Option Pricing with Heavy Tailed Distribution : Application to Barrier Options

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Abstract

Although many market participants use the Black-Scholes (BS) model, there is a lot of evidence that the BS model does not reflect the skewness or kurtosis of the return distribution correctly. Under the Generalized Extreme Value (GEV) model, Markose and Alerton [11] derives the closed-form solutions for vanilla options, and also removes the distortion of the market only with an additional parameter. In this paper, we use the technique in Rubinstein and Reiner [13] to get the closed-form solutions for barrier options by introducing the Corrected BS (CBS) model – the GEV model closed to the BS model. Also, by introducing CBS volatility we show that barrier option prices are continuous to barriers under GEV model. Then, we compare those solutions with the BS model during the global credit crisis. As the BS model underestimate the probability of barrier hit while the credit crisis, we may infer that the traditional BS model undervalues in-type barrier options and overvalues out-type barrier options.

Keywords: Barrier option pricing, Heavy tailed distribution, Generalized extreme value (GEV) distribution, Global credit crisis
JEL Classification: C58, G01, G13

Highlights

- We derive the explicit formulas for barrier options under the assumption that the return distributions of underlying asset follow the Generalized Extreme Value (GEV) distribution.
- We find a way to connect Markose and Alerton [11]'s work on GEV model and Rubinstein and Reiner [13]'s technique to derive closed-form solutions of barrier options by introducing the Corrected BS (CBS) model – the GEV model closed to the BS model.
- We show that the introduction of CBS volatility makes the prices of barrier options continuous on the barrier level.
- We derive missing equations and rectify errors in the work of Markose and Alerton [11], which has some equations unsolved and contains some minor mistakes.

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1. Introduction

According to Jackwerth and Rubinstein [8], the stock market crash of October 1987 was an extreme event with probability of 10^{-160}, which is virtually impossible with the Gaussian type thin-tailed distribution models. Two years later, on October 13, 1989, the S&P 500 index fell about 6%, which has a probability of 2.7×10^{-7} and should happen only once in 14,756 years under the same hypothesis. Despite that Gaussian type thin-tailed distributions play an essential role in mathematical finance such as Black-Scholes option pricing or the calculation of Value at Risk, it is time to rethink the early suggestion by Mandelbrot [10] on the possibility of heavy tails in financial data (Kim and Kim [9]).

The celebrated Black-Scholes (BS) model [1] has been widely used in derivatives markets, because the model can evaluate plain vanilla options in markets with simple intuitive parameters. However, when the markets fluctuate sharply, as the BS model is unable to reflect the skewness and kurtosis of the markets, the difference between the values of the vanilla call options based on the BS model and the actual prices of the markets grows rapidly. For example, when the distribution is skewed to the left, there are serious mismatch in valuating down-and-in barrier options based on the BS model assumption since it underestimates the probability of reaching the barrier.

Recently, Markose and Alentorn [11] proposed a new vanilla option pricing model based on Generalized Extreme Value (GEV) distribution. Extreme value theory has been widely used in hydrology, climatology and insurance areas (See [2], [3], [5]). At the heart of the theory, there are two types of distributions, GEV and Generalized Pareto (GPD) distributions, which can be justified by the some limit theorems[1].

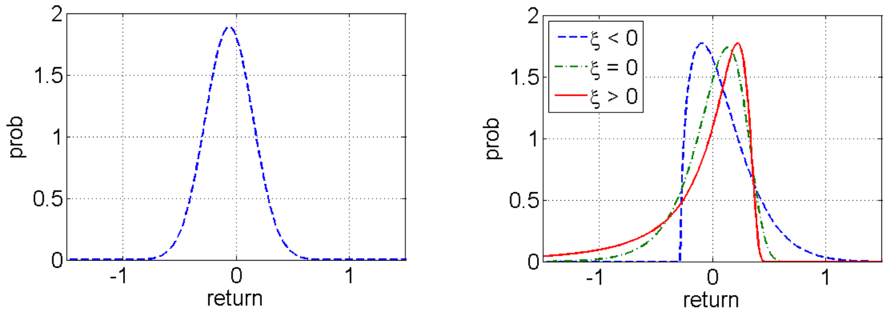
The barrier option pricing formula, on the other hand, was first solved by Merton [12] as an application of the Black-Scholes-Merton differential equation in section 9 of his paper. In his paper, he used the so called method of image for solving the partial differential equation, which requires quite advanced mathematics. Later on, Rubinstein and Reiner [13] found a much easier way to obtain the same solutions by breaking down the integrations in the risk neutral valuation.

This paper takes the aforementioned two ideas together, GEV-based option pricing and breaking down the integrations, to derive explicit barrier option pricing formulas based on the GEV distribution. The derived formulas are numerically comparable to the barrier option pricing formulas based on BS assumptions when the markets are stable. However, they over/under-price the BS based barrier option prices when the markets move sharply, indicating that they are capable of capturing the skewness and kurtosis of the underlying distributions of the market participants.

[1] Fisher–Tippett–Gnedenko theorem and Pickands–Balkema–de Haan theorem

2. A Brief Review of GEV distribution

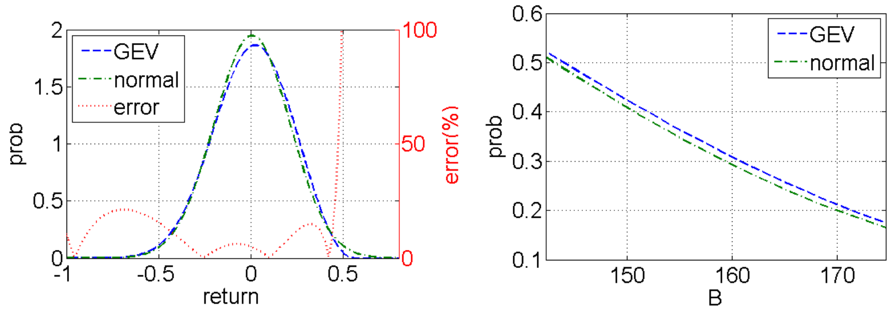
To introduce the GEV briefly, Figure 1(b) shows the probability density function of GEV distributions with various shape parameter ξ. Note that we can reflect the skewness[2] and kurtosis[3] of the return distribution as we control the parameter ξ.



(a) normal distribution (b) GEV distribution

Figure 1: pdfs of normal and GEV Distribution

The similarity between normal distribution and appropriately chosen GEV distribution[4] is verified by Figure 2(a). Figure 2(b) shows the probability of hitting barrier according to each barrier when current underlying asset is 142 in the Figure 2(a) distribution.



(a) pdfs and error (b) prob to hit upper barriers

Figure 2: prob to hit barriers when GEV distribution is approximated to normal distribution

Even though the normal distribution is similar to GEV distribution, the probability of normal distribution to breach the upper barrier is underestimated to GEV distribution. It means there is a chance that the BS model can undervalue or overvalue OTC derivatives like barrier options, despite of similar fitting results in exchange-traded derivatives. Furthermore, there is a higher risk to make wrong valuation when two distributions are not similar.

