau}$ at time τ, where $0 \leq \tau < T_1 - \Delta$t;

Seventh, if M buys up the shares worth $\alpha \cdot u(\alpha)$ at time $T_1 - \Delta$t, he can push up

[5] There is no point in manipulating the price at a non-reference date(time), and time T_1 is the common reference date (time) for all futures contracts with k reference dates.

S_{T_1} by α, where $u(\alpha)$ is the average cost of \$1 manipulation of S_{T_1} given α and $S_{T_1-\Delta t}$;

Finally, we assume $\frac{\partial u(\alpha)}{\partial \alpha} > 0$ because the cost of \$1 manipulation increases as α increases *ceteris paribus*.[6] In particular, we assume $u(\alpha) = c \cdot \alpha$, where $c > 0$;

Suppose that the share price to be realized at T_1 without M's manipulation is S_{T_1}. The conditional mean, S_{T_1} at $T_1 - \Delta t$, is $E_{T_1-\Delta t}[S_{T_1}] = S_{T_1-\Delta t} \cdot e^{\mu \cdot \Delta t}$, as shown in Section III. If $F_k = F_1$ and M manipulates S_{T_1} by spending $\alpha \cdot u(\alpha) = c \cdot \alpha^2$, his wealth (W) at T_1 will be: $W = N(F_{1,T_1} - F_{1,\tau}) - c \cdot \alpha^2 = N(S_{T_1} + \alpha - F_{1,\tau}) - c \cdot \alpha^2 = N \cdot S_{T_1} + N(\alpha - F_{1,\tau}) - c \cdot \alpha^2$. Thus,

$E_{T_1-\Delta t}[W] = E_{T_1-\Delta t}[N \cdot S_{T_1} + N(\alpha - F_{1,\tau}) - c \cdot \alpha^2]$

$= N \cdot E_{T_1-\Delta t}[S_{T_1}] + N(\alpha - F_{1,\tau}) - c \cdot \alpha^2 = N \cdot S_{T_1-\Delta t} \cdot e^{\mu \cdot \Delta t} + N(\alpha - F_{1,\tau}) - c \cdot \alpha^2$.

To maximize the effect, the first order condition with respect to α is $N - 2c\alpha = 0$ or $\alpha^* = \frac{N}{2c}$. The maximized *incremental* profit for M due to the price manipulation by α is

$$N\alpha^* - c \cdot (\alpha^*)^2 = N \cdot \frac{N}{2c} - c \cdot \left(\frac{N}{2c}\right)^2 = \frac{N^2}{2c} - \frac{N^2}{4c} = \frac{N^2}{4c} > 0.$$

Next, suppose everything is the same but $F_k = F_n$, where $2 \leq n$. If M manipulates S_{T_1} by spending $\alpha \cdot u(\alpha) = c \cdot \alpha^2$, $F_{n,T_1} = \frac{(S_{T_1}+\alpha) + \sum_{i=2}^{n}(S_{T_i})}{n}$, and

$W = N \cdot (F_{n,T_1} - F_{n,\tau}) - c \cdot \alpha^2 = N \cdot \left[\frac{(S_{T_1}+\alpha)+\sum_{i=2}^{n}(S_{T_i})}{n} - F_{n,\tau}\right] - c \cdot \alpha^2$. Thus

$E_{T_1-\Delta t}[W] = E_{T_1-\Delta t}\left[N\left(\frac{(S_{T_1}+\alpha)+\sum_{i=2}^{n}(S_{T_i})}{n} - F_{n,\tau}\right) - c \cdot \alpha^2\right]$

---

[6] M can manipulate S_{T_1} only by buying the shares up. Then a higher α leads to a higher *average* price of the shares he should buy up than does a lower α does, which implies a higher average cost of \$1 manipulation of S_{T_1}.

$$= \frac{N}{n} \cdot E_{T_1-\Delta t}[S_{T_1}] + N\left(\frac{\alpha + \sum_{i=2}^{n}(S_{T_i})}{n} - F_{n,\tau}\right) - c \cdot \alpha^2$$

$$= \frac{N \cdot S_{T_1-\Delta t} \cdot e^{\mu \cdot \Delta t}}{n} + N\left(\frac{\alpha + \sum_{i=2}^{n}(S_{T_i})}{n} - F_{n,\tau}\right) - c \cdot \alpha^2.$$

The first order condition with respect to α is $\frac{N}{n} - 2c\alpha = 0$ or $\alpha^* = \frac{1}{n} \cdot \frac{N}{2c}$, and the maximized incremental profit for M from this price manipulation is

$$\frac{N}{n} \cdot \alpha^* - c \cdot (\alpha^*)^2 = \frac{N \cdot \frac{1}{n} \cdot \frac{N}{2c}}{n} - c \cdot \left(\frac{1}{n} \cdot \frac{N}{2c}\right)^2 = \frac{1}{n^2} \cdot \frac{N^2}{2c} - \frac{1}{n^2} \cdot \frac{N^2}{4c} = \frac{1}{n^2} \cdot \frac{N^2}{4c} > 0.$$

Accordingly, the size of the manipulation of F_{n,T_1} will be

$$\frac{S_{T_1} + \alpha^* + \sum_{i=2}^{n}(S_{T_i})}{n} - \frac{S_{T_1} + \sum_{i=2}^{n}(S_{T_i})}{n} = \frac{\alpha^*}{n} = \frac{\frac{1}{n} \cdot \frac{N}{2c}}{n} = \frac{1}{n^2} \cdot \frac{N}{2c}.$$

Table 1 summarizes these results.

Table 1 (Price Manipulation Risk of Two Futures Contracts)

Type of Futures(F_{k,T_1})	Manipulation of S_{T_1}	Manipulation of F_{k,T_1}	Manipulation Profit
Plain vanilla (k=1)	$\frac{N}{2c}$	$\frac{N}{2c}$	$\frac{N^2}{4c}$
AV (k=n)	$\frac{1}{n} \cdot \frac{N}{2c}$	$\frac{1}{n^2} \cdot \frac{N}{2c}$	$\frac{1}{n^2} \cdot \frac{N^2}{4c}$

As in Table 1, when the AV contract F_n is traded, price manipulation decreases in many respects, compared with the case of F_1. The sizes of price manipulation of stock and futures decrease to $\frac{1}{n}$ times and $\frac{1}{n^2}$ times, respectively, while M's manipulation profit also decreases to $\frac{1}{n^2}$ times. For instance, if n is set at 4, the manipulation sizes of stock and futures and M's manipulation profit will decrease to $\frac{1}{4}$ times, $\frac{1}{16}$ times, and $\frac{1}{16}$ times, respectively, compared with the plain vanilla futures case. This implies that investors in an AV futures contract and its underlying asset would suffer much less from price manipulation than they would in a plain vanilla futures contract and its underlying asset.

So far, M's manipulation target is set at expiry $(= S_{T_1})$ only. Even with a more flexible approach, however, the risk of price manipulation decreases in a similar manner. For instance, suppose M manipulates S_{T_k}, for his existing position in F_n at time τ, where $1 < k < \tau \leq$ n. Then the results will be fundamentally the same as above.[7] Also, M might have to pay an invisible cost at each attempt of price manipulation since, in reality, price manipulation takes not only money but also time and energy. Then, with a higher total cost per $1 manipulation, M's optimal manipulation size and profit would decrease further. A smaller profit would, in its turn, reduce the manipulation incentive for M. M might give up manipulation for a profit lower than a certain level since, in reality, price manipulators tend to act for a significant amount of profit only. In this sense, AV futures contracts could reduce not only bad effects of price manipulation ex post but also attempts of price manipulation ex ante. Lastly, the most important result will still hold for index arbitrage, another major cause of expiration day effects. Like price manipulation at T_1, unwinding positions in an index arbitrage at T_1 can change the stock index drastically, but F_{n,T_1} will change only $\frac{1}{n}$ times as much.

3. Features of AV Futures as Financial Instruments

[7] I make similar assumptions. For instance, if M spends $\alpha \cdot v(\alpha)$ at time $T_k - \Delta\text{t}$, he can push up S_{T_k} by α given α and $S_{T_k - \Delta\text{t}}$, where $v(\alpha) = d \cdot \alpha \ (d > 0)$.

In this section, we explore important features of AV futures as financial instruments. Investors with long or short positions in a futures contract do not usually hold them to expiry; rather, they often unwind them at a future time prior to its maturity. But the market price of an AV futures contract at a future time is a stochastic process and cannot be predicted perfectly. This is why the expected value and the variance of an AV futures contract's possible price at a future time might matter to investors. Besides, they might be more interested in future *profits* than in future prices. In this section, I solve for the mean and the variance of an AV futures contract's possible price or profit at a future time. First, I find that the variance of either the price or the profit is decreasing in *n*, the number of reference dates of the AV futures. Second, expected future prices or profits of AV futures are also decreasing in *n*. Third, the current price of an AV futures contract decreases with *n* until its earliest reference date. To explore all these features of AV futures, I review the log-normal distribution first.

