Robust Portfolio Management with Risk Limits*

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Abstract

This paper studies robust portfolio management of an ambiguity-averse fund manager who has risk limits. Contrary to traditional investment rules, optimal portfolio management takes different forms in which it is significantly affected by the presence of risk limits and ambiguity aversion. With reasonably calibrated market parameters, we find that the loss amount induced by a Value-at-Risk (VaR) constraint or an expected shortfall (ES) constraint decreases as ambiguity aversion increases. When we consider the VaR constraint together with ambiguity aversion, the risky investment is substantially lower compared to that without ambiguity aversion. Furthermore, the robust portfolio management amplifies the effect of the ES constraint, and, thus, the investment strategy of ambiguity-averse fund managers tends to become increasingly similar to the one without the ES constraint as ambiguity aversion increases.

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1 Introduction

> Ambiguity is an uninsurable risk. We can purchase insurance to help deal with loss from most risky events. We can lay-off risk, spread risk and avoid risky situations. Although ambiguity is a form of risk, there is no way to insure against it. ... The key to overcoming ambiguity is for participants to adopt a risk appropriate asset allocation and stick to it. Ambiguous events come and go. If participants let the fear of ambiguous outcomes affect their decision-making, they will never implement an allocation strategy that is aggressive enough. (Employee Benefit News, February 25, 2014)

The above quote emphasizes the importance of developing a robust portfolio management strategy in which the effects of ambiguity are appropriately considered. Since the seminal paper of Hansen and Sargent (1995) robust portfolio choice problems against uncertainty about the equity return process have been actively studied. Gilboa and Schmeidler (1989) axiomatize max-min utility, and Hansen and Sargent (2001) and Hansen *et al.* (2006) extend the model of Gilboa and Schmeidler (1989) to a continuous time robust consumption and portfolio choice problem. Maenhout (2004) introduce homothetic robustness through which analytical results of robust consumption and portfolio rules are derived and the equity premium puzzle can be resolved. Various types of robustness such as recursive smooth ambiguity have resolved various puzzles in finance (Liu *et al.*, 2005; Zhu, 2001; Ju and Miao, 2012).

Fund managers are exposed to uncertainty and risk about equity returns. In particular, Anderson *et al.* (2009) find that uncertainty is a key determinant of the equity returns than risk, and verify that there exists a significant uncertainty-return trade-off. An ambiguity-averse fund manager's portfolio selection problem can be formulated within the standard mean-variance portfolio optimization and empirical analysis (Garlappi *et al.*, 2007) shows that robust portfolio rules against the ambiguity aversion about the expected rate of stock returns are more efficient than classical models in terms of stability and Shapre ratio.

Fund managers are significantly affected by various portfolio constraints. There is a strand of literature regarding optimal portfolio strategy with portfolio constraints. Cvitanić and Karatzas (1992), Cuoco (1997), and Cuoco and Liu (2000) show that the optimal trading strategy of fund managers heavily depends on portfolio constraints. Moreover, Grossman and

Vila (1989), Basak (1995), and Grossman and Zhou (1996) constrain the optimal portfolio choice to consider portfolio insurance in which fund manager's terminal wealth is always maintained above pre-specified level. Recently, Dai *et al.* (2011) investigate how optimal trading strategy is affected by the joint of portfolio constraints characterized by position limits and transaction costs.

We make significant contributions to the fund manager's portfolio management by finding an optimal portfolio strategy in the presence of both portfolio constraints and ambiguity. We focus on the joint considerations of risk limits represented by Value-at-Risk (VaR) constraint or expected shortfall (ES) constraint[1] and ambiguity. VaR is the lowest tail percentile for losses from the distribution of profit and loss, and ES imposes the percentile for the first moment of losses. Portfolio management based on VaR has been a popular standard choice by industry regulations (Jorion, 2006). Hull (2005) states that fund managers and financial institutions use VaR as the relevant risk measure. Empirically, Berkowitz and O'brien (2002) show the superiority of the VaR estimates relative to the 99th percentile from the distribution of profit and loss in terms of the performance by examining the statistical accuracy of the VaR forecasts. However, there exist some literature which find that VaR might not be appropriate as a risk measure because it would generate large losses during economic downturns (Basak and Shapiro, 2001). Instead of using VaR, ES has been used as an alternative risk measure in the context of modern portfolio management (Artzner *et al.*, 1997; Acerbi and Tasche, 2002; Frey and McNeil, 2002; Tasche, 2002; Inui and Kijima, 2005; Yamai and Yoshiba, 2005).