[2] skewness of the GEV distribution $= \left(g_3 - 3g_1g_2 + 2g_1^3\right) / \left(g_2 - g_1^2\right)^{3/2}$, where $g_k = \Gamma\left(1 - k\xi\right)$

[3] kurtosis of the GEV distribution $= \left(g_4 - 4g_1g_3 + 6g_2g_1^2 - 3g_1^4\right) / \left(g_2 - g_1^2\right)^2 - 3$, where $g_k = \Gamma\left(1 - k\xi\right)$

[4] approximation error = | (pdf of the GEV distribution) - (pdf of the normal distribution) | ÷ (pdf of the GEV distribution)

3. Assumptions of the Valuation Model

To model extreme economic losses, Markose and Alerton [11] suppose that simple negative returns have the GEV distribution. Under the supposition, they show that the parameters in the model should satisfy a particular equation in order to rule out arbitrage possibilities.

Assumption 1

Let a time interval $[t,T]$ be given: in this paper, all options below are assumed to be priced at t and expire at T. We consider an underlying stock with price process S_u $(u \in [t,T])$. We posit that the simple negative return $L_t^T = -(S_T - S_t)/S_t$ has a GEV distribution, that is,

$$L_t^T \sim GEV(\mu,\sigma,\xi)$$

In an arbitrage-free economy, where we place ourselves henceforward, the three parameters μ, σ, and ξ should satisfy the equation

$$\mu = 1 - \frac{F_t^T}{S_t} - \left[\frac{\Gamma(1-\xi)-1}{\xi}\right]\sigma \;\; (\xi \neq 0)$$

Here F_t^T is the stock futures price at t with maturity T(see Markose and Alerton [11]. Moreover, a relation among the parameters can be also suggested in the case of $\xi = 0$ by $\mu = 1 - F_t^T/S_t - \tilde{\gamma}\sigma$, where $\tilde{\gamma}$ is the Euler- Mascheroni constant.

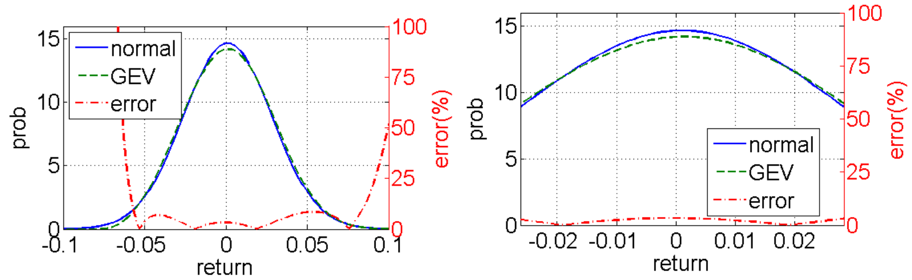
The price for a barrier option[5] is significantly sensitive to the distributions of stock returns[6] before maturity as well as to the one at maturity, models of which the stochastic process plays a crucial role in establishing. Unlike the BS model, however, if any, a stochastic differential equation which the process satisfies has been hardly identified, to the best of the authors' knowledge.

By and large, the implied return distribution at maturity obtained from a model other than the BS model in a market for exchange-traded derivatives does not necessarily have a normal distribution. But a central neighborhood of the implied one still can be approximated by the one derived from the BS model and therein lies our major premise: we presuppose that for a barrier option, a central neighborhood of the return distribution at maturity under the condition that the underlying stock price crosses the barrier before maturity also can be fit by the one under the same condition except that S_u would follow a geometric Brownian motion. For the sake of conciseness, it would be timely for us to name the return distributions present in our postulation: from here on, the former is called a GEV-based conditional return distribution and the latter distribution a normal-based conditional return distribution. Further, we let τ_B denote the first instant that the underlying breaches the barrier.

The above hypothesis enables us to do without any stochastic process by dint of works of Rubinstein and Reiner [13] and Shreve [14], which shows that the density function of a normal-based conditional return distribution is

$$g(x;\sigma) = \frac{e^{2v\alpha/\sigma^2}}{\sigma\sqrt{2\pi(T-t)}} \exp\left\{-\left[\frac{x-2\alpha-v(T-t)}{\sigma\sqrt{2(T-t)}}\right]^2\right\},\ \alpha = \ln\left(\frac{B}{S_t}\right),\ v = r - \frac{1}{2}\sigma^2$$

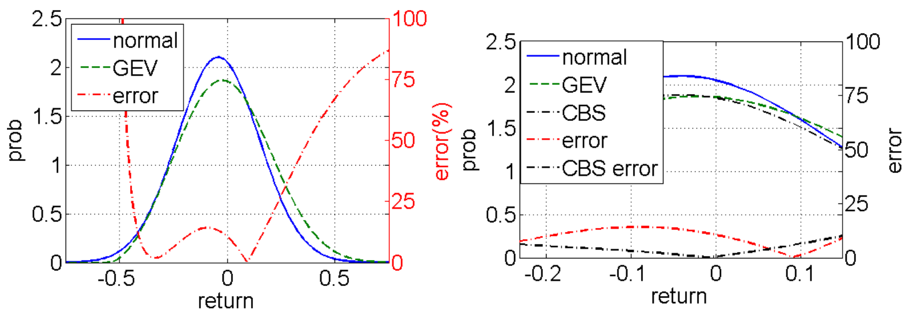
Figure 3(a) depicts the density function of the stock return from the GEV model along with that from the BS model in a stable market for exchange-traded derivatives[7] , whereas Figure 4(a) does in an unstable



(a) whole interval

(b) central neighborhood

Figure 3: PDFs of the GEV model and the BS model when a market is stable



(a) whole interval

(b) central neighborhood

Figure 4: PDFs of the GEV model and the BS model when a market is unstable

one[8]. In the stable market, those functions look alike. On the other hand, the unstable market gives rise to an obtrusive gap between them.

The same is true for their central neighborhoods which Figures 3(b) and 4(b) illustrate. We can, however, adjust the volatility of the BS model in order that the approximation error in the unstable market be as small as possible. Hereafter we refer to such an adjusted BS volatility as a Corrected BS volatility (CBS volatility) and such an adjusted BS model as a Corrected BS model (CBS model). At first sight, these might seem vague, but as will be described in section 5, they will have been derived from specific formulas and thus will rather turn out to be intrinsic. We sum up and restate the aforementioned discussion as Assumption 2 for future reference.

Assumption 2

For a barrier option, a central neighborhood of the distribution of the log-return at maturity $r_t^T = \ln\left(S_T/S_t\right)$ conditional on having hit the barrier also can be approximated by one under the same condition except that the underlying stock price process would act as the counterpart in the BS model:

$$\frac{dP\left(r_t^T < r\,|\tau_B \leq T\right)}{dr} \approx g\left(r, \sigma_{CBS}\right) \quad \left(|r| \ll 1\right)$$

Here σ_{CBS} denotes the CBS volatility.