3.1. Log-normal Distribution

As in the Black-Scholes option pricing model, I assume that S_t follows an Ito process:

$$\boldsymbol{dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot dB}, \qquad (1)$$

where μ and σ are the mean and the standard deviation of the continuously compounded returns of S_t, and B_t is a Wiener process or a standard Brownian motion. I assume that μ and r are constants, where $0 < r < \mu$. The current time is 0, and the stochastic process of the natural logarithm of S_t follows (2) by Ito's Lemma:

$$\boldsymbol{d[lnS] = \left(\mu - \frac{1}{2}\,\sigma^2\right)dt + \sigma \cdot dB.} \tag{2}$$

(2) implies (3).

$$\boldsymbol{d[ln(S)] = ln(S_{t+dt}) - ln(S_t) \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)dt,\ \sigma^2 \cdot dt\right), \forall t.} \tag{3}$$

Then,

$[ln(S_{dt}) - ln(S_0)] + [ln(S_{2dt}) - ln(S_{dt})] + \cdots + [ln(S_{t-dt}) - ln(S_{t-2dt})]$

$+[ln(S_t) - ln(S_{t-dt})]$

$$= \boldsymbol{ln(S_t) - ln(S_0) \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t,\ \sigma^2 \cdot t\right)}.^{8} \tag{4}$$

S_0 and $ln(S_0)$ are known constants at t = 0, so (4-1) and (4-2) hold.

$$ln(S_t) \sim N\left(ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t,\ \sigma^2 \cdot t\right). \tag{4-1}$$

$$ln(S_t) = ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma(B_t - B_0). \tag{4-2}$$

Meanwhile, a lognormal random variable X with mean m and variance v^2 has the following moments.

$$\ln(X) \sim N(m, v^2) \ \rightarrow\ E[X] = e^{m+\frac{v^2}{2}} \text{and } Var[X] = e^{2m+v^2}(e^{v^2} - 1). \tag{5}$$

By (4-1) and (5), the expected value and the variance of S_T as of time zero are as follows.

$$E[S_T] = S_0 \cdot e^{\mu \cdot T}, \text{and } Var[S_T] = (S_0)^2 \cdot e^{2\mu \cdot T}\left(e^{\sigma^2 \cdot T} - 1\right), \forall T > 0. \tag{6}$$

Also, the covariance between any two future prices is as follows.

Lemma 1 (Covariance between two futures prices at T_k and T_m):

$$Cov\left[S_{T_m}, S_{T_k}\right] = (S_0)^2 e^{\mu(T_m+T_k)}\left(e^{\sigma^2 \cdot \mathrm{Min}[T_m, T_k]} - 1\right), \forall\, T_m,\ T_k > 0.$$

[8] This result is due to the fact that dB is i.i.d. N(0, dt).

As a corollary of (6) and Lemma 1, I have some methods to compare variances or covariances.

Corollary 1 (Comparison of variances and covariances):

$Var[S_{T_k}]$ <$Var[S_{T_m}]$ if $T_k < T_m$, and

$$Cov\left[S_{T_m}, S_{T_k}\right] < Cov\left[S_{T_p}, S_{T_q}\right] \text{ if } \begin{Bmatrix} T_m + T_k < T_p + T_q \text{ and } Min[T_m, T_k] \leq Min\left[T_p, T_q\right] \\ or \\ T_m + T_k \leq T_p + T_q \text{ and } Min[T_m, T_k] < Min\left[T_p, T_q\right] \end{Bmatrix}.$$

Corollary 1 will be used to compare the variances of the prices of AV futures with different numbers of reference dates. Although Corollary 1 does not list all possible cases where $Cov\left[S_{T_m}, S_{T_k}\right] < Cov\left[S_{T_p}, S_{T_q}\right]$, it is sufficient for the proofs in this paper.

3.2. Volatility of AV Futures

Here I calculate and compare variances of AV futures contracts' possible prices at a future time t. For convenience, first I refer to Var[X] as $V[X]$ and Cov[X, Y] as $C_v[X, Y]$, hereafter. Second, I assume that the market price at time *t* of an AV futures is equal to its theoretical value at *t* in Proposition 1 by the no-arbitrage principle. Hence $F_{X,t,TH} = F_{X,t}$. Third, plain vanilla futures contracts are a special case of AV futures contracts with n = 1, so I specify X of F_X in terms of "*n*" only. That is, $F_{n,t}$ denotes the market price at *t* of an AV futures, for which the final settlement price is $\frac{1}{n} \cdot \sum_{i=1}^{n} \left(S_{T_i}\right)$. Fourth, the variance of a stock price at a future time or the covariance between stock prices at two different times is always calculated as of now or time 0 unless stated otherwise. That is, $V[S_t|I_0] = V[S_t]$, $V[F_{n,t}|I_0] = V[F_{n,t}]$, and $C_v[S_{T_m}, S_{T_k}|I_0] = C_v[S_{T_m}, S_{T_k}]$, where $t, T_k, T_m > 0$ and I_0 = the information set available at time 0.

Next, for Proposition 2, I need Lemma 2, the proof of which is provided in the Appendix.

Lemma 2 (Variance of (the average of) a series of random variables):

$Let\ F_n = \frac{1}{n}\left[\sum_{i=1}^{n} a_i\right]$, where a_i and a_j have covariance $C_v[a_i, a_j], \forall i, j \leq n$.

Then,

$$V[F_n] - V[F_{n-1}]$$

$$= \frac{1}{n^2(n-1)^2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_n] - C_v[a_i, a_j]\right] + 2(n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_j] - C_v[a_i, a_j]\right]\right].$$

Using Lemma 2, I obtain Proposition 2. The proposition compares variances of two AV futures contracts with different reference dates at a future time *t,* and its proof is provided in the Appendix.

Proposition 2 (Variances of time-t prices of AV futures):

The variance of the price of an AV futures contract with more reference dates at any future time *t* is smaller than that with fewer reference dates given $\boldsymbol{T_{k+1} \leq t < T_k}$.

$$V[F_{n,t}] - V[F_{n-1,t}]$$

$$= \frac{1}{n^2(n-1)^2}$$

$$\times \left[\sum_{j=1}^{n-1} \sum_{i=1}^{n-1} \left[C_v[a_n, a_n] - C_v[a_i, a_j] \right] + 2(n-1) \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} \left[C_v[a_n, a_j] - C_v[a_i, a_j] \right] \right] < 0,$$

$$\text{where } C_v[a_i, a_j] = \begin{cases} C_v[S_t, S_t] e^{r(T_i + T_j - 2t)}, & i, j \le k \\ C_v\left[S_t, S_{T_j}\right] e^{r(T_i - t)}, & i \le k, j \ge k+1 \\ C_v\left[S_{T_i}, S_t\right] e^{r(T_j - t)}, & i \ge k+1, j \le k \\ C_v\left[S_{T_i}, S_{T_j}\right], & i, j \ge k+1 \end{cases}.$$

Meanwhile, investors in AV futures might be interested in not only the volatility of prices but also that of *profits*. Determining the variance of trading profits, however, cannot be done with complete accuracy as of today since there are uncertainties in cash flows to investors' margin accounts while they hold futures positions(from today to liquidation time). Specifically, the profit from trading a futures contract is a function of not only the change in its price but also (the opportunity costs of) its initial margin and future daily cash flows from marking-to-market, which is unknown today. Therefore, in order to compare profits of trading two different AV futures, one assumption is needed: we ignore (uncertain) future daily cash flows in investors' margin accounts and focus on two prices: the current one and the one at a future liquidation time. That is, I compare the variance of $F_{n,t} - F_{n,0}$ with that of $F_{n-1,t} - F_{n-1,0}$. This could be at least partly justified if daily cash flows to an investor's margin account, including initial margins, are small relative to $F_{n,t}, F_{n-1,t}, F_{n,0},$ and $F_{n-1,0}$. More importantly, these daily cash flows themselves are not a relevant factor in comparing profits from trading different AV futures because they will all be eventually recovered when positions in AV futures are liquidated(except for the case of a margin call).[9] The *relevant* cash flows in comparing profits from two AV futures are the *incremental* cash flows, which are investors' *opportunity costs* (*benefits*) of those daily cash flows, i.e., the interest expenses (revenues) from the cash flows. These opportunity costs are *much* smaller than $F_{n,t}, F_{n-1,t}, F_{n,0}$, or $F_{n-1,0}$, and almost negligible.[10] In this sense, comparing two profits from trading two different AV futures without taking into account (opportunity costs of) daily cash flows might not be very problematic.

Next, the volatility of the profit from trading an AV futures contract is the same for a long or a short position since $Var[F_{n,t} - F_{n,0}] = Var[F_{n,0} - F_{n,t}] = Var[F_{n,t}]$, where $F_{n,0}$ is known as of today. Thus assuming either a long or a short position would not matter. Hence, Corollary 2 is presented here as a corollary of Proposition 2:

Corollary 2 (Variance of profit at a future time t):

$$V\left[F_{n,t} - F_{n,0}\right] - V\left[F_{n-1,t} - F_{n-1,0}\right] = V\left[F_{n,t}\right] - V\left[F_{n-1,t}\right] < 0,$$

where $2 \leq n$ and $0 < t \leq T_1$.