This paper follows the framework of Basak and Shapiro (2001) in that we include the VaR constraint or the ES constraint in the portfolio optimization problem and that a fund manager must satisfy the constraint at the terminal time. In particular, we do not consider any reevaluation of VaR or discrete reporting VaR.[2] The large difference between our model and the model of Basak and Shapiro (2001) is to be attributed to the fund manager's am-

[1]Expected shortfall is well known the so-called *conditional value at risk, average value at risk, and expected tail loss.*

[2]Cuoco and Liu (2006) determine an optimal investment strategy for financial institutions in which they must report their VaR given some discrete reporting periods. They show that VaR-based capital requirements contribute significantly to reducing financial institutions' portfolio risk. Cuoco et al. (2008) overcome the assumption of static or dynamically inconsistent VaR constraints as existing literature considered and conclude that a trader subject to the VaR constraints takes lower risky investment than an unconstrained trader does.

biguity aversion against the uncertainty about the first moment of stock returns. We derive analytical results about the robust portfolio management strategy for the ambiguity-averse fund manager. This is our technical contribution. Contrary to traditional investment rules, optimal portfolio management takes different forms in which it is significantly affected by the presence of risk limits and ambiguity aversion.

Basak and Shapiro (2001) have a negative point of view of using VaR-based risk management, because it leads to severe losses during economic downturns. However, this is not necessarily true when one considers the ambiguity-averse fund manager. With reasonably calibrated market parameters, we find that the loss amount due to our robust portfolio management with the VaR constraint significantly decreases as the ambiguity aversion increases. Hence, in that case the VaR constraint can be an effective tool in risk management. In the presence of the ES constraint, the fund manager cares about big losses, so that in unfavorable states the losses are significantly smaller than the case of the VaR constraint. Our numerical experiments confirm the intuition of Basak and Shapiro (2001) that the fund manager constrained by the ES constraint is more reluctant to allow the shortfall than the fund manager with the VaR constraint is, and show that when we consider the ambiguity aversion of the fund manager, the loss amount decreases as ambiguity aversion increases.

The interesting observation of our model is that in Basak and Shapiro (2001) the fund manager with the VaR constraint attempts to have her wealth to be above the constraint by investing aggressively in the stock, but our robust portfolio management suggests that if the fund manager formulates her investment strategy by considering the VaR constraint together with her ambiguity aversion, then her risky investment is substantially lower compared to that of Basak and Shapiro (2001). Relative to the ES constraint, in Basak and Shapiro (2001) the fund manager formulates an investment strategy that reduces the stock investment to satisfy the ES constraint. Our robust portfolio management amplifies the effect of the ES constraint, and, thus, the strategy of the ambiguity-averse fund managers tends to become increasingly similar to the one without the ES constraint as ambiguity aversion increases. In this sense, the ambiguity aversion may be a more significant factor in portfolio management than are risk limits such as the VaR and the ES constraint.

This paper is organized as follows. In Section 2 we clarify our problem concerning the robust portfolio management with risk limits. Section 3 suggests the solution to the problem and provides analytical results relative to optimal strategies. Section 4 shows the numerical implications with reasonably calibrated market parameters. We conclude in Section 5.

2 The Model

2.1 Benchmark Portfolio Management

We consider a fund manager who wants to find the maximal score of her constant relative risk aversion utility preference for terminal wealth at time $T \in (0, \infty)$. The fund manager can trade two assets in a financial market: a risk-free asset (e.g., a bond) and a risky asset (e.g., a stock). The bond price grows at a continuously compounded, constant rate $r > 0$. The stock price S_t follows

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $\mu > r$ is the expected rate of the stock return, $\sigma > 0$ is the volatility of the return on the stock, and B_t is a standard Brownian motion defined on an appropriate probability space. We assume that r, μ, σ are constant, i.e., that investment opportunity is constant.

The wealth process W_t of the fund manager with an initial wealth $W_0 = w > 0$ is given by

$$dW_t = \{r + \pi_t(\mu - r)\} W_t dt + \pi_t \sigma W_t dB_t,$$

where π_t is the fraction of wealth invested in the stock at time t. Then, the objective function (or value function) is to maximize

$$\max_{\pi} E\Big[\frac{W_T^{1-\gamma}}{1-\gamma}\Big],$$

where $\gamma > 0$ is the coefficient of relative risk aversion.