[5] All barrier options involved herein are assumed to offer no rebate.
[6] Throughout the article, unless otherwise specified, here 'return' always means the log-return.
[7] Based on the data on KOSPI 200 index call option prices, April 2014
[8] Based on the data on KOSPI 200 index call option prices, October 2008

4. Pricing Formulas of Barrier Options

The goal of this section is to show how the we tried to combine the accuracy of GEV model and the accessibility of the BS model as regards pricing barrier options. To start with, as mentioned earlier, Markose and Alerton [11] produces the closed-form solutions for the GEV-based vanilla option pricing model for $\xi > 0$. Documented in Appendices A and B are the proof of valuation for $\xi < 0$ and slight rectification of the pricing formula for $\xi = 0$.

(a) the pricing formula for vanilla call options[9]

$$C_t(K)_{GEV} = \begin{cases} e^{-r(T-t)}\left[S_t\left\{\left(1-\mu+\frac{\sigma}{\xi}\right)e^{-H^{-1/\xi}} - \frac{\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)\right\} - Ke^{-H^{-1/\xi}}\right] & (\xi \neq 0) \\ e^{-r(T-t)}\left[S_t\left\{\left(1-\mu+\bar{H}\sigma\right)e^{-\exp(\bar{H})} + \sigma\Gamma\left(0, \exp\left(\bar{H}\right)\right)\right\} - Ke^{-\exp(\bar{H})}\right] & (\xi = 0) \end{cases}$$

Here $H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_t} - \mu\right)$, $\bar{H} = -\left(\frac{1-K/S_t-\mu}{\sigma}\right)$, and $C_t(K)_{GEV}$ is the price of a vanilla call option with strike price K for the GEV model at t.

In a similar manner to their calculation, we can draw a closed-form expression of the below integral:

(b) an item in aid of the closed-form valuation of barrier call options under the GEV model

$$\begin{aligned} \widetilde{C}_t(K,B)_{GEV} &= e^{-r(T-t)}\int_{\ln(B/S_t)}^{\infty}\left(S_te^u - K\right)^+ f(u)\,du \\ &= e^{-r(T-t)}\left[S_t\left\{\left(1-\mu+\frac{\sigma}{\xi}\right)e^{-H_1^{-1/\xi}} - \frac{\sigma}{\xi}\Gamma\left(1-\xi, H_1^{-1/\xi}\right)\right\} - Ke^{-H_1^{-1/\xi}}\right] (\xi \neq 0) \end{aligned}$$

Here $H_1 = 1 + \frac{\xi}{\sigma}\left(1 - \frac{B}{S_t} - \mu\right)$ and f is the density function of the log-return r_T^t deduced from $L_t^T \sim GEV(\mu, \sigma, \xi)$ in Assumption 1. Note that $C_t(K)_{GEV} = \widetilde{C}_t(K,K)_{GEV}$.

Assumption 2 allows us to apply techniques in Rubinstein and Reiner [13] to the valuation problems. To be more specific, we are going to express the situation where the payoff of a barrier option is non-zero as a combination of several simple cases and partition the domain of the integration along the expression. The below formulas are a portion of his results, which will help in constructing the pricing formulas of barrier options later.

(c) ad-hoc terms which will be used on barrier option pricing

$$e^{-r(T-t)}\int_{\ln(K/S_t)}^{\infty}\left(S_te^u - K\right)^+ g(u;\sigma)\,du = CD\left(y;\sigma;\phi=1,\eta=1\right)$$

$$e^{-r(T-t)}\int_{\ln(B/S_t)}^{\infty}\left(S_te^u - K\right)^+ g(u;\sigma)\,du = CD\left(y_1;\sigma;\phi=1,\eta=1\right)$$

$$e^{-r(T-t)}\int_{-\infty}^{\ln(K/S_t)}\left(S_te^u - K\right)^+ g(u;\sigma)\,du = CD\left(y;\sigma;\phi=1,\eta=-1\right)$$

$$e^{-r(T-t)}\int_{-\infty}^{\ln(B/S_t)}\left(S_te^u - K\right)^+ g(u;\sigma)\,du = CD\left(y_1;\sigma;\phi=1,\eta=-1\right)$$

[9] $\Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt$: (upper) incomplete gamma function

Here the function CD and the constants y, y_1 are as follows.

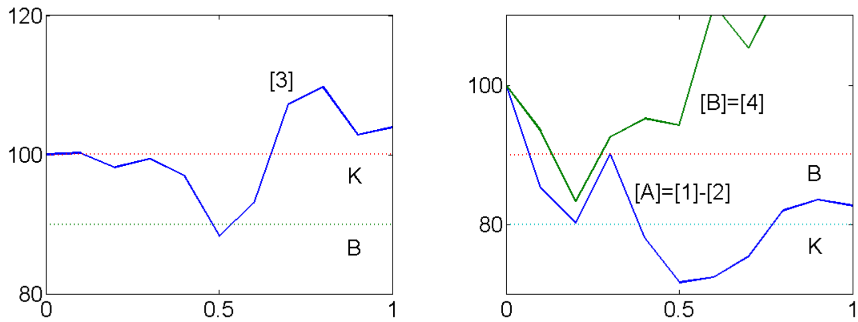
$$CD\left(x;\sigma;\phi,\eta\right) = \phi S_t \left(B/S_t\right)^{2\lambda} N\left(\eta x\right) - \phi K e^{-r(T-t)} \left(B/S_t\right)^{2\lambda-2} N\left(\eta x - \eta\sigma\sqrt{T-t}\right)$$

$$y = \frac{\ln\left(B^2/S_t K\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t},\ y_1 = \frac{\ln\left(B/S_t\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t},\ \lambda = \frac{1}{2} + \frac{r}{\sigma^2}$$

Our first example is a down-and-in barrier call option. Firstly, we consider the case that its barrier B is smaller than its strike price K. To receive a positive payoff from the call, the underlying stock price must cross the barrier before maturity and end up above the strike price. Henceforward we refer to such a situation, that is, $\{S_T > K\,|\tau_B \leq T\}$, as Scenario [3]. This scenario renders g cut out for the density function of the stock return at maturity. Keeping these in mind, we can calculate the price for a down-and-in barrier call option at time t under Scenario [3] for $\xi \neq 0$ (The calculation for $\xi = 0$ goes parallel, which we leave out).

$$\begin{aligned} C_t^{B(DI)}\left(K\right)_{GEV} &= e^{-r(T-t)} \int_{\{profit\ is\ nonzero\}} \left(S_t e^u - K\right)^+ dP \\ &= e^{-r(T-t)} \int_{[3]} \left(S_t e^u - K\right)^+ dP \\ &= e^{-r(T-t)} \int_{\ln(K/S_t)}^{\infty} \left(S_t e^u - K\right)^+ g\left(u;\sigma_{CBS}\right) du \\ &= CD\left(y;\sigma_{CBS};\phi = 1, \eta = 1\right) \quad (B < K) \end{aligned}$$

Here $C_t^{B(DI)}\left(K\right)_{GEV}$ represents the price of a down-and-in call option with strike price K and the barrier B at t for the GEV model.