3.3. Expected Prices of AV Futures

Here I first compute and compare current or projected future prices of the futures of

[9] Suppose that an investor takes a long position in an AV futures with *n* reference dates and that the future liquidation time *t* is *m* (trading) days from now. If δ denotes the length of one day or $\delta = \frac{1}{260}$, the cumulative cash flows to her margin account up to t would be $\left(F_{n,\delta} - F_{n,0}\right) + \left(F_{n,2\delta} - F_{n,\delta}\right) + \cdots + \left(F_{n,(m-1)\delta} - F_{n,(m-2)\delta}\right) + \left(F_{n,m\delta} - F_{n,(m-1)\delta}\right) = F_{n,m\delta} - F_{n,0} = F_{n,t} - F_{n,0}$, which is the trading profit in our discussion. The initial margin would be similarly recovered at *t*, too.

[10] For instance, the three-month LIBOR is 0.0023 per annum as of July 1, 2014. Note that 0.0023 per annum of daily cash flows is "much much" smaller than $F_{n,t}$ since daily cash flows to margin accounts are already *much* smaller than $F_{n,t}$.

different reference dates as in Proposition 3. Later I work on expected prices or profits of the futures. As before, the expected value of S_t or $F_{n,t}$ is calculated as of time 0 unless stated otherwise. That is, $E[S_t|I_0] = E[S_t]$ and $E\left[F_{n,t}|I_0\right] = E\left[F_{n,t}\right]$, where $0 < t$, and I_0 is the information set available at time 0.

In particular, $i)$ of Proposition 3 means that, of the two AV futures being compared, an AV futures contract with fewer reference dates is more expensive than the other at any time prior to T_n, the earliest reference date of the two futures. Accordingly, any AV futures with n reference dates costs less than its corresponding plain vanilla futures prior to T_n, where $\mathrm{n} > 1$. This can be a meaningful feature of AV futures since investors are more likely to take an initial position in an AV futures before T_n for an effective hedge than after T_n. For convenience, this paper is based on the assumption that an investor takes an initial position at time 0 unless specified otherwise.

Proposition 3 (Time-t prices of AV futures with different reference dates):

An AV futures contract with more reference dates is cheaper than one with fewer reference dates prior to the earliest reference date (case i).

$$F_{n,t} - F_{n-1,t}$$

$$= \left\{ \begin{array}{ll} i)\ \dfrac{S_t e^{r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right) < 0, & where\ 0 \leq t < T_n \\ ii)\ \dfrac{1}{n}\left(S_{T_n} - \dfrac{1}{n-1} \cdot \sum_{i=1}^{n-1} \left(S_t e^{r(T_i - t)}\right)\right), & where\ T_n \leq t < T_{n-1} \\ iii)\ \dfrac{1}{n}\left(S_{T_n} - \dfrac{1}{n-1}\left(\sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} \left(S_t e^{r(T_i - t)}\right)\right)\right), & where\ T_{n-1} \leq t < T_1 \\ & or\ T_{k+1} \leq t < T_k\ and\ 1 \leq k \leq n-2 \\ iv)\ \dfrac{1}{n}\left(S_{T_n} - \dfrac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_i}\right), & where\ t = T_1 \end{array} \right\}.$$

Next, expected prices of AV futures can be shown as in Proposition 4. I find that AV futures contracts' expected prices always decrease with *n*.

Proposition 4 (Expected price at time *t* of an AV futures):

The expected price at any future time t of an AV futures contract with more reference dates is lower than that with fewer reference dates.

$E[F_{n,t}] - E[F_{n-1,t}]$

$$= \begin{Bmatrix} i)\, \dfrac{S_0 \cdot e^{\mu \cdot t + r(T_n - t)}}{n(n-1)} \cdot \displaystyle\sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right) < 0, & where\ 0 \leq\ t < T_n \\ ii)\ \dfrac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \displaystyle\sum_{i=1}^{n-1} \left(1 - e^{\mu(t - T_n) + r(T_i - t)}\right) < 0, & where\ T_n \leq\ t < T_{n-1} \\ iii)\ \dfrac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \left[\displaystyle\sum_{i=k+1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k} \left(1 - e^{\mu(t - T_n) + r(T_i - t)}\right) \right] < 0, & where\ T_{n-1} \leq t < T_1 \\ & or\ T_{k+1} \leq\ t < T_k\ and\ 1 \leq k \leq n-2 \\ iv)\ \dfrac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \displaystyle\sum_{i=1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) < 0, & where\ t = T_1 \end{Bmatrix}.$$

Lastly, let me define the profit from trading an AV futures contract with n reference dates as $F_{n,t} - F_{n,0}$, assuming a long position. Its expected value is $E[F_{n,t}|I_0] - F_{n,0}$. Then Proposition 5 holds, the proof for which is provided in the Appendix.

Proposition 5 (Expected profits in AV futures trading):

$$[E[F_{n,t}] - F_{n,0}] - [E[F_{n-1,t}] - F_{n-1,0}] < 0, \qquad where\ 2 \leq n.$$

4. Conclusion

Most plain vanilla futures, whose settlement prices are set for a specific time or day, are subject to expiration day effects, and will therefore fail to meet various hedging needs of many investors. AV futures, on the other hand, are much less risky in this regard. Compared with arithmetic or geometric average *options*, AV *futures* are also easy to understand, price, and use, and they can be designed to meet the flexible needs of investors. In this paper, I create and price arithmetic average futures as a complement to existing futures, show how these contracts will reduce the risk of price manipulation, explore their features as investment choices, and discuss their market implications. I find that the mean and the variance of an arithmetic average futures' price or profit at a future time are lower than those of its corresponding existing futures and that, in particular, its price or profit volatility is decreasing in the number of reference dates. I also believe that the pricing formulas can shed some light on the pricing of currently available futures such as CAC40 futures, Ibex35 futures, HIS futures, and MSCI TW futures, each of which is a kind of arithmetic average futures contract.

To my knowledge, this paper is the first attempt to price arithmetic average futures contracts, and I hope that further academic and practical research on this subject will continue until a range of these futures are actively traded in over-the-counter or exchange-traded markets.

Appendix

Proof of Proposition 1 (no arbitrage price)

The current time is t, where $T_{k+1} \leq t < T_k$ and $1 \leq k \leq n-1$. Suppose the current market price of an AV futures contract with n reference dates, $F_{n,t}$, is lower than in Proposition 1 by $d(> 0)$. That is $F_{n,t} = \frac{1}{n}\left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t \cdot e^{r(T_i - t)}\right) - d$. In order to take advantage of this opportunity, an arbitrager can take a long position in the futures, whose settlement payoff at T_1 will be (assuming 1 as the contract size of $F_{n,t}$)

$$F_{n,T_1} - F_{n,t} = \frac{1}{n}\left(\sum_{i=1}^{n} S_{T_i}\right) - \left(\frac{1}{n}\left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t \cdot e^{r(T_i - t)}\right) - d\right)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{k} S_{T_i}\right) - \left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{r(T_i - t)}\right) + d.$$

To lock in the arbitrage profit, d, he should trade the underlying shares at time t such that his payoff at T_1 from the shares will be $-\frac{1}{n}\left(\sum_{i=1}^{k} S_{T_i}\right) + \left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{r(T_i - t)}\right) = (let)\ A + B$. For this purpose, he should sell short $\frac{1}{n}\left(\sum_{i=1}^{k} e^{-r(T_1 - T_i)}\right)$ shares at time t, invest the proceeds, $\frac{1}{n}\left(\sum_{i=1}^{k} e^{-r(T_1 - T_i)}\right) S_t$, in a risk-free asset and hold it until T_1. This risk-free investment will generate $\frac{1}{n}\left(\sum_{i=1}^{k} e^{-r(T_1 - T_i)}\right) S_t \cdot e^{r(T_1 - t)} = \frac{1}{n}\left(\sum_{i=1}^{k} e^{r(T_i - t)}\right) S_t = B$ dollars at T_1 and a short position in $\frac{1}{n}\left(\sum_{i=1}^{k} e^{-r(T_1 - T_i)}\right)$ shares after time t. Next, to eliminate this short position by T_1 and create A at T_1, he borrows $\frac{1}{n} S_{T_i} \cdot e^{-r(T_1 - T_i)}$ dollars at each T_i $(1 \leq i \leq k)$, immediately buys $\frac{1}{n} \cdot e^{-r(T_1 - T_i)}$ shares, and delivers them to the counterparty of his short position in the $\frac{1}{n} e^{-r(T_1 - T_i)}$ shares. This series of transactions at each T_i (= borrowing, buying and delivering, $1 \leq i \leq k$) will collectively eliminate his short position in $\frac{1}{n}\left(\sum_{i=1}^{k} e^{-r(T_1-T_i)}\right)$ shares completely, and make him liable for the total debt of $\frac{1}{n}\left[\sum_{i=1}^{k}\left(S_{T_i} \cdot e^{-r(T_1-T_i)}\right) e^{r(T_1-T_i)}\right] = \frac{1}{n}\left(\sum_{i=1}^{k} S_{T_i}\right)$ dollars at T_1. That is, his payoff at T_1 from this series of transactions at each T_i will be collectively $-\frac{1}{n}\left(\sum_{i=1}^{k} S_{T_i}\right) = A$. This completes the proof.