Merton (1969) derived the following Hamilton-Jacobi-Bellman (HJB) equation to solve the optimization problem:

$$0 = \max_{\pi} \Big[V_t + \{r + \pi(\mu - r)\} w V_w + \frac{1}{2}\pi^2 \sigma^2 w^2 V_{ww}\Big], \tag{1}$$

where $V(t, w)$ is the value function, and V_t, V_w, and V_{ww} are its partial derivatives with respect to time t and initial wealth w with boundary condition

$$V(T, w) = \frac{w^{1-\gamma}}{1-\gamma}. \tag{2}$$

Then the value function $V(t,w)$ follows

$$V(t,w) = f(t)\frac{w^{1-\gamma}}{1-\gamma},$$

where

$$f(t) = e^{a(T-t)}, \quad a = (1-\gamma)\Big(r + \frac{\kappa^2}{2\gamma}\Big),$$

where $\kappa = (\mu - r)/\sigma$ represents the *Sharpe ratio*. Moreover, the optimal fraction π_t^* of wealth invested in the stock is given by

$$\pi_t^* = \frac{\kappa}{\gamma\sigma},$$

which is exactly same as in Merton (1969).

2.2 Robust Portfolio Management in the Absence of Risk Limits

To incorporate ambiguity aversion (or model uncertainty) we adopt the robust preference structure of Anderson *et al.* (2003). Specifically, the wealth process of the fund manager who prefers robustness is given by

$$dW_t = \{r + \pi_t(\mu - r + \sigma h_t)\}W_t dt + \pi_t \sigma W_t dB_t, \tag{3}$$

where π_t is the fraction of wealth invested in the stock at time t, and h_t is an endogenous drift adjustment. The drift adjustment is chosen to minimize the sum of three terms; the first term is equation (1), the second one is the additional drift component in the wealth process (3), and the third one is an entropy penalty (Anderson *et al.*, 2003). Specifically, h_t is chosen as the following:

$$h_t = \arg\min_h \Big[V_t + \{r + \pi(\mu - r)\}wV_w + \frac{1}{2}\pi^2\sigma^2 w^2 V_{ww} + \pi\sigma w h V_w + \frac{1}{2\hat{\theta}}\pi\sigma w h^2\Big],$$

where $\hat{\theta} \geq 0$ measures the strength of the preference for robustness.[3] We replace $\hat{\theta}$ by $\Psi(w,t)$ as a first step to get the homothetic property of robustness (Maenhout, 2004). Then HJB (1) can be

$$0 = \max_\pi \inf_h \Big[V_t + \{r + \pi(\mu - r)\}wV_w + \frac{1}{2}\pi^2\sigma^2 w^2 V_{ww} + \pi\sigma w h V_w + \frac{1}{2\Psi(w,t)}\pi\sigma w h^2\Big]. \tag{4}$$

The optimality condition for h gives

$$h^* = -\Psi V_w.$$

[3] See Anderson *et al.* (2003) for the details of entropy and $\hat{\theta}$.

Substituting h^* into HJB (4) yields

$$0 = \max_{\pi} \Big[V_t + \{r + \pi(\mu - r)\} w V_w + \frac{1}{2}\pi^2\sigma^2 w^2 V_{ww} - \frac{1}{2}\Psi\pi^2\sigma^2 w^2 (V_w)^2 \Big],$$

with the boundary condition (2). Optimal risky investment π^* is given by the necessary optimality condition as

$$\pi^* = -\frac{V_w}{w\big(V_{ww} - \Psi(V_w)^2\big)} \frac{\kappa}{\sigma}.$$

Following Maenhout (2004), we use

$$\Psi(t, w) = \frac{\theta}{(1-\gamma)V(w,t)} > 0,$$

which is called the *homothetic robustness* (or ambiguity aversion). Then HJB (4) becomes

$$0 = V_t + rwV_w - \frac{1}{2}\kappa^2 \frac{(V_w)^2}{\big(V_{ww} - \Psi(V_w)^2\big)} - \frac{1}{2}\Psi\kappa^2 \frac{(V_w)^4}{\big(V_{ww} - \Psi(V_w)^2\big)^2}. \tag{5}$$

If we conjecture the solution form of V as

$$V(t, w) = g(t)\frac{w^{1-\gamma}}{1-\gamma}, \tag{6}$$

then $g(t)$, h^* and π^* are obtained

$$g(t) = e^{b(T-t)}, \quad b = (1-\gamma)\Big[r + \frac{1}{2}\frac{\gamma}{(\gamma+\theta)^2}\kappa^2\Big],$$

$$h^* = -\frac{\kappa\theta}{\gamma+\theta}, \quad \text{and} \quad \pi^* = \frac{\kappa}{\sigma(\gamma+\theta)}. \tag{7}$$

Note that the optimal risky investment π^* is exactly same as the one of Maenhout (2004) with the coefficient γ of relative risk aversion is replaced by $\gamma + \theta$.