(a) $B < K$ (b) $B \geq K$

Figure 5: scenarios related to a down-and-in call option

Next we derive the pricing formula for the option when $B \geq K$. This leads us to think of three (not necessarily exclusive) scenarios. We call $\{S_T > K\}$ as Scenario [1], $\{S_T > B\}$ as Scenario [2], and $\{S_T > B\,|\tau_B \leq T\}$ as Scenario [4], the first two of which have been set with no regard for barrier hit. Then the call generates a payoff exactly when

$$\begin{aligned} & S_T > K\,|\tau_B \leq T \\ \Leftrightarrow\quad & [A]\left(K < S_T \leq B|\tau_B \leq T\right)\ or\ [B]\left(S_T > B\,|\tau_B \leq T\right) \\ \Leftrightarrow\quad & \left(K < S_T \leq B\right)\ or\ \left(S_T > B\,|\tau_B \leq T\right) \\ \Leftrightarrow\quad & \{[1]\left(S_T > K\right)\ \&\ \sim[2]\left(S_T > B\right)\}\ or\ [4]\left(S_T > B\,|\tau_B \leq T\right) \end{aligned}$$

In Scenario [4], g plays the role of the density function of the stock return at maturity. However, the Scenarios [1] and [2] are extraneous to barrier hit, so it is f that corresponds to the density function of the return at maturity in the scenarios.

$$\begin{aligned}
C_t^{B(DI)}(K)_{GEV} &= e^{-r(T-t)} \int_{\{profit\ is\ nonzero\}} (S_t e^u - K)^+ \, dP \\
&= e^{-r(T-t)} \left[\int_{[1]} (S_t e^u - K)^+ \, dP - \int_{[2]} (S_t e^u - K)^+ \, dP + \int_{[4]} (S_t e^u - K)^+ \, dP \right] \\
&= e^{-r(T-t)} \left[\int_{\ln(K/S_t)}^{\infty} (S_t e^u - K)^+ f(u) \, du - \int_{\ln(B/S_t)}^{\infty} (S_t e^u - K)^+ f(u) \, du \right. \\
&\qquad \left. + \int_{\ln(B/S_t)}^{\infty} (S_t e^u - K)^+ g(u; \sigma_{CBS}) \, du \right] \\
&= C_t(K)_{GEV} - \widetilde{C}_t(K,B)_{GEV} + CD(y_1; \sigma_{CBS}; \phi = 1, \eta = 1) \quad (B \geq K)
\end{aligned}$$

Taking advantage of in-out parity, we find the pricing formula of a down-and-out barrier call option from those of a vanilla call option and of a down-and-in barrier call option.

$$C_t^{B(DO)}(K)_{GEV} = C_t(K)_{GEV} - C_t^{B(DI)}(K)_{GEV} = \begin{cases} C_t(K)_{GEV} - CD(y; \sigma_{CBS}; \phi = 1, \eta = 1) & (B < K) \\ \widetilde{C}_t(K,B)_{GEV} - CD(y_1; \sigma_{CBS}; \phi = 1, \eta = 1) & (B \geq K) \end{cases}$$

Our next barrier option is an up-and-in call. For $B < K$, the option acts exactly the same as the corresponding vanilla call option does, so all we need to ponder is just estimating the option price for $B \geq K$. As before, the option admits a positive payoff if and only if the underlying price lies above the strike price at maturity while having hit the barrier. So we think about three cases, namely, $\{S_T > B\}$ (Scenario [2]), $\{S_T \leq K | \tau_B \leq T\}$ (Scenario [5]), and $\{S_T \leq B | \tau_B \leq T\}$ (Scenario [6]).

$$\begin{aligned}
& S_T > K | \tau_B \leq T \\
\Leftrightarrow \quad & [A]\,(K < S_T \leq B | \tau_B \leq T) \ \ or \ \ [B]\,(S_T > B | \tau_B \leq T) \\
\Leftrightarrow \quad & (K < S_T \leq B | \tau_B \leq T) \ \ or \ \ (S_T > B) \\
\Leftrightarrow \quad & \{\sim [5]\,(S_T \leq K | \tau_B \leq T) \ \& \ [6]\,(S_T \leq B | \tau_B \leq T)\} \ \ or \ \ [2]\,(S_T > B)
\end{aligned}$$

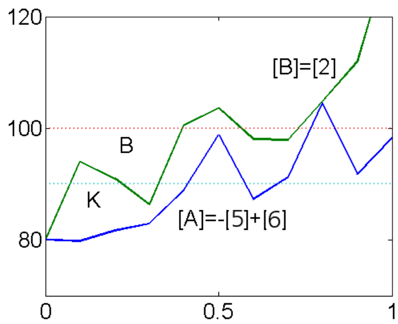


Figure 6: scenarios related to a up-and-in call option

Being either in Scenario [5] or in Scenario [6] implies that the underlying stock price breaches the barrier before maturity, so g is right for the density function of the stock return at maturity. On the other hand, since Scenario [2] has nothing to do with barrier breach, f is pertinent to the function.

$$
\begin{aligned}
C_t^{B(UI)}(K)_{GEV} &= e^{-r(T-t)} \int_{\{profit\ is\ nonzero\}} (S_t e^u - K)^+ \, dP \\
&= e^{-r(T-t)} \left[-\int_{[5]} (S_t e^u - K)^+ \, dP + \int_{[6]} (S_t e^u - K)^+ \, dP + \int_{[2]} (S_t e^u - K)^+ \, dP \right] \\
&= e^{-r(T-t)} \left[\int_{\ln(B/S_t)}^{\infty} (S_t e^u - K)^+ f(u) \, du - \int_{-\infty}^{\ln(K/S_t)} (S_t e^u - K)^+ g(u; \sigma_{CBS}) \, du \right. \\
&\qquad \left. + \int_{-\infty}^{\ln(B/S_t)} (S_t e^u - K)^+ g(u; \sigma_{CBS}) \, du \right] \\
&= \widetilde{C}_t (K, B)_{GEV} - CD(y; \sigma_{CBS}; \phi = 1, \eta = -1) + CD(y_1; \sigma_{CBS}; \phi = 1, \eta = -1) \quad (B \geq K)
\end{aligned}
$$

Again, by invoking in-out parity once more, we can obtain the pricing formula of an up-and-out call option from those of a vanilla call option and of an up-and-in call option, which implies that the price of an up-and-out call option should be zero for $B < K$. In a similar manner, we can draw closed-form solutions for barrier put option prices, which are elaborated on in Appendix D.

$$
\begin{aligned}
C_t^{B(UO)}(K)_{GEV} &= C_t(K)_{GEV} - C_t^{B(UI)}(K)_{GEV} \\
&= C_t(K)_{GEV} - \widetilde{C}_t (K, B)_{GEV} + CD(y; \sigma_{CBS}; \phi = 1, \eta = -1) \quad (B \geq K)
\end{aligned}
$$

5. CBS Volatility

This section aims to show that the CBS model serves as a bridge between the GEV model and the BS model. Thus it is essential that we make a formal definition of the concept of a CBS volatility.

As has seen earlier, a vanilla option price is virtually determined by a central neighborhood of the return distribution. This suggests that the price obtained with the GEV model is almost the same as that with the CBS model, owing to Assumption 2. Thus, according to the fact that any vanilla option has a positive vega in the BS model, there is, once the existence is guaranteed, a unique solution σ_{CBS} for $C_t\left(K\right)_{GEV} = C_t\left(K\right)_{CBS}$ with respect to the variable σ. It is the very solution that we refer to as CBS volatility henceforward. We remark that upon determined, σ_{CBS} is also the unique solution of $P_t\left(K\right)_{GEV} = P_t\left(K\right)_{CBS}$ by virtue of put-call parity. The CBS volatility defined this way depends on the strike price of the option.