Meanwhile, if $F_{n,t} = \frac{1}{n}\left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t \cdot e^{r(T_i-t)}\right) + d$, where $d > 0$, an arbitrager can take advantage of this arbitrage opportunity by taking a short position in one AV futures contract at t. His short position in the futures will lead to the cash flow at T_1 of

$$-F_{n,T_1} + F_{n,t} = -\frac{1}{n}\left(\sum_{i=1}^{n} S_{T_i}\right) + \left(\frac{1}{n}\left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t \cdot e^{r(T_i-t)}\right) + d\right)$$

$$= -\frac{1}{n}\left(\sum_{i=1}^{k} S_{T_i}\right) + \left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{r(T_i-t)}\right) + d.$$

To lock in the arbitrage profit, d, he should also trade the underlying shares at t such that his payoff at T_1 from the shares will be $\frac{1}{n}\left(\sum_{i=1}^{k} S_{T_i}\right) - \left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{r(T_i-t)}\right) = (let)\ A + B$. For this, he should borrow $\left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{-r(T_1-T_i)}\right)$ dollars and buy $\left(\frac{1}{n}\sum_{i=1}^{k} e^{-r(T_1-T_i)}\right)$ shares at time t. As a result, he should pay back $\left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{-r(T_1-T_i)}\right) e^{r(T_1-t)} = \left(\frac{1}{n}\sum_{i=1}^{k} S_t \cdot e^{r(T_i-t)}\right)$ dollars at T_1, which will create B successfully. Meanwhile, he has $\left(\frac{1}{n}\sum_{i=1}^{k} e^{-r(T_1-T_i)}\right)$ shares at t. At each time $T_i (1 \leq i \leq k)$, he sells $\frac{1}{n} e^{-r(T_1-T_i)}$ shares for $\frac{1}{n} e^{-r(T_1-T_i)} \cdot S_{T_i}$ dollars and invests the proceeds in a risk-free asset and hold it until T_1. These transactions enable him to consume all the shares he has and receive

$\frac{1}{n}\left[\sum_{i=1}^{k}\left(e^{-r(T_1-T_i)}\cdot S_{T_i}\right)e^{r(T_1-T_i)}\right]=\frac{1}{n}\left(\sum_{i=1}^{k}S_{T_i}\right)$ dollars at T_1, which is A. Therefore, his ultimate total cash flow at T_1 from these transactions of shares will be $\mathrm{A}+B=\frac{1}{n}\left(\sum_{i=1}^{k}S_{T_i}\right)-\left(\frac{1}{n}\sum_{i=1}^{k}S_t\cdot e^{r(T_i-t)}\right)$ dollars. This completes the proof.■

Proof of Lemma 2

$F_n=\frac{1}{n}\left[\sum_{i=1}^{n}a_i\right]$. Then $V[F_n]=\frac{1}{n^2}\left[\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]\right]$. Therefore,

$V[F_n]-V[F_{n-1}]$

$$=\frac{1}{n^2}\left[\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]\right]-\frac{1}{(n-1)^2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}C_v[a_i,a_j]\right]$$

$$=\frac{1}{n^2(n-1)^2}\left[(n-1)^2\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]-n^2\cdot\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}C_v[a_i,a_j]\right]$$

$$=(let)\frac{1}{n^2(n-1)^2}[A].$$

Now,

$$A=(n-1)^2\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]-n^2\cdot\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}C_v[a_i,a_j]$$

$$=(n^2-2n+1)\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]-n^2\cdot\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}C_v[a_i,a_j]$$

$$=(n^2)\left[\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]-\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}C_v[a_i,a_j]\right]+(1-2n)\sum_{j=1}^{n}\sum_{i=1}^{n}C_v[a_i,a_j]$$

$$= (n^2)\left[\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j] + \sum_{j=1}^{n} C_v[a_n, a_j] + \sum_{i=1}^{n} C_v[a_i, a_n] - C_v[a_n, a_n]\right]\right.$$

$$\left. - \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j]\right] + (1 - 2n)\sum_{j=1}^{n}\sum_{i=1}^{n} C_v[a_i, a_j]$$

$$= (n^2)\left[\sum_{j=1}^{n} C_v[a_n, a_j] + \sum_{i=1}^{n} C_v[a_i, a_n] - C_v[a_n, a_n]\right] + (1 - 2n)\sum_{j=1}^{n}\sum_{i=1}^{n} C_v[a_i, a_j]$$

$$= 2n^2 \cdot \sum_{j=1}^{n} C_v[a_n, a_j] - n^2 \cdot C_v[a_n, a_n] + (1 - 2n)\sum_{j=1}^{n}\sum_{i=1}^{n} C_v[a_i, a_j]$$

$$\left(\text{since} \sum_{j=1}^{n} C_v[a_n, a_j] = \sum_{i=1}^{n} C_v[a_i, a_n]\right)$$

$$= 2n^2\left[C_v[a_n, a_n] + \sum_{j=1}^{n-1} C_v[a_n, a_j]\right] - n^2 \cdot C_v[a_n, a_n] + (1 - 2n)\sum_{j=1}^{n}\sum_{i=1}^{n} C_v[a_i, a_j]$$

$$= n^2 \cdot C_v[a_n, a_n] + 2n^2 \cdot \sum_{j=1}^{n-1} C_v[a_n, a_j]$$

$$+(1 - 2n)\left[\sum_{j=1}^{n}\sum_{i=1}^{n-1} C_v[a_i, a_j] + \sum_{j=1}^{n-1} C_v[a_n, a_j] + C_v[a_n, a_n]\right]$$

$$= (n^2 + 1 - 2n)C_v[a_n, a_n] + (2n^2 + 1 - 2n)\sum_{j=1}^{n-1} C_v[a_n, a_j] + (1 - 2n)\left[\sum_{j=1}^{n}\sum_{i=1}^{n-1} C_v[a_i, a_j]\right]$$

$$= (n - 1)^2 \cdot C_v[a_n, a_n] + (2n^2 + 1 - 2n)\sum_{j=1}^{n-1} C_v[a_n, a_j]$$

$$+(1 - 2n)\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j] + \sum_{i=1}^{n-1} C_v[a_i, a_n]\right]$$

$$= (n-1)^2 \cdot C_v[a_n, a_n] + (2n^2 + 1 - 2n) \sum_{j=1}^{n-1} C_v[a_n, a_j]$$

$$+(1-2n)\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j] + \sum_{j=1}^{n-1} C_v[a_n, a_j]\right]$$

$$= (n-1)^2 \cdot C_v[a_n, a_n] + (2n^2 + 1 - 2n + 1 - 2n) \sum_{j=1}^{n-1} C_v[a_n, a_j]$$

$$+(1-2n)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j]$$

$$= (n-1)^2 \cdot C_v[a_n, a_n] + 2(n-1)^2 \sum_{j=1}^{n-1} C_v[a_n, a_j] - (2n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j]$$

$$= \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_n, a_n] + 2(n-1)\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_n, a_j]\right] - (2n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j]$$

($\because C_v[a_n, a_n]$ is a constant with respect to i or j, and so is $C_v[a_n, a_j]$ with respect to i)

$$= \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_n, a_n] + 2(n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_n, a_j] - [2(n-1)+1]\sum_{j=1}^{n-1}\sum_{i=1}^{n-1} C_v[a_i, a_j]$$

$$= \sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_n] - C_v[a_i, a_j]\right] + 2(n-1)\left(\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_j] - C_v[a_i, a_j]\right]\right).$$

Hence,

$$V[F_n] - V[F_{n-1}] = \frac{1}{n^2(n-1)^2}[A]$$

$$= \frac{1}{n^2(n-1)^2} \times \left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_n] - C_v[a_i, a_j]\right] + 2(n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_j] - C_v[a_i, a_j]\right]\right].$$

■

Proof of Proposition2

① Suppose that k+2 $\le n$. Given $T_{k+1} \le t < T_k$, this implies $T_{n-1} \le t < T_1$. And

$$F_{n,t} = \frac{1}{n}\left[\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i - t)}\right] = (let)\frac{1}{n}\left[\sum_{i=1}^{n} a_i\right],$$

$$where\ a_i = \begin{Bmatrix} S_t e^{r(T_i-t)}, & 1 \le i \le k \\ S_{T_i}, & k+1 \le i \le n \end{Bmatrix}$$ by Proposition 1. Then,

$$V[F_{n,t}] = \frac{1}{n^2}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} C_v[a_i, a_j]\right],$$

$$where\ C_v[a_i, a_j] = \begin{Bmatrix} C_v[S_t, S_t]e^{r(T_i+T_j-2t)}, & i, j \le k \\ C_v\left[S_t, S_{T_j}\right]e^{r(T_i-t)}, & i \le k, j \ge k+1 \\ C_v\left[S_{T_i}, S_t\right]e^{r(T_j-t)}, & i \ge k+1, j \le k \\ C_v\left[S_{T_i}, S_{T_j}\right], & i, j \ge k+1 \end{Bmatrix}.$$