2.3 Robust Portfolio Management In the Presence of Risk Limits

In the previous section, we derived robust portfolio management as Maenhout (2004) suggested. Now we consider robust portfolio management with two risk limits: VaR and ES.

2.3.1 With Value-at-Risk constraints

We consider the VaR constraint in the robust portfolio management. Basak and Shapiro (2001) consider the following VaR constraint:

$$P\{W_T \geq \underline{W}\} \geq 1 - \alpha, \text{[4]} \tag{8}$$

[4] The VaR constraint considered in this paper reduces to the case of portfolio insurance, which constrains the terminal wealth to be above the level $\underline{W}$ always (Grossman and Vila, 1989; Basak, 1995; Grossman and

which means that the fund manager will lose more than $w - \underline{W}$ with probability $0 \le \alpha < 1$.[5] Then the ambiguity-averse fund manager with VaR constraint would like to maximize her expected utility of terminal wealth

$$V(t,w) = \max_{\pi} E\Big[\frac{W_T^{1-\gamma}}{1-\gamma}\Big], \tag{9}$$

$$\text{subject to} \quad P\{W_T \le \underline{W}\} \le \alpha,$$

which is equivalent to the representation of the VaR constraint in (8). The value function $V(t,w)$ still satisfies HJB (5), but the separation in the conjectured solution V of (6) is no longer applied to this robust portfolio management problem in the presence of VaR constraint.

2.3.2 With expected shortfall constraints

Although financial risk management frequently uses the VaR constraint, it has severe shortcomings in terms of the appropriateness of risk measure and in that VaR-based risk management would result in large losses during economic downturns.[6] Instead of using the VaR constraint, now we introduce the expected shortfall (ES) constraint as an alternative risk measure. The ES constraint focuses on the first moment of losses, not on their amount. Specifically, under the shortfall constraint the fund manager is constrained as the following:

$$E\big[\tilde{\xi}_T\big(\underline{W} - W_T\big)\mathbf{1}_{\{W_T \le \underline{W}\}}\big] \le \alpha, \tag{10}$$

where $\tilde{\xi}$ is a state price density (or stochastic discount factor) induced by market completeness. Then the ambiguity-averse fund manager with ES constraint has the following value function:

$$V(t,w) = \max_{\pi} E\Big[\frac{W_T^{1-\gamma}}{1-\gamma}\Big], \tag{11}$$

Zhou, 1996), when $\alpha = 0$. The case of $0 < \alpha < 1$ allows the terminal wealth to be below $\underline{W}$ with a probability α. The VaR constraint is never binding when $\alpha = 1$.

[5] See Basak and Shapiro (2001) for the details of modeling the VaR constraint.

[6] Following Artzner *et al.* (1999), VaR is not a *coherent* risk measure because the *subadditivity property* does not hold. Furthermore, the VaR-based risk management would generate large losses in unfavorable states (Basak and Shapiro, 2001). Alexander and Baptista (2002, 2004) imply that economic agents could take portfolios with larger standard deviations when they use VaR as a risk measure in certain conditions. Hence, many researchers have suggested to use ES as an alternative risk measure to remedy the shortcomings in VaR (Artzner *et al.*, 1997; Acerbi and Tasche, 2002; Frey and McNeil, 2002; Tasche, 2002; Inui and Kijima, 2005; Yamai and Yoshiba, 2005).

which is subject to the ES constraint (10).

Basak and Shapiro (2001) characterize optimal strategy of a fund manager as a function of the unique state price density, but in this paper the state price density is difficult to characterize explicitly, due to the ambiguity aversion. Therefore, the *martingale approach* is not easily applicable to our problem. Instead, we utilize the *dynamic programming approach* developed by Kraft and Steffensen (2013) for the portfolio selection problem with various portfolio constraints to solve our robust portfolio management problem with risk limits.

3 Solution

3.1 Optimal Wealth Process without Risk Limits

The fund manager's optimal wealth process X_t in the absence of risk limits can be

$$dX_t = (r + \frac{\kappa\kappa_\theta}{\gamma+\theta})X_t dt + \frac{\kappa}{\gamma+\theta}X_t dB_t, \quad X_0 = x > 0.$$

where $\kappa_\theta = \kappa - \frac{\kappa\theta}{\gamma+\theta}$ is the *ambiguity-adjusted* Sharpe ratio. It is followed by substituting the drift adjustment h^* and optimal risky investment π^* given as in (7) into the wealth process (3). We then can express the optimal wealth process W_t^* in the presence of the risk limits constraint as a function $\phi(t,x)$ of time t and wealth x (Kraft and Steffensen, 2013) as