In the same vein, we can also devise the CBS volatility when it comes to barrier option pricing. But in general, a naive application of the above volatility may incur the discontinuity of the fair pricing solution in B. Hence so as to reflect the continuity of fair prices in a market, we ought to adapt the definition of CBS volatility for practical use. As will be seen later, such alteration takes place by means of the quantities $C_t\left(K\right)$, $P_t\left(K\right)$ plus $\widetilde{C}_t\left(K,B\right)$, $\widetilde{P}_t\left(K,B\right)$. Fortunately, the CBS volatility defined for a barrier option also functions as a link between the two models, and the intuitive interpretation of the definition sounds rather plausible because $\widetilde{C}_t\left(K,B\right)$, $\widetilde{P}_t\left(K,B\right)$ are barely different from vanilla option prices provided that B is near K. In the rest of this section, we will prove that the pricing functions for barrier options based on the CBS volatility are continuous with respect to B regardless of barrier option type.

(a) Definition of the CBS volatility

The CBS volatility is defined to be the solution of one of the following equations: it depends on the barrier, the strike price, and the type of the barrier option concerned.

the type of barrier option	$K < S_t$	$K \geq S_t$
down type call	$\widetilde{C}_t\left(K,S_t\right)_{GEV} = \widetilde{C}_t\left(K,S_t\right)_{CBS}$	$C_t(K)_{GEV} = C_t\left(K\right)_{CBS}$
up type call	$C_t(K)_{GEV} - \widetilde{C}_t\left(K,S_t\right)_{GEV} = C_t\left(K\right)_{CBS} - \widetilde{C}_t\left(K,S_t\right)_{CBS}$	
down type put	$P_t(K)_{GEV} - \widetilde{P}_t\left(K,S_t\right)_{GEV} = P_t\left(K\right)_{CBS} - \widetilde{P}_t\left(K,S_t\right)_{CBS}$	
up type put	$P_t(K)_{GEV} = P_t\left(K\right)_{CBS}$	$\widetilde{P}_t\left(K,S_t\right)_{GEV} = \widetilde{P}_t\left(K,S_t\right)_{CBS}$

The definitions of $P_t\left(K\right)_{GEV}$, $\widetilde{P}_t\left(K,B\right)_{GEV}$ are introduced in Appendices C and D.

(b) Theorem on the CBS volatility

Regardless of the type, the barrier option pricing function is continuous at all barrier values.

(Proof) Here we only verify the continuity of the pricing function for a down-and-in barrier call option. The ways to prove our statement for the other types are basically similar to this.

Because the CBS model is one of BS models, the pricing formula for a down-and-in call option in the CBS model could be deduced from that in the BS model.

$$C_t^{B(DI)}\left(K\right)_{CBS} = \begin{cases} CD\left(y;\sigma_{CBS};\phi=1,\eta=1\right) & (B < K) \\ C_t\left(K\right)_{CBS} - \widetilde{C}_t\left(K,B\right)_{CBS} + CD\left(y_1;\sigma_{CBS};\phi=1,\eta=1\right) & (B \geq K) \end{cases}$$

Because of the continuity of $C_t^{B(DI)}\left(K\right)_{CBS}$, it is enough to show that $C_t^{B(DI)}\left(K\right)_{GEV} - C_t^{B(DI)}\left(K\right)_{CBS}$ is continuous. Since the continuity of $C_t^{B(DI)}\left(K\right)_{CBS}$, except at $B = K$ and $B = S_t$ is already known, we need to check whether the difference is continuous at both the points.

Calculating $C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}$,

$$C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS} = \begin{cases} 0 & (B < K) \\ \left[C_t(K)_{GEV} - C_t(K)_{CBS}\right] - \left[\widetilde{C}_t(K,B)_{GEV} - \widetilde{C}_t(K,B)_{CBS}\right] & (B \geq K) \end{cases}$$

When $B = K$, $C_t(K) = \tilde{C}_t(K,B)$, so

$$\lim_{B \to K-} \left[C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}\right] = 0 = \left[C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}\right]_{B=K}$$

Therefore $C_t^{B(DI)}(K)_{GEV}$ is continuous at $B = K$. Now let us show the continuity of $C_t^{B(DI)}(K)_{GEV}$ at $B = S_t$, is continuous . Note that $B = S_t \Longrightarrow C_t^{B(DI)}(K) = C_t(K)$, so if $K < S_t$,

$$\lim_{B \to S_t-} \left[C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}\right] = \left[C_t(K)_{GEV} - C_t(K)_{CBS}\right] - \left[\widetilde{C}_t(K,B)_{GEV} - \widetilde{C}_t(K,B)_{CBS}\right]$$

$$= C_t(K)_{GEV} - C_t(K)_{CBS} = \left[C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}\right]_{B=S_t}$$

where the penultimate equality follows from our definition of CBS volatility for $K < S_t$. This proves our assertion for $K < S_t$. When $K \geq S_t$,

$$\lim_{B \to S_t-} \left[C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}\right] = 0$$
$$= C_t(K)_{GEV} - C_t(K)_{CBS} = \left[C_t^{B(DI)}(K)_{GEV} - C_t^{B(DI)}(K)_{CBS}\right]_{B=S_t}$$

where the last equality but one also holds due to the definition. These concludes our proof. □

6. Empirical Test

As explained in the introduction, the BS model suffices in a stable market, so there might seem to be no justification for studying the GEV model. This chapter is to corroborate the results so far achieved by aid of the sources from a market for exchange-traded derivatives (in this paper, vanilla options whose underlying is KOSPI200 index as in Figure 12) during the global credit risk. For this, we scrutinize the data on October 23, 2008 and use the market quotes of call options with 2 months to maturity.

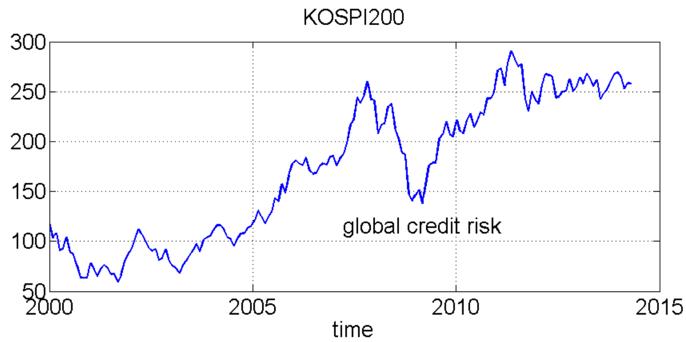
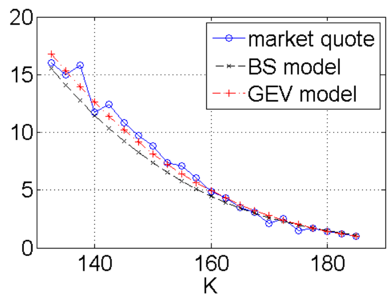
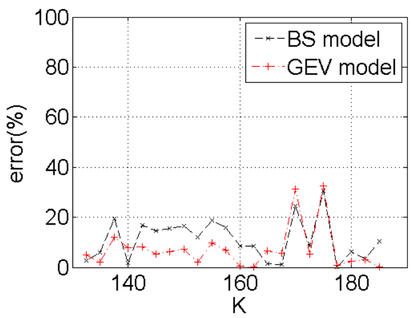