Also, by Lemma 2,

$$V[F_{n,t}] - V[F_{n-1,t}]$$

$$= \frac{1}{n^2(n-1)^2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_n] - C_v[a_i, a_j]\right] + 2(n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_j] - C_v[a_i, a_j]\right]\right].$$

Then, given $T_n < T_{n-1} < \cdots < T_{k+1} \le t < T_k < \cdots < T_2 < T_1$ and by Corollary 1,

$$C_v[a_n, a_n] - C_v[a_i, a_j]$$

$$= \begin{Bmatrix} C_v\left[S_{T_n}, S_{T_n}\right] - C_v[S_t, S_t]e^{r(T_i+T_j-2t)}, & i, j \le k \\ C_v\left[S_{T_n}, S_{T_n}\right] - C_v\left[S_t, S_{T_j}\right]e^{r(T_i-t)}, & i \le k, k+1 \le j \le n-1 \\ C_v\left[S_{T_n}, S_{T_n}\right] - C_v\left[S_{T_i}, S_t\right]e^{r(T_j-t)}, & k+1 \le i \le n-1, j \le k \\ C_v\left[S_{T_n}, S_{T_n}\right] - C_v\left[S_{T_i}, S_{T_j}\right], & k+1 \le i, j \le n-1 \end{Bmatrix} < 0.$$

Similarly,

$$C_v[a_n, a_j] - C_v[a_i, a_j]$$

$$
= \begin{Bmatrix} C_v[S_{T_n}, S_t]e^{r(T_j-t)} & - \; C_v[S_t, S_t]e^{r(T_i+T_j-2t)}, & i,j \; \leq k \\ C_v\left[S_{T_n}, S_{T_j}\right] & - \; C_v\left[S_t, S_{T_j}\right]e^{r(T_i-t)}, & i \leq k, k+1 \leq j \leq n-1 \\ C_v[S_{T_n}, S_t]e^{r(T_j-t)} & - \; C_v[S_{T_i}, S_t]e^{r(T_j-t)}, & k+1 \leq i \leq n-1, j \leq k \\ C_v\left[S_{T_n}, S_{T_j}\right] & - \; C_v\left[S_{T_i}, S_{T_j}\right], & k+1 \leq i,j \leq n-1 \end{Bmatrix} < 0.
$$

Therefore, $V[F_{n,t}] - V[F_{n-1,t}] < 0 \; where \; n \geq k+2$.

② If $\mathrm{n} = \mathrm{k} + 1$, which implies $T_n \leq t < T_{n-1}$ given $T_{k+1} \leq t < T_k$,

$$
F_{n,t} = \frac{1}{k+1}\left[S_{T_{k+1}} + \sum_{i=1}^{k} S_t e^{r(T_i-t)}\right] \text{and } F_{n-1,t} = \frac{1}{k}\left[\sum_{i=1}^{k} S_t e^{r(T_i-t)}\right]. \text{ Then,}
$$

$$
V\left[F_{n,t}\right] = \frac{1}{(k+1)^2}\left[\sum_{j=1}^{k+1}\sum_{i=1}^{k+1} C_v[a_i, a_j]\right],
$$

$$
where \; C_v[a_i, a_j] = \begin{Bmatrix} C_v[S_t, S_t]e^{r(T_i+T_j-2t)}, & i,j \; \leq k \\ C_v\left[S_t, S_{T_{k+1}}\right]e^{r(T_i-t)}, & i \leq k, j = k+1 \\ C_v\left[S_{T_{k+1}}, S_t\right]e^{r(T_j-t)}, & i = k+1, j \leq k \\ C_v\left[S_{T_{k+1}}, S_{T_{k+1}}\right], & i,j = k+1 \end{Bmatrix}.
$$

Also,

$$
V\left[F_{n-1,t}\right] = \frac{1}{k^2}\left[\sum_{j=1}^{k}\sum_{i=1}^{k} C_v[a_i, a_j]\right] = \frac{1}{k^2}\left[\sum_{j=1}^{k}\sum_{i=1}^{k} C_v[S_t, S_t]e^{r(T_i+T_j-2t)}\right].
$$

Then, by Lemma 2,

$$
V[F_{n,t}] - V[F_{n-1,t}]
$$

$$
= \frac{1}{(k+1)^2 \cdot k^2}\left[\sum_{j=1}^{k}\sum_{i=1}^{k}\left[C_v[a_{k+1}, a_{k+1}] - C_v[a_i, a_j]\right] + 2k\sum_{j=1}^{k}\sum_{i=1}^{k}\left[C_v[a_{k+1}, a_j] - C_v[a_i, a_j]\right]\right].
$$

Then, given $T_{k+1} \leq t < T_k < \cdots < T_2 < T_1$ and by Corollary 1,

$C_v[a_{k+1}, a_{k+1}] - C_v[a_i, a_j] = C_v\left[S_{T_{k+1}}, S_{T_{k+1}}\right] - \; C_v[S_t, S_t]e^{r(T_i+T_j-2t)} < 0$, where $i,j \; \leq k$.

Also,

$C_v[a_{k+1}, a_j] - C_v[a_i, a_j] = C_v[S_{T_{k+1}}, S_t]e^{r(T_j - t)} - C_v[S_t, S_t]e^{r(T_i + T_j - 2t)} < 0$, where $i, j \le k$.

Therefore, $V[F_{n,t}] - V[F_{n-1,t}] < 0, where\ n = k + 1$.

③ Suppose that n ≤ k, which implies $t < T_n$ given $T_{k+1} \le t < T_k$.

Then, by Proposition 1,

$$F_{n,t} = \frac{1}{n}\sum_{i=1}^{n} S_t e^{r(T_i - t)} = (let)\frac{1}{n}\sum_{i=1}^{n} a_i.$$

And,

$$V[F_{n,t}] = \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} C_v[a_i, a_j]\right], \text{where} C_v[a_i, a_j] = C_v[S_t, S_t]e^{r(T_i + T_j - 2t)}.$$

Also, by Lemma 2,

$$V[F_{n,t}] - V[F_{n-1,t}]$$

$$= \frac{1}{n^2(n-1)^2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_n] - C_v[a_i, a_j]\right] + 2(n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_j] - C_v[a_i, a_j]\right]\right].$$

Given $T_n < T_{n-1} < \cdots < T_{k+1} \le t < T_k < \cdots < T_2 < T_1$ and by Corollary 1,

$$C_v[a_n, a_n] - C_v[a_i, a_j]$$

$$= C_v[S_t, S_t]e^{r(T_n + T_n - 2t)} - C_v[S_t, S_t]e^{r(T_i + T_j - 2t)} < 0, \text{where } 1 \le i, j \le n - 1 \le k - 1.$$

Similarly,

$$C_v[a_n, a_j] - C_v[a_i, a_j]$$

$$= C_v[S_t, S_t]e^{r(T_n + T_j - 2t)} - C_v[S_t, S_t]e^{r(T_i + T_j - 2t)} < 0, \text{where } 1 \le i, j \le n - 1 \le k - 1.$$

Therefore,

$$V[F_{n,t}] - V[F_{n-1,t}] < 0, \text{where } n \le k.$$

④ Suppose $t = T_1$. Then

$F_{n,t} = \frac{1}{n}\left[\sum_{i=1}^{n} S_{T_i}\right]$ and $F_{n-1,t} = \frac{1}{n-1}\left[\sum_{i=1}^{n-1} S_{T_i}\right]$ by Proposition 1. Then,

$$V\left[F_{n,t}\right] = \frac{1}{n^2}\left[\sum_{j=1}^{n}\sum_{i=1}^{n} C_v\left[S_{T_i}, S_{T_j}\right]\right].$$

Also, by Lemma 2,

$V[F_{n,t}] - V[F_{n-1,t}]$

$$= \frac{1}{n^2(n-1)^2}\left[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_n] - C_v[a_i, a_j]\right] + 2(n-1)\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\left[C_v[a_n, a_j] - C_v[a_i, a_j]\right]\right].$$

Then, given $T_n < T_{n-1} < \cdots < T_2 < T_1$ and by Corollary 1,

$$C_v[a_n, a_n] - C_v[a_i, a_j] = C_v\left[S_{T_n}, S_{T_n}\right] - C_v\left[S_{T_i}, S_{T_j}\right] < 0, \forall i, j \leq n-1.$$

Similarly,

$$C_v[a_n, a_j] - C_v[a_i, a_j] = C_v\left[S_{T_n}, S_{T_j}\right] - C_v\left[S_{T_i}, S_{T_j}\right] < 0, \forall i \leq n-1.$$

Therefore, $V[F_{n,t}] - V[F_{n-1,t}] < 0 \; where \; t = T_1$.