$$W_t^* = \phi(t,x) = E_{t,x}^Q[e^{-r(T-t)}g(X_T)],$$

which satisfies

$$\phi_t(t,x) = r\phi(t,x) - rx\phi_x(t,x) - \frac{1}{2}\Big(\frac{\kappa}{\gamma+\theta}\Big)^2 x^2\phi_{xx}, \quad \phi(T,x) = g(x), \tag{12}$$

where Q is a risk-neutral measure and $g(x)$ is a claim of x to be determined depending upon the risk limits. Equation (12) implies that optimal wealth process W_t^* is the option value corresponding to the claim g at time t. Accordingly, the initial wealth w of the fund manager is equivalent to the option price $\phi(0,x)$. Basak and Shapiro (2001) interpret the wealth process with the VaR constraint from two points of view: a combination of a portfolio insurance and a short position in binary options; a portfolio strategy without the VaR constraint and an appropriate position in corridor options. They also show that the wealth with the ES constraint is equivalent to an option contingent upon a minimum of two securities; one is risky and the other is riskless. Determining the appropriate form of $g(x)$ corresponding to the risk limits is equivalent to deriving a robust portfolio management strategy according to the risk limits.

3.2 Robust Portfolio Management with VaR Constraint

In this section, we specify the risk limits. The first consideration for the risk limits is the VaR constraint. By purchasing a put option on the terminal wealth, the fund manager maintains her terminal wealth to be above the level $\underline{W}$ in all states. This is the so-called *portfolio insurance strategy.* However, this strategy is somewhat expensive because in favorable states the probability that wealth is significantly lower than $\underline{W}$ is very small. Therefore, we can consider robust portfolio management that applies VaR constraint. Specifically, in addition to purchasing the put option selling a put option with a lower strike price than the purchased put option gives cheaper price than purchasing the portfolio insurance, and satisfies the VaR constraint. In this sense, the appropriate form of $g(x)$ that corresponds to the VaR constraint is determined by

$$g(x) = x + (\underline{W} - x)\mathbf{1}_{\{x<\underline{W}\}} - (k_\alpha - x)\mathbf{1}_{\{x<k_\alpha\}} - (\underline{W} - k_\alpha)\mathbf{1}_{\{x<k_\alpha\}}, \tag{13}$$

where k_α is a constant to be determined and is regarded as the strike price of the put option in which the VaR constraint is satisfied. The third and fourth terms of the right hand side in (13) have negative values and imply that a large loss can happen during unfavorable states. Furthermore, the losses due to the robust portfolio management may be significantly increased or decreased due to the ambiguity aversion of fund managers. Thus, how and how much ambiguity aversion affects the portfolio management must be investigated, especially in unfavorable states.

By utilizing the Lagrangian method, we can obtain the unconstrained portfolio selection problem; its objective function is

$$V(t, w) = U(t, x) = E_{t,x}\big[\tilde{u}\big(g(X_T)\big)\big],$$

where

$$\tilde{u}(x) = \frac{x^{1-\gamma}}{1-\gamma} - \lambda_\alpha \mathbf{1}_{\{x<\underline{W}\}}.$$

Here, λ_α is a Lagrange multiplier that accounts for the VaR constraint.

We state a theorem concerning value function V, optimal wealth process ϕ, and optimal risky investment π_t^*. First, we introduce the classical Black and Scholes (1973) model. We denote the call option value by

$$Call(t,x,r,\sigma,K) = N\big(d_1(t,x,r,\sigma,K)\big)x - N\big(d_2(t,x,r,\sigma,K)\big)Ke^{-r(T-t)},$$

where

$$d_1(t,x,r,\sigma,K) = \frac{1}{\sigma\sqrt{T-t}}\Big[\ln\Big(\frac{x}{K}\Big)+\Big(r+\frac{\sigma^2}{2}\Big)(T-t)\Big],\ d_2(t,x,r,\sigma,K) = d_1(t,x,r,\sigma,K)-\sigma\sqrt{T-t},$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. Here, x is the price of the underlying asset at time t, r is the risk-free interest rate, σ is the volatility of returns of the underlying asset, and K is the strike price. The put option value is given by put-call parity as

$$Put(t,x,r,\sigma,K) = N\big(-d_2(t,x,r,\sigma,K)\big)Ke^{-r(T-t)} - N\big(-d_1(t,x,r,\sigma,K)\big)x.$$