Figure 7: time series of KOSPI200 index (unit : month)

Figure 8(a) exhibits the fitting results of the BS model and of the GEV model on the exchange-traded derivatives and the method used to fit the models to the market is to minimize the square sum of relative errors [10] between market quotes and estimated prices. In the GEV model fits better than the BS model does. This is because when the market is unstable, the skewness and the kurtosis of the return distribution become quite different from those of the normal distribution. Generally, the bigger the strike price of a call option gets, the cheaper the option price comes to be. The graph of market in the figure looks more or less fluctuating though. As to this fluctuation, we can speculate that the instability of the market created an arbitrage opportunity for a little while. Figure 8(b) shows the relative errors of the fitting results, which demonstrate that the GEV option pricing model outperforms the BS model.



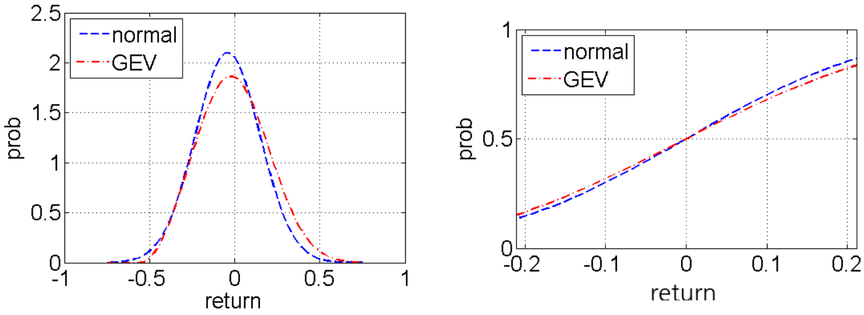
(a) theoretical and market prices



(b) relative fitting errors

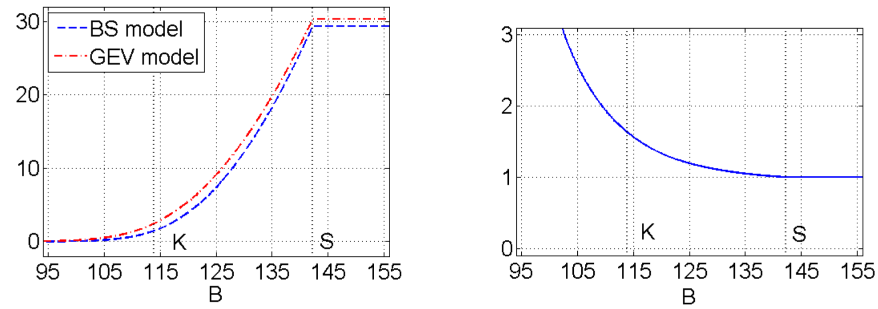
Figure 8: fitting results of the BS model and of the GEV model on the derivatives(vanilla option) data

[10]relative error = | (theoretical price) - market quote) | ÷ (market quote)



(a) PDFs of returns from the models (b) CDFs of returns from the models

Figure 9: the density functions of the BS model and GEV model fitted on market

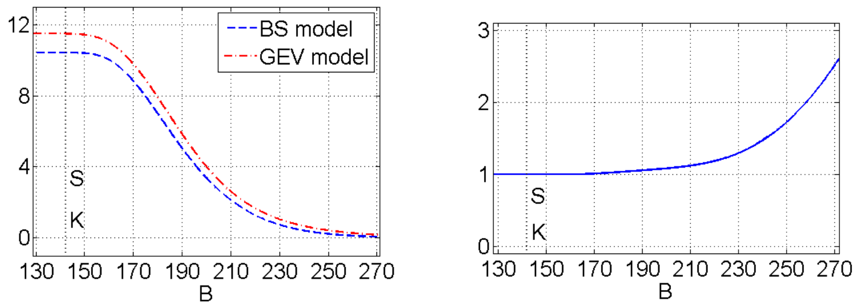


(a) down-and-in call prices from the models (b) GEV prices to BS prices (DIC)

Figure 10: down-in-call prices from each distribution

Figure 9(a) depicts the density functions of the return distributions derived from both the models. Since the one for the BS model has its graph symmetric, it cannot properly reflect the skewness and the kurtosis of the actual return distribution, only to get sharper than the counterpart of the GEV model. If we presume that the GEV model might be more suitable for the market during the period on the basis of the out-performance of the model, there is a high probability that the implied volatility of the BS model is smaller than the standard deviation of the actual return distribution. Under such a situation, whoever uses the BS model for pricing the derivatives cannot help but underestimate the existent market risk. Figure 9(b) draws the cumulative distribution functions of both the pricing models derived from the actual distribution, whose graphs are translated horizontally so as to pass the point $(0, 0.5)$. Since, at that point, the slope of the GEV-based graph is less steep than that of the BS-based graph, if the barrier is not located far from the current stock price, it implies that the BS model underestimates the probability of hitting the barrier irrespective of the barrier type.

Figure 10(a) documents the price of a down-and-in call option over barrier on both the models. First recall that when its barrier price is bigger than the stock price, the option is the same as the corresponding vanilla call option. Therefore you can be convinced that the GEV-based vanilla option price is more expensive than the BS-based one at this time. As said earlier, because in times of the credit crisis the BS model underestimates the standard deviation of the actual return distribution, it seems to also undervalue the vanilla call option. This locates the whole graph for the BS model below the one for the GEV model. Nevertheless, it is rash to jump to the conclusion that the BS model underestimates the probability of barrier hit only because of the price gap of the corresponding vanilla call option. To figure it out more precisely,



(a) up-and-in call prices from the models

(b) GEV prices to BS prices (UIC)

Figure 11: up-in-call prices from each distribution

as in Figure 10(b), we have scaled the ratio of GEV-based price of the vanilla option to its BS-based price to 1 and draw the graph of (the price of the down-and-in call option for the GEV model)/(the price of the down-and-in call option for the BS model) over barrier. The whole values in the figure are not less than 1, which ensures that the barrier is hit with higher possibility under the GEV model than under the other model. This explains why the GEV-based price of the down-and-in call option is higher than the BS-based price. Such a phenomenon becomes more intensive as the barrier gets deeper, and on a very deep barrier the value of the barrier option under the GEV model is over three times that the price under the BS model. This lets us infer that the phenomenon may be attributed to the previous observation that the BS model underestimates the probability of barrier breach. Figure 11(a) and 11(b) are about the operations on an up-and-in call option, whose results are analogous to those of the down-and-in call option.