And this completes the proof. ■

Proof of Proposition 3

i) $0 \leq t < T_n$:

$F_{n,t} = \frac{1}{n}\cdot\sum_{i=1}^{n} S_t \cdot e^{r(T_i - t)} = \frac{1}{n}\cdot\left(S_t \cdot e^{r(T_n - t)} + \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}\right)$. Thus,

$$F_{n,t} - F_{n-1,t} = \frac{1}{n}\cdot\left(S_t \cdot e^{r(T_n - t)} + \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}\right) - \frac{1}{n-1}\cdot\sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}$$

$$= \frac{S_t \cdot e^{r(T_n - t)}}{n} + \left(\frac{1}{n} - \frac{1}{n-1}\right) \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}$$

$$= \frac{S_t \cdot e^{r(T_n - t)}}{n} - \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}$$

$$= \frac{(n-1) \cdot S_t \cdot e^{r(T_n - t)}}{n(n-1)} - \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}$$

$$= \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_n - t)} - \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}$$

$(since\ S_t \cdot e^{r(T_n - t)}\ is\ a\ constant\ with\ respect\ to\ i)$

$$= \frac{S_t e^{r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right) < 0, \text{where } t < T_n < T_i, \forall i \le n-1.$$

ii) $T_n \le t < T_{n-1}$:

$$F_{n,t} - F_{n-1,t} = \frac{1}{n}\left(S_{T_n} + \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}\right) - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}$$

$$= \frac{1}{n}\left(S_{T_n} + \left(1 - \frac{n}{n-1}\right) \cdot \sum_{i=1}^{n-1} S_t \cdot e^{r(T_i - t)}\right)$$

$$= \frac{1}{n}\left(S_{T_n} - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} \left(S_t e^{r(T_i - t)}\right)\right), \text{where} T_i - t > 0, \forall i \le n-1.$$

The sign of $F_n - F_{n-1}$ can be anything, depending on the realized values of S_{T_n} and S_t.

iii) $T_{k+1} \le t < T_k (1 \le k \le n-2)$:

$$F_{n,t} - F_{n-1,t} = \frac{1}{n}\left(\sum_{i=k+1}^{n} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i - t)}\right) - \frac{1}{n-1}\left(\sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i - t)}\right)$$

$$= \frac{1}{n}\left(S_{T_n} + \sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i-t)}\right) - \frac{1}{n-1}\left(\sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i-t)}\right)$$

$$= \frac{1}{n}\left(S_{T_n} + \left(1 - \frac{n}{(n-1)}\right) \cdot \sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i-t)}\right)$$

$$= \frac{1}{n}\left(S_{T_n} - \frac{1}{(n-1)}\left(\sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i-t)}\right)\right).$$

The sign of $F_n - F_{n-1}$ can be anything, depending on the realized values of S_{T_i}'s and S_t, where $k + 1 \leq i \leq n$.

iv) $t = T_1$:

$$F_{n,t} - F_{n-1,t} = \frac{1}{n} \cdot \sum_{i=1}^{n} S_{T_i} - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_i}$$

$$= \frac{1}{n}\left(S_{T_n} + \sum_{i=1}^{n-1} S_{T_i}\right) - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_i}$$

$$= \frac{1}{n}\left(S_{T_n} + \left(1 - \frac{1}{n-1}\right) \cdot \sum_{i=1}^{n-1} S_{T_i}\right)$$

$$= \frac{1}{n}\left(S_{T_n} - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_i}\right).$$

Again, the sign of this can be anything, depending on the realized values of S_{T_i}'s, where $1 \leq i \leq n$, and this completes the proof. ■

Proof of Proposition 4

i) $0 \leq t < T_n$:

$$E[F_{n,t}] - E\big[F_{n-1,t}\big] = E[F_{n,t} - F_{n-1,t}]$$

$$= \ E\left[\frac{S_{\mathrm{t}}e^{r(T_n-t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1}\left(1 - e^{r(T_i-T_n)}\right)\right] \text{ (by Proposition 3)}$$

$$= E[S_{\mathrm{t}}] \cdot \frac{e^{r(T_n-t)}}{n(n-1)} \cdot \left(\sum_{i=1}^{n-1}\left(1 - e^{r(T_i-T_n)}\right)\right)$$

$$= \ \frac{S_0 \cdot e^{\mu \cdot t + r(T_n-t)}}{n(n-1)} \cdot \left(\sum_{i=1}^{n-1}\left(1 - e^{r(T_i-T_n)}\right)\right) < 0, \qquad where\ T_n < T_i, \ \ \forall i \leq n-1.$$

$ii)\ T_n \leq \ t < T_{n-1}$:

$$E[F_{n,t}] - E\big[F_{n-1,t}\big] = E\left[\frac{1}{n}\left(S_{T_n} \ - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1}\left(S_{\mathrm{t}}e^{r(T_i-t)}\right)\right)\right] \text{(by Proposition 3)}$$

$$= E\left[\frac{1}{n}\left(\frac{1}{n-1} \cdot (n-1)S_{T_n} \ - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1}\left(S_{\mathrm{t}}e^{r(T_i-t)}\right)\right)\right]$$

$$= E\left[\frac{1}{n}\left(\frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_n} \ - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1}\left(S_{\mathrm{t}}e^{r(T_i-t)}\right)\right)\right]$$

$$= E\left[\frac{1}{n}\left(\frac{1}{n-1} \cdot \sum_{i=1}^{n-1}\left(S_{T_n} - S_{\mathrm{t}}e^{r(T_i-t)}\right)\right)\right]$$

$$= \frac{1}{n(n-1)} \cdot E\left[\sum_{i=1}^{n-1}\left(S_{T_n} - S_{\mathrm{t}}e^{r(T_i-t)}\right)\right]$$

$$= \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} E\left(S_{T_n} - S_{\mathrm{t}}e^{r(T_i-t)}\right)$$

$$= \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1}\left[E\left(S_{T_n}\right) - E\left(S_{\mathrm{t}}e^{r(T_i-t)}\right)\right]$$

$$= \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left[S_0 e^{\mu \cdot T_n} - S_0 e^{\mu \cdot t + r(T_i - t)} \right]$$

$$= \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left[S_0 e^{\mu \cdot T_n} \left(1 - e^{\mu(t - T_n) + r(T_i - t)} \right) \right]$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(t - T_n) + r(T_i - t)} \right) < 0, where\ T_n \le t < T_{n-1} < T_{n-2} < \cdots < T_1.$$

iii) $T_{n-1} \le t < T_1$:

In other words, the range of t is $T_{k+1} \le t < T_k, where\ 1 \le k \le n-2.$

$E[F_{n,t}] - E[F_{n-1,t}]$

$$= E\left[\frac{1}{n} \left(S_{T_n} - \frac{1}{n-1} \left(\sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i - t)} \right) \right) \right] \text{ (by Proposition 3)}$$

$$= E\left[\frac{1}{n} \left(\frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_n} - \frac{1}{n-1} \left(\sum_{i=k+1}^{n-1} S_{T_i} + \sum_{i=1}^{k} S_t e^{r(T_i - t)} \right) \right) \right]$$

$$= E\left[\frac{1}{n} \left(\frac{1}{n-1} \left(\sum_{i=1}^{n-1} S_{T_n} - \sum_{i=k+1}^{n-1} S_{T_i} - \sum_{i=1}^{k} S_t e^{r(T_i - t)} \right) \right) \right]$$

$$= \frac{1}{n(n-1)} \cdot E\left[\sum_{i=1}^{n-1} S_{T_n} - \sum_{i=k+1}^{n-1} S_{T_i} - \sum_{i=1}^{k} S_t e^{r(T_i - t)} \right]$$

$$= \frac{1}{n(n-1)} \left[\sum_{i=1}^{n-1} E[S_{T_n}] - \sum_{i=k+1}^{n-1} E[S_{T_i}] - \sum_{i=1}^{k} E[S_t e^{r(T_i - t)}] \right]$$

$$= \frac{1}{n(n-1)} \left[\sum_{i=1}^{n-1} S_0 e^{\mu \cdot T_n} - \sum_{i=k+1}^{n-1} S_0 e^{\mu \cdot T_i} - \sum_{i=1}^{k} S_0 e^{\mu \cdot t + r(T_i - t)} \right]$$

$$= \frac{1}{n(n-1)} \left[\left(\sum_{i=k+1}^{n-1} S_0 e^{\mu \cdot T_n} + \sum_{i=1}^{k} S_0 e^{\mu \cdot T_n} \right) - \sum_{i=k+1}^{n-1} S_0 e^{\mu \cdot T_i} - \sum_{i=1}^{k} S_0 e^{\mu \cdot t + r(T_i - t)} \right]$$

$$= \frac{1}{n(n-1)}\left[\sum_{i=k+1}^{n-1} S_0 e^{\mu \cdot T_n} - \sum_{i=k+1}^{n-1} S_0 e^{\mu \cdot T_i} + \sum_{i=1}^{k} S_0 e^{\mu \cdot T_n} - \sum_{i=1}^{k} S_0 e^{\mu \cdot t + r(T_i - t)}\right]$$

$$= \frac{1}{n(n-1)}\left[\sum_{i=k+1}^{n-1} S_0 e^{\mu \cdot T_n}\left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k} S_0 e^{\mu \cdot T_n}\left(1 - e^{\mu(t - T_n) + r(T_i - t)}\right)\right]$$

$$= \frac{1}{n(n-1)}\left[S_0 e^{\mu \cdot T_n}\left(\sum_{i=k+1}^{n-1}\left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k}\left(1 - e^{\mu(t - T_n) + r(T_i - t)}\right)\right)\right]$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)}\left[\sum_{i=k+1}^{n-1}\left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k}\left(1 - e^{\mu(t - T_n) + r(T_i - t)}\right)\right] < 0,$$

where $T_n < t, T_n < T_i, \forall i \leq n-1,$ and $t < T_i, \forall i \leq k$.