Theorem 3.1 *The value function $V(t,w)$ is characterized by*

$$\begin{aligned} U(t,x) = {} & \frac{e^{\tilde{r}(T-t)}}{1-\gamma}\Big[x^{1-\gamma} + Put\Big(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, \underline{W}^{1-\gamma}\Big) - Put\Big(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, k_\alpha^{1-\gamma}\Big)\Big] \\ & - \frac{e^{\tilde{r}(T-t)}}{1-\gamma}(\underline{W}^{1-\gamma} - k_\alpha^{1-\gamma} + (1-\gamma)\lambda_\alpha)e^{-\tilde{r}(T-t)}P_{t,x}\{X_T < k_\alpha\}, \end{aligned} \tag{14}$$

where

$$\tilde{r} = (1-\gamma)\Big(r + \frac{1}{2}\frac{\kappa\kappa_\theta}{\gamma+\theta}\Big),$$

and k_α and λ_α are constants to be determined. The optimal wealth process W_t^* is given by

$$\begin{aligned} \phi(t,x) = {} & x + Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W}\Big) - Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big) \\ & - (\underline{W} - k_\alpha)e^{-r(T-t)}P^Q_{t,x}\{X_T < k_\alpha\}. \end{aligned} \tag{15}$$

Moreover, if we assume that

$$-\frac{xU_{xx}}{U_x} = -\frac{x\phi_{xx}}{\phi_x} + \gamma, \tag{16}$$

then optimal risky investment π_t^* is given by

$$\pi_t^* = \frac{\mu - r}{\sigma^2}\frac{\phi_x}{\phi}\frac{1}{\Big[\frac{\gamma}{x} + \frac{\theta}{(1-\gamma)U(t,x)}U_x\Big]}. \tag{17}$$

Proof. See Appendix 6.1. **Q.E.D.**

Note that

$$d_1(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, K^{1-\gamma}) = d_1(t, x, r, \frac{\kappa}{\gamma+\theta}, K).$$

Then

$$\begin{aligned}&\frac{\partial}{\partial x}\Big[N\big(-d_2(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, \underline{W}^{1-\gamma})\big)\underline{W}^{1-\gamma}e^{-r(T-t)} - N\big(-d_1\big(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, \underline{W}^{1-\gamma}\big)\big)x^{1-\gamma}\Big] \\ &= (1-\gamma)x^{-\gamma}\frac{\partial}{\partial x}\Big[N\big(-d_2(t, x, r, \frac{\kappa}{\gamma+\theta}, \underline{W})\big)\underline{W}e^{-r(T-t)} - N\big(-d_1(t, x, r, \frac{\kappa}{\gamma+\theta}, \underline{W})\big)x\Big].\end{aligned} \tag{18}$$

A straightforward calculation gives

$$e^{-r(T-t)}\frac{\partial}{\partial x}P^Q_{t,x}\{X_T < k_\alpha\} = \frac{x}{k_\alpha}N'\big(d_1(t, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha)\big)d_1'(t, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha), \tag{19}$$

and

$$e^{-\tilde{r}(T-t)}\frac{\partial}{\partial x}P_{t,x}\{X_T < k_\alpha\} = \frac{1}{k_\alpha^{-\gamma}}x^{-\gamma}e^{-r(T-t)}\frac{\partial}{\partial x}P^Q_{t,x}\{X_T < k_\alpha\}. \tag{20}$$

Substituting equations (18), (19), and (20) into the derivatives of optimal wealth (15) and the fund manager's value function (14) with respect to x yields

$$U_x(t, x) = e^{\tilde{r}(T-t)}x^{-\gamma}\phi_x(t, x), \tag{21}$$

where

$$\frac{\underline{W}^{1-\gamma}}{1-\gamma} - \frac{k_\alpha^{1-\gamma}}{1-\gamma} + \lambda_\alpha = (\underline{W} - k_\alpha)k_\alpha^{-\gamma}.$$

With the above relationship, the two unknown constants k_α and λ_α are attained by $P\{g(X_T) < \underline{W}\} = \alpha$. Note that relationship (21) implies assumption (16).

Optimal wealth process ϕ formulated by (15) is equivalent to the option price corresponding to claim g at time t. At the optimum, the ambiguity-averse fund manager has wealth x at time t in the absence of the VaR constraint, which is the first term of the right hand side in (15). To satisfy the VaR constraint, the manager should take a long position in the put option with strike $\underline{W}$ and a short position in the put option with strike k_α, which is lower than $\underline{W}$. These actions are reflected in the second and third terms. The last term stems from the fact that the VaR constraint allows the situation in which the terminal wealth can be lower than $\underline{W}$ with a probability α. Contrary to portfolio insurance, the fund manager does not mind large losses, in which case $X_T < k_\alpha$ is allowed with probability α. The manager can then be exposed to large losses during unfavorable states; this is the meaning of the last term in the optimal wealth (15).