7. Concluding Remarks

Markose and Alerton [11] derived no-arbitrage conditions and pricing equations of vanilla options under the assumption that the negative returns of underlying asset follow GEV distribution. They left some equations unsolved and made some minor mistakes. In the appendix, we derive missing equations and rectify errors. We use Rubinstein and Reiner [13]'s technique and the GEV model to derive closed-form solutions of barrier options.

During the derivation, we suppose that the central neighborhood of the GEV distribution and the normal distribution are similar by introducing the Corrected BS (CBS) model – the GEV model closed to the BS model – to make our assumption reasonable. The CBS volatility can be chosen arbitrarily depending on the situation but we fixed it to make the price of the ATM vanilla option under CBS model same as the price under the GEV model. It uniquely exists on high probability. Also, by introducing CBS volatility we show that barrier option prices are continuous to barriers under GEV model.

People still use the BS model believing that the BS model reflects the market, even though the BS model does not reflect the market perfectly, . However, when the market is unstable, the BS model has low fitting ability due to the skewness and kurtosis of the return distribution. In this paper, we adjust the skewness and kurtosis of the distribution of stocks in the global credit crisis by only adding one parameter with the GEV model. We prove that the BS model undervalues vanilla call options and more undervalues in-type barrier options. We also find that the probability of large pricing gap of OTC derivatives between two probability distributions increases, the one derived from the BS model and the other from the GEV model, even if the evaluation of exchange-traded derivatives are similar. These analyses may imply that the BS model overvalues or undervalues prices of OTC derivatives regardless of a market situation and can even bring wrong prices of derivatives when the market is unstable.

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Appendix A - the price formula of vanilla call options for the GEV model where $\xi < 0$ [11]

The density function for the GEV model where $\xi \neq 0$ is as below.

* the density function $\tilde{f}\left(L_t^T\right)$ when $L_t^T \sim GEV\left(\mu, \sigma, \xi\right)$ for $\xi \neq 0$

$$\tilde{f}\left(L_t^T\right) = \frac{1}{\sigma}\left[1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \exp\left\{-\left[1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} \left(L_t^T > \mu - \frac{\sigma}{\xi}\right)$$

If we define $h\left(S_T\right)$ as the density function on S_T, then

$$\begin{aligned} h\left(S_T\right) &= \tilde{f}\left(L_t^T\right)\left|\frac{\partial L_t^T}{\partial S_T}\right| = \tilde{f}\left(L_t^T\right)\frac{1}{S_t} \\ &= \frac{1}{S_t\sigma}\left[1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \exp\left\{-\left[1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} \left(S_T > S_t\left(1-\mu-\frac{\sigma}{-\xi}\right)\right) \end{aligned}$$

Provided that a vanilla call is not deep in the money option, $K > S_t\left(1-\mu-\frac{\sigma}{-\xi}\right)$,

$$C_t\left(K\right)_{GEV} = e^{-r(T-t)}\int_K^{\infty}\left(S_T-K\right)\frac{1}{S_t\sigma}\left[1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}-1} \exp\left\{-\left[1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\} dS_T$$

Let $\gamma = 1+\xi\left(\frac{L_t^T-\mu}{\sigma}\right)$. Then,

$$S_T = S_t\left(1-\mu-\frac{\sigma}{\xi}\left(\gamma-1\right)\right),\ dS_T = -S_t\frac{\sigma}{\xi}d\gamma,\ h\left(\gamma\right) = \frac{1}{S_t\sigma}\left(-\gamma^{-1-\frac{1}{\xi}}\right)\exp\left(-\gamma^{-\frac{1}{\xi}}\right)$$

Putting that $H = 1+\xi\left(\frac{1-K/S_t-\mu}{\sigma}\right)$,[12]

$$\begin{aligned} C_t\left(K\right)_{GEV} &= e^{-r(T-t)}\int_H^{\infty}\left(S_t\left(1-\mu-\frac{\sigma}{\xi}\left(\gamma-1\right)\right)-K\right)\frac{1}{S_t\sigma}\left(-\gamma^{-1-\frac{1}{\xi}}\right)\exp\left(-\gamma^{-\frac{1}{\xi}}\right)\left(-S_t\frac{\sigma}{\xi}d\gamma,\right) \\ &= e^{-r(T-t)}\frac{1}{\xi}\left[\frac{S_t\sigma}{\xi}\int_H^{\infty}\gamma^{-\frac{1}{\xi}}\exp\left(-\gamma^{-\frac{1}{\xi}}\right)d\gamma\right. \\ &\qquad \left. -\left(S_t\left(1-\mu+\frac{\sigma}{\xi}\right)-K\right)\int_H^{\infty}\left(\gamma^{-1-\frac{1}{\xi}}\right)\exp\left(-\gamma^{-\frac{1}{\xi}}\right)d\gamma\right] \\ &= e^{-r(T-t)}\left[S_t\left(\left(1-\mu+\frac{\sigma}{\xi}\right)e^{-H^{-1/\xi}}-\frac{\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)\right)-Ke^{-H^{-1/\xi}}\right] \end{aligned}$$

[11] The technique used in appendices A and B is based on Markose(2011).

[12] The price formula where $\xi < 0$ is basically the same as that where $\xi > 0$ induced by Markose(2011).

Appendix B - the price formula of vanilla call options for the GEV model where $\xi = 0$

The density function for the GEV model where $\xi = 0$ is as below.

* the density function $\tilde{f}\left(L_t^T\right)$ where gross loss L_t^T follows a GEV distribution ($\xi = 0$)

$$\tilde{f}\left(L_t^T\right) = \frac{1}{\sigma}\exp\left(-\frac{L_t^T-\mu}{\sigma}\right)\exp\left[-\exp\left(-\frac{L_t^T-\mu}{\sigma}\right)\right]\ \left(-\infty < L_t^T < \infty\right)$$

If we define $h\left(S_T\right)$ as the density function on S_t, then

$$\begin{aligned} h\left(S_T\right) &= \tilde{f}\left(L_t^T\right)\left|\frac{\partial L_t^T}{\partial S_T}\right| = \tilde{f}\left(L_t^T\right)\frac{1}{S_t} \\ &= \frac{1}{S_t\sigma}\exp\left(-\frac{L_t^T-\mu}{\sigma}\right)\exp\left[-\exp\left(-\frac{L_t^T-\mu}{\sigma}\right)\right]\ \left(-\infty < S_T < \infty\right) \end{aligned}$$

Thus the fair price of a given option is

$$C_t\left(K\right)_{GEV} = e^{-r(T-t)}\int_K^\infty \left(S_T-K\right)\frac{1}{S_t\sigma}\exp\left(-\frac{L_t^T-\mu}{\sigma}\right)\exp\left[-\exp\left(-\frac{L_t^T-\mu}{\sigma}\right)\right]dS_T$$

Let $\gamma = -\frac{L_t^T-\mu}{\sigma} = -\frac{1-S_T/S_t-\mu}{\sigma}$. Then,

$$S_T = S_t\left(1-\mu+\sigma\gamma\right),\ dS_T = -S_t\sigma d\gamma,\ h\left(\gamma\right) = \frac{1}{S_t\sigma}\exp\left(\gamma - \exp\gamma\right)$$