$iv)$ $t = T_1$:

$$E[F_{n,t}] - E[F_{n-1}, t] = E\left[\frac{1}{n}\left(S_{T_n} - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_i}\right)\right] \text{ (by Proposition 3)}$$

$$= E\left[\frac{1}{n}\left(\frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_n} - \frac{1}{n-1} \cdot \sum_{i=1}^{n-1} S_{T_i}\right)\right]$$

$$= E\left[\frac{1}{n}\left(\frac{1}{n-1} \cdot \sum_{i=1}^{n-1}\left(S_{T_n} - S_{T_i}\right)\right)\right] = \frac{1}{n(n-1)} \cdot E\left[\sum_{i=1}^{n-1}\left(S_{T_n} - S_{T_i}\right)\right]$$

$$= \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1}\left(E[S_{T_n}] - E[S_{T_i}]\right) = \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1}\left(S_0 e^{\mu \cdot T_n} - S_0 e^{\mu \cdot T_i}\right)$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1}\left(1 - e^{\mu(T_i - T_n)}\right) < 0, \qquad \text{where } T_n < T_i, \forall i \leq n-1.$$

This completes the proof of Proposition 4. ■

Proof of Proposition 5

Suppose $T_{k+1} \leq t < T_k$, where $1 \leq k \leq n-2$. This means $k+2 \leq n$ and $T_{n-1} \leq t < T_1$, and this case belongs to case $iii)$ of Proposition4. Then

$$[E[F_{n,t}] - F_{n,0}] - [E[F_{n-1,t}] - F_{n-1,0}] = E[F_{n,t}] - E[F_{n-1,t}] - [F_{n,0} - F_{n-1,0}]$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \left(\sum_{i=k+1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k} \left(1 - e^{\mu(t-T_n)+r(T_i-t)}\right) \right)$$

$$- \frac{S_0 e^{r \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right) \text{ (by } iii) \text{ of Proposition 4 and } i) \text{ of Proposition 3)}$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \left(\sum_{i=k+1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k} \left(1 - e^{\mu(t-T_n)+r(T_i-t)}\right) \right)$$

$$+ \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} (S_0 e^{r \cdot T_i} - S_0 e^{r \cdot T_n})$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \times$$

$$\left(\sum_{i=k+1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) + \sum_{i=1}^{k} \left(1 - e^{\mu(t-T_n)+r(T_i-t)}\right) + \sum_{i=1}^{n-1} (e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}) \right)$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \left(\sum_{i=k+1}^{n-1} \left(1 - e^{\mu(T_i - T_n)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}\right) \right.$$

$$\left. + \sum_{i=1}^{k} \left(1 - e^{\mu(t-T_n)+r(T_i-t)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}\right) \right)$$

$$= (let) \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \left[\sum_{i=k+1}^{n-1} A_i + \sum_{i=1}^{k} B_i \right],$$

$where\ A_i = 1 - e^{\mu \cdot (T_i - T_n)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}$

$and\ B_i = 1 - e^{\mu(t-T_n)+r(T_i-t)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}$.

I will show $\left[E[F_{n,t}]-F_{n,0}\right]-\left[E[F_{n-1,t}]-F_{n-1,0}\right]<0$ by demonstrating that $A_i<0\ and\ B_i<0, \forall i$. To prove $A_i<0$, let us say $\mu=r+\alpha, and\ T_i=T_n+\beta_i$, where $0<\alpha$ and $0<\beta_i$ given $k+1\le i\le n-1$. Then,

$A_i = 1-e^{(r+\alpha)\beta_i}+e^{-(r+\alpha)T_n+r(T_n+\beta_i)}-e^{-(r+\alpha)T_n+r\cdot T_n}$

$= 1-e^{(r+\alpha)\beta_i}+e^{-\alpha\cdot T_n+r\cdot\beta_i}-e^{-\alpha\cdot T_n}$

$= e^{r\cdot\beta_i}\left(-e^{\alpha\cdot\beta_i}+e^{-\alpha\cdot T_n}\right)-e^{-\alpha\cdot T_n}+1$

$< e^{r\cdot\beta_i}\left(-e^{\alpha\cdot\beta_i}+e^{-\alpha\cdot T_n}\right)-e^{-\alpha\cdot T_n}+e^{\alpha\cdot\beta_i}=\left(e^{r\cdot\beta_i}-1\right)\left(-e^{\alpha\cdot\beta_i}+e^{-\alpha\cdot T_n}\right)<0,$

since $e^{r\cdot\beta_i}>1, \left(-e^{\alpha\cdot\beta_i}\right)<(-1)$, and $e^{-\alpha\cdot T_n}<1$.

Next, I prove $B_i<0$, where $1\le i\le k$. Let us say

$T_i-T_n=(T_i-t)+(t-T_n)=\beta_{i1}+\beta_{i2}= \beta_i$, where$T_i-t=\beta_{i1}>0, \forall i\le k$, and

$t-T_n=\beta_{i2}>0$. Then

$B_i = 1-e^{\mu(t-T_n)+r(T_i-t)}+e^{-\mu\cdot T_n+r\cdot T_i}-e^{-\mu\cdot T_n+r\cdot T_n}$

$= 1-e^{(r+\alpha)\beta_{i2}+r\cdot\beta_{i1}}+e^{-(r+\alpha)\cdot T_n+r\cdot(T_n+\beta_i)}-e^{-(r+\alpha)\cdot T_n+r\cdot T_n}$

$= 1-e^{\alpha\cdot\beta_{i2}+r\cdot\beta_i}+e^{-\alpha\cdot T_n+r\cdot\beta_i}-e^{-\alpha\cdot T_n}$

$= \left(1-e^{\alpha\cdot\beta_{i2}+r\cdot\beta_i}\right)+e^{-\alpha\cdot T_n}\left(e^{r\cdot\beta_i}-1\right)$

$< \left(1-e^{\alpha\cdot\beta_{i2}+r\cdot\beta_i}\right)+e^{-\alpha\cdot T_n}\left(e^{r\cdot\beta_i+\alpha\cdot\beta_{i2}}-1\right)=\left(1-e^{\alpha\cdot\beta_{i2}+r\cdot\beta_i}\right)\left(1-e^{-\alpha\cdot T_n}\right)<0,$

where $\alpha, \beta_{i2}, r, \beta_i$, and $T_n>0$.

Next, suppose $T_n\le t<T_{n-1}$, which belongs to case ii) of Proposition 4. Then

$\left[E[F_{n,t}]-F_{n,0}\right]-\left[E[F_{n-1,t}]-F_{n-1,0}\right]=E[F_{n,t}]-E[F_{n-1,t}]-\left[F_{n,0}-F_{n-1,0}\right]$

$$= \frac{S_0e^{\mu\cdot T_n}}{n(n-1)}\cdot\sum_{i=1}^{n-1}\left(1-e^{\mu(t-T_n)+r(T_i-t)}\right)-\frac{S_0e^{r\cdot T_n}}{n(n-1)}\cdot\sum_{i=1}^{n-1}\left(1-e^{r(T_i-T_n)}\right)$$

(by ii) of Proposition 4 and i) of Proposition 3)

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(t-T_n)+r(T_i-t)}\right) + \frac{1}{n(n-1)} \cdot \sum_{i=1}^{n-1} (S_0 e^{r \cdot T_i} - S_0 e^{r \cdot T_n})$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(t-T_n)+r(T_i-t)}\right) - \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} (e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n})$$

$$= \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(t-T_n)+r(T_i-t)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}\right)$$

$$= (let) \frac{S_0 e^{\mu \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} B_i, \text{ where} B_i = 1 - e^{\mu(t-T_n)+r(T_i-t)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}.$$

I will show $[E[F_{n,t}] - F_{n,0}] - [E[F_{n-1,t}] - F_{n-1,0}] < 0$ by showing that $B_i < 0, \forall i$.