Optimal risky investment π_t^* suggested by (17) takes different forms compared to the classical risky investment obtained from Merton (1969). The value of π_t^* may be significantly affected by the presence of both the VaR constraint and the ambiguity aversion θ. Without the preference for robustness ($\theta = 0$) risky investment is generated mainly by the optimal wealth ϕ and reduces to the one formulated by Maenhout (2004), whereas in the presence of the robustness ($\theta > 0$) the value function U directly affects the risky investment.

3.3 Robust Portfolio Management with ES Constraint

Under the risk-neutral measure Q, the optimal wealth process X_t without the risk limits follows

$$dX_t = rX_t dt + \frac{\kappa}{\gamma + \theta} X_t dB_t, \quad X_0 = x > 0.$$

Because the financial market is complete, a unique state price density[7] ξ_t exists; it follows

$$d\xi_t = -r\xi_t dt - \kappa_\theta dB_t, \quad \xi_0 = \xi > 0.$$

Then the fund manager is constrained as

$$E[\xi_T(\underline{W} - X_T)\mathbf{1}_{\{X_T \le \underline{W}\}}] \le \alpha.$$

Utilizing the dynamic programming approach to the constrained portfolio (Kraft and Steffensen, 2013) gives the following optimal wealth process W_t^* in the presence of the ES constraint:

$$W_t^* = \phi(t, x) = E_{t,x}^Q[e^{-r(T-t)} g(X_T)],$$

where $g(x)$ is a claim of x depending upon the ES constraint and should be determined corresponding to that constraint.[8] We guess the claim $g(x)$ as the following:

$$g(x) = x + (\underline{W} - x)\mathbf{1}_{\{x < \underline{W}\}} - \frac{\underline{W}}{k_\alpha}(k_\alpha - x)\mathbf{1}_{\{x < k_\alpha\}},$$

[7]In the previous subsection, we argued that formulating the state price density explicitly is difficult due to the ambiguity aversion. Specifically, because the endogenous drift term h_t in the wealth process (3) should be chosen, before that we cannot know the exact form of the state price density. However, given that the drift term is determined to be h_t^* we can easily characterize the unique state price density.

[8]For notational simplicity, we use the same notations of optimal wealth process $W^*(t)$, $\phi(t, x)$, the claim $g(x)$ according to the ES constraint, and the constants k_α, λ_α, which are described in Section 3.2.

where k_α is a constant to be determined according to the ES constraint. In the presence of the ES constraint, the fund manager cares about big losses, so that in unfavorable states the losses are significantly smaller than in the presence of the VaR constraint. The robust portfolio management strategy with the ES constraint has lower costs than does the portfolio insurance strategy, and allows the fund manager to effectively cope with the big losses that occur during the unfavorable states.

The fund manager's problem in the presence of ES constraint is formulated as

$$U(t,x,\xi) = E_{t,x}\big[\tilde{u}\big(g(X_T)\big)\big],$$

where

$$\tilde{u}(x,\xi) = \frac{x^{1-\gamma}}{1-\gamma} - \lambda_\alpha \xi(\underline{W} - x)\mathbf{1}_{\{x \le \underline{W}\}},$$

and λ_α is a Lagrangian multiplier induced by the ES constraint.

We provide a theorem relative to value function U, optimal wealth process ϕ, and optimal risky investment π_t^* in the presence of the ES constraint.

Theorem 3.2 *The value function $U(t,x,\xi)$ follows*

$$U(t,x,\xi) = \frac{e^{\tilde{r}(T-t)}}{1-\gamma}\Big[x^{1-\gamma} + Put\Big(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, \underline{W}^{1-\gamma}\Big) - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{1-\gamma} Put\Big(t, x^{1-\gamma}, \tilde{r}, \frac{(1-\gamma)\kappa}{\gamma+\theta}, k_\alpha^{1-\gamma}\Big)\Big]$$
$$- \lambda_\alpha \frac{\underline{W}}{k_\alpha}\xi Put\Big(t, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha\Big),$$

where k_α and λ_α are constants to be determined. The optimal wealth process $\phi(t,x)$ is given by

$$\phi(t,x) = x + Put\Big(t, x, r, \frac{\kappa}{\gamma+\theta}, \underline{W}\Big) - \frac{\underline{W}}{k_\alpha} Put\Big(t, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha\Big). \tag{22}$$

If we assume that

$$-\frac{xU_{xx}}{U_x - U_{\xi x}\xi} = -\frac{x\phi_{xx}}{\phi_x} + \gamma, \tag{23}$$

then optimal risky investment, π_t^*, is given by

$$\pi_t^* = \frac{\mu - r}{\sigma^2}\frac{\phi_x}{\phi}\frac{1}{\Big[\frac{\gamma}{x} + \frac{\theta}{(1-\gamma)}U_x\Big]}.$$