Putting that $\bar{H} = -\frac{1-K/S_t-\mu}{\sigma}$,

$$\begin{aligned} C_t\left(K\right)_{GEV} &= e^{-r(T-t)}\int_{\bar{H}}^\infty \left(S_t\left(1-\mu+\sigma\gamma\right)-K\right)\frac{1}{S_t\sigma}\exp\left(\gamma-\exp\gamma\right)\left(-S_t\sigma d\gamma\right) \\ &= e^{-r(T-t)}\left[S_t\sigma\int_{\bar{H}}^\infty \gamma\exp\left(\gamma-\exp\gamma\right)d\gamma + \left(S_t\left(1-\mu\right)-K\right)\int_{\bar{H}}^\infty \exp\left(\gamma-\exp\gamma\right)d\gamma\right] \\ &= e^{-r(T-t)}\left[S_t\left(\left(1-\mu+\bar{H}\sigma\right)e^{-\exp\bar{H}} + \sigma\Gamma\left(0,\exp\bar{H}\right)\right) - Ke^{-\exp\bar{H}}\right] \end{aligned}$$

Appendix C - the price formula of vanilla put options for the GEV model [13]

Because we already know the price formula of vanilla call options for the GEV model, by using put-call parity, we can easily derive the price formula of vanilla put options as follows.

$$P_t(K)_{GEV} = C_t(K)_{GEV} - S_t + e^{-r(T-t)}K$$

Here $P_t(K)_{GEV}$ is the price of a vanilla put option with the strike price K at t for the GEV model.

(i) the price formula of vanilla put options for the GEV model where $\xi \neq 0$

$$P_t(K)_{GEV} = e^{-r(T-t)}\left[K\left(1 - e^{-H^{-1/\xi}}\right) - S_t\left(e^{r(T-t)} - \left(1 - \mu + \frac{\sigma}{\xi}\right)e^{-H^{-1/\xi}} + \frac{\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)\right)\right]$$

$$H = 1 + \frac{\xi}{\sigma}\left(1 - \frac{K}{S_t} - \mu\right),\ \mu = 1 - \frac{F_t^T}{S_t} - \left(\frac{\Gamma(1-\xi)-1}{\xi}\right)\sigma$$

(ii) the price formula of vanilla put options for the GEV model where $\xi = 0$

$$P_t(K)_{GEV} = e^{-r(T-t)}\left[K\left(1 - e^{-\exp\bar{H}}\right) - S_t\left(e^{r(T-t)} - (1 - \mu + \bar{H}\sigma)e^{-\exp\bar{H}} - \sigma\Gamma(0, \exp\bar{H})\right)\right]$$

$$\bar{H} = -\frac{1 - K/S_t - \mu}{\sigma},\ \mu = 1 - \frac{F_t^T}{S_t} - \tilde{\gamma}\sigma\ \ (\tilde{\gamma}:\ \text{the Euler} - \text{Mascheroni constant})$$

[13]Markose(2011) derives the price formula of vanilla put options for the GEV model only if $\boldsymbol{\xi > 0}$.

Appendix D - the price formula of barrier put options for the GEV model where $\xi \neq 0$

Applying the similar method used to derive the price formula of vanilla put options for the GEV model, the below integrals could be explicitly calculated. Let the result be defined as $\tilde{P}\left(K,B\right)_{GEV}$.

* an ancillary term for the closed-form valuation of barrier put options for the GEV model

$$\begin{aligned}\tilde{P}_t\left(K,B\right)_{GEV} &= e^{-r(T-t)}\int_{-\infty}^{\ln(B/S_t)}\left(K-S_te^{u}\right)^{+}f\left(u\right)du\\ &= e^{-r(T-t)}\left[K\left(1-e^{-H_1^{-1/\xi}}\right)\right.\\ &\qquad \left.-S_t\left(e^{r(T-t)}-\left(1-\mu+\frac{\sigma}{\xi}\right)e^{-H_1^{-1/\xi}}+\frac{\sigma}{\xi}\Gamma\left(1-\xi,H_1^{-1/\xi}\right)\right)\right] \quad (\xi\neq 0)\end{aligned}$$

Here $H_1 = 1+\frac{\xi}{\sigma}\left(1-\frac{B}{S_t}-\mu\right)$. Note that $P_t\left(K\right)=\tilde{P}_t\left(K,K\right)$.

And the function CD and the constants y, y_1 defined in the section 3 are also used here.

$$CD\left(x;\sigma;\phi,\eta\right)=\phi S_t\left(B/S_t\right)^{2\lambda}N\left(\eta x\right)-\phi K e^{-r(T-t)}\left(B/S_t\right)^{2\lambda-2}N\left(\eta x-\eta\sigma\sqrt{T-t}\right)$$

$$y=\frac{\ln\left(B^2/S_tK\right)}{\sigma\sqrt{T-t}}+\lambda\sigma\sqrt{T-t},\ y_1=\frac{\ln\left(B/S_t\right)}{\sigma\sqrt{T-t}}+\lambda\sigma\sqrt{T-t},\ \lambda=\frac{1}{2}+\frac{r}{\sigma^2}$$

Since the detail of the derivation is similar to that of call options, it is omitted here. If one likes to derive the price formula of vanilla put options, he is advised to refer to the chapter 3 and Rubinstein(1991) together.

(i) the price formula of down-in put options for the GEV model where $\xi \neq 0$

$$P_t^{B(DI)}\left(K\right)_{GEV}=\begin{cases}\tilde{P}_t\left(K,B\right)_{GEV}+CD\left(y;\sigma_{CBS};\phi=-1,\eta=1\right)-CD\left(y_1;\sigma_{CBS};\phi=-1,\eta=1\right) & (B<K)\\ P_t\left(K\right)_{GEV} & (B\geq K)\end{cases}$$

(ii) the price formula of down-out put options for the GEV model where $\xi \neq 0$

$$P_t^{B(DO)}\left(K\right)_{GEV}=\begin{cases}P_t\left(K\right)_{GEV}-\tilde{P}_t\left(K,B\right)_{GEV}-CD\left(y;\sigma_{CBS};\phi=-1,\eta=1\right)+CD\left(y_1;\sigma_{CBS};\phi=-1,\eta=1\right) & (B<K)\\ 0 & (B\geq K)\end{cases}$$

(iii) the price formula of up-in put options for the GEV model where $\xi \neq 0$

$$P_t^{B(UI)}\left(K\right)_{GEV}=\begin{cases}P_t\left(K\right)_{GEV}-P_t\left(B\right)_{GEV}-CD\left(y_1;\sigma_{CBS};\phi=-1,\eta=-1\right) & (B<K)\\ CD\left(y;\sigma_{CBS};\phi=-1,\eta=-1\right) & (B\geq K)\end{cases}$$

(iv) the price formula of up-out put options for the GEV model where $\xi \neq 0$

$$P_t^{B(UO)}\left(K\right)_{GEV}=\begin{cases}P_t\left(B\right)_{GEV}+CD\left(y_1;\sigma_{CBS};\phi=-1,\eta=-1\right) & (B<K)\\ P_t\left(K\right)_{GEV}-CD\left(y;\sigma_{CBS};\phi=-1,\eta=-1\right) & (B\geq K)\end{cases}$$