Again, $\mu = r + \alpha, T_i - t = \beta_{i1} > 0, t - T_n = \beta_{i2} > 0$

and $T_i - T_n = (T_i - t) + (t - T_n) = \beta_{i1} + \beta_{i2} = \beta_i > 0$ given $1 \leq i \leq n-1$. Then,

$$B_i = 1 - e^{\mu(t-T_n)+r(T_i-t)} + e^{-\mu \cdot T_n + r \cdot T_i} - e^{-\mu \cdot T_n + r \cdot T_n}$$

$$= 1 - e^{(r+\alpha)\beta_{i2} + r \cdot \beta_{i1}} + e^{-(r+\alpha) \cdot T_n + r \cdot (T_n + \beta_i)} - e^{-(r+\alpha) \cdot T_n + r \cdot T_n}$$

$$= 1 - e^{\alpha \cdot \beta_{i2} + r \cdot \beta_i} + e^{-\alpha \cdot T_n + r \cdot \beta_i} - e^{-\alpha \cdot T_n}$$

$$= \left(1 - e^{\alpha \cdot \beta_{i2} + r \cdot \beta_i}\right) + e^{-\alpha \cdot T_n}\left(e^{r \cdot \beta_i} - 1\right)$$

$$< \left(1 - e^{\alpha \cdot \beta_{i2} + r \cdot \beta_i}\right) + e^{-\alpha \cdot T_n}\left(e^{r \cdot \beta_i + \alpha \cdot \beta_{i2}} - 1\right) = \left(1 - e^{\alpha \cdot \beta_{i2} + r \cdot \beta_i}\right)(1 - e^{-\alpha \cdot T_n}) < 0,$$

where $\alpha, \beta_{i2}, r, \beta_i$, and $T_n > 0$.

Next suppose $0 \leq t < T_n$, which belongs to case i) of Proposition 4. Then

$$[E[F_{n,t}] - F_{n,0}] - [E[F_{n-1,t}] - F_{n-1,0}] = E[F_{n,t}] - E[F_{n-1,t}] - [F_{n,0} - F_{n-1,0}]$$

$$= \frac{S_0 e^{\mu \cdot t + r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right) - \frac{S_0 e^{r \cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right)$$

(by i) of Proposition 4 and i) of Proposition 3)

$$= \frac{S_0 e^{\mu\cdot t + r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right) - \frac{S_0 e^{\mu\cdot t + r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(e^{-\mu\cdot t + r\cdot t} - e^{-\mu\cdot t + r\cdot t - r(T_n - T_i)}\right)$$

$$= \frac{S_0 e^{\mu\cdot t + r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)} - e^{-\mu\cdot t + r\cdot t} + e^{-\mu\cdot t + r\cdot t - r(T_n - T_i)}\right)$$

$$= (let) \frac{S_0 e^{\mu\cdot t + r(T_n - t)}}{n(n-1)} \cdot \sum_{i=1}^{n-1} A_i, \text{ where} A_i = 1 - e^{r(T_i - T_n)} - e^{-\mu\cdot t + r\cdot t} + e^{-\mu\cdot t + r\cdot t - r(T_n - T_i)}.$$

Again, $\mu = r + \alpha$ and $T_i = T_n + \beta_i, where\ 0 < \alpha$ and $0 < \beta_i \ \ given\ 1 \le i \le n-1$. Then,

$$A_i = 1 - e^{r(T_i - T_n)} + e^{r(T_i - T_n) + t(r-\mu)} - e^{t(r-\mu)}$$

$$= 1 - e^{r\cdot\beta_i} + e^{r\cdot\beta_i - t\alpha} - e^{-t\alpha}$$

$$= \left(1 - e^{r\cdot\beta_i}\right) - e^{-t\alpha}\left(1 - e^{r\cdot\beta_i}\right)$$

$$= \left(1 - e^{r\cdot\beta_i}\right)(1 - e^{-t\alpha}) < 0, since\ 1 - e^{r\cdot\beta_i} < 0, \text{and } 1 - e^{-t\alpha} > 0.$$

Last, suppose $t = T_1$, which belongs to case $iv)$ of Proposition 4. Then

$$\left[E[F_{n,t}] - F_{n,0}\right] - \left[E[F_{n-1,t}] - F_{n-1,0}\right] = E[F_{n,t}] - E[F_{n-1,t}] - \left[F_{n,0} - F_{n-1,0}\right]$$

$$= \frac{S_0 e^{\mu\cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) - \frac{S_0 e^{r\cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{r(T_i - T_n)}\right)$$

(by $iv)$ of Proposition 4 and $i)$ of Proposition 3)

$$= \frac{S_0 e^{\mu\cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(T_i - T_n)}\right) - \frac{S_0 e^{\mu\cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} (e^{-\mu\cdot T_n + r\cdot T_n} - e^{-\mu\cdot T_n + r\cdot T_i})$$

$$= \frac{S_0 e^{\mu\cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} \left(1 - e^{\mu(T_i - T_n)} + e^{-\mu\cdot T_n + r\cdot T_i} - e^{-\mu\cdot T_n + r\cdot T_n}\right)$$

$$= (let) \frac{S_0 e^{\mu\cdot T_n}}{n(n-1)} \cdot \sum_{i=1}^{n-1} B_i, \text{ where } B_i = 1 - e^{\mu(T_i - T_n)} + e^{-\mu\cdot T_n + r\cdot T_i} - e^{-\mu\cdot T_n + r\cdot T_n}.$$

Again, $\mu = r + \alpha$ and $T_i = T_n + \beta_i$, where $0 < \alpha$ and $0 < \beta_i$given $1 \le i \le n-1$. Then

$$B_i = 1 - e^{\mu(T_i - T_n)} + e^{-\mu\cdot T_n + r\cdot T_i} - e^{-\mu\cdot T_n + r\cdot T_n}$$

$$= 1 - e^{(r+\alpha)\beta_i} + e^{-(r+\alpha)\cdot T_n + r\cdot(T_n+\beta_i)} - e^{-(r+\alpha)\cdot T_n + r\cdot T_n}$$

$$= 1 - e^{\alpha\cdot\beta_i + r\cdot\beta_i} + e^{-\alpha\cdot T_n + r\cdot\beta_i} - e^{-\alpha\cdot T_n}$$

$$= \left(1 - e^{\alpha\cdot\beta_i + r\cdot\beta_i}\right) + e^{-\alpha\cdot T_n}\left(e^{r\cdot\beta_i} - 1\right)$$

$$< \left(1 - e^{\alpha\cdot\beta_i + r\cdot\beta_i}\right) + e^{-\alpha\cdot T_n}\left(e^{r\cdot\beta_i + \alpha\cdot\beta_i} - 1\right) = \left(1 - e^{\alpha\cdot\beta_i + r\cdot\beta_i}\right)(1 - e^{-\alpha\cdot T_n}) < 0,$$

where $\alpha, \beta_i, r,$ and $T_n > 0$.

Then, $\left[E[F_{n,t}] - F_{n,0}\right] - \left[E[F_{n-1,t}] - F_{n-1,0}\right] < 0$ for any future time t and for any AV futures with $2 \leq$ n. This concludes the proof. ■

References

Chamberlain, Trevor, Sherman Cheung, and Clarence Kwan, 1989, Expiration-day effects of Index Futures and Options: Some Canadian Evidence, *Financial Analyst Journal,* 45, 67-71

Hsieh, Shu-Fan, and Tai Ma, 2009, Expiration-day effects: Does settlement price matter?, *International Review of Economics and Finance,* 18, 290-300

Hsieh, Wen-Liang, 2009, Expiration-day effects on individual stocks and the overall market: Evidence from Taiwan, *The Journal of Futures Markets*, 29, 920-945

Hull, John, 2012, *Risk Management and Financial Institutions*, third edition, John Wiley & Sons, Hoboken, New Jersey

Hull, John, 2009, *Options, futures, and other derivatives*, seventh edition, Pearson Education, Upper Saddle River, New Jersey

Illueca, M., and J. Lafuente, 2006, New evidence on expiration-day effects using realized volatility: an intraday analysis for the Spanish stock market, *The Journal of Futures Markets,* 26, 923-938

Kemna, A., and A. Vorst (1990), A Pricing model for options based on average asset values, *Journal of Banking and Finance*, 14, 113-129

Milevsky, Moshe, and Steven Posner, 1998, Asian options, the sum of lognormals, and the reciprocal gamma distribution, *Journal of Financial and Quantitative Analysis,* 33, 409-422

Schlag, Christian, 1996, Expiration-day effects of stock index derivatives in Germany, *European Financial Management*, 1, 69-95

Stoll, Hans, and Robert Whaley, 1987, Program trading and expiration-day effects, *Financial Analyst Journal*, 16-28

Stoll, Hans, and Robert Whaley, 1997, Expiration-day effects of the all ordinary share price index futures; empirical evidence and alternative settlement procedures, *Australian Journal of Management*, 22, 139-174

Turnbull, Stuart, and Lee Wakeman, 1991, A quick algorithm for pricing European Average options, *Journal of Financial and Quantitative Analysis,* 26, 377-389

Vipul, 2005, Futures and options expiration-day effects: The Indian evidence, *The Journal of Futures Markets,* 25, 1045-1065