Proof. See Appendix 6.2. **Q.E.D.**

By calculating the derivatives of U, one can obtain the following relationship:

$$U_x - U_{\xi x}\xi = e^{\tilde{r}(T-t)}x^{-\gamma}\Big[1 + \frac{\partial}{\partial x}Put\Big(t, x, r, \frac{\kappa}{\gamma+\theta}, \underline{W}\Big) - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{1-\gamma}\frac{\partial}{\partial x}Put\Big(t, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha\Big)\Big].$$

Furthermore, using the derivatives of ϕ assumption (23) can be rewritten as

$$0 = \Big(\frac{\gamma}{x}(U_x - U_{\xi x}\xi) + U_{xx}\Big)\phi_x - \phi_{xx}U_x$$
$$= \Big[\frac{\partial}{\partial x}Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big)\frac{\partial^2}{\partial x^2}Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W}\Big) - \frac{\partial^2}{\partial x^2}Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big)$$
$$- \frac{\partial^2}{\partial x^2}Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big)\frac{\partial}{\partial x}Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W}\Big)\Big]\Big[\lambda_\alpha\frac{\underline{W}}{k_\alpha}\xi - e^{\tilde{r}(T-t)}\xi^{-\gamma}\Big\{\frac{\underline{W}}{k_\alpha} - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{1-\gamma}\Big\}\Big].$$

Then, due to the relationship $\xi_t = \xi_0 e^{-\tilde{r}t}(X_t/x)^{-\gamma}$ we can obtain

$$\lambda_\alpha = x^{-\gamma}e^{\tilde{r}T}\Big\{1 - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{-\gamma}\Big\}. \tag{24}$$

Because the unknown constant k_α is obtained by $P\{g(X_T) < \underline{W}\} = \alpha$, the unknown constant λ_α is also attained using (24).

Theorem 3.2 explains how at optimum the fund manager takes robust portfolio management strategy by taking positions of put options with different strike prices, and shows the dependence of risky investment on the fund manager's ambiguity aversion θ. More specifically, relative to the optimal wealth $\phi(t,x)$ the fund manager has the wealth x at time t. Because she is constrained by the ES constraint, the second term of the right hand side in (22) represents the purchase of a put option with strike price $\underline{W}$. This is the portfolio insurance strategy. This strategy comes expensive because the losses are not frequent; according to the ES constraint the fund manager takes a short position of $\underline{W}/k_\alpha$ put options with strike price k_α, which is reflected in the last term in (22). The sum of wealth x, a long and a short position in put options is the robust portfolio management with the ES constraint.

4 Implications

4.1 Optimal Strategies with VaR constraint

Optimal terminal wealth with the VaR constraint is a discontinuous function of terminal wealth X_T without the VaR constraint (Figure 1). A discontinuity occurs at k_α; this discontinuity implies that at maturity the fund manager with the VaR constraint protects a shortfall of her wealth compared to the minimum wealth level $\underline{W}$ only if the terminal wealth X_T without the VaR constraint lies between k_α and $\underline{W}$. The manager allows the shortfall for the case where X_T is below k_α and this means that she lets her wealth be exposed to large losses with probability α. In this sense, k_α can be regarded as a tolerance for loss.[9] When

[9] Note that k_α is attained by $P\{g(X_T) < \underline{W}\} = \alpha$.

we consider the ambiguity aversion of the fund manager, the fund manager with the VaR constraint is willing to protect against more losses, or equivalently, has a higher tolerance for loss k_α (Figure 1 (B)). The amount of loss is less severe for the ambiguity-averse fund manager than for one who is not ambiguity-averse.

[Insert Figure 1 here.]

Figure 1 seems to imply that the severe losses may happen in the bad-states when fund management uses the VaR constraint. To quantify expected losses, we define *a loss amount* as the ratio between the present value of expected losses and the product of the pre-specified probability α and the initial wealth w. The fund manager can be exposed to large losses with probability α. We use $W_{k_\alpha} \equiv \phi(t, k_\alpha)$ to denote the fund manager's tolerance for loss with probability α.

Theorem 4.1 *We define a loss amount as*

$$L(x; w, \theta) = E\Big[\frac{\xi_T}{\xi_0}\Big(W_{k_\alpha} - g(X_T)\Big)\boldsymbol{1}_{\{g(X_T)<k_\alpha\}}\Big]\Big/(\alpha \times w).$$

Then $L(\cdot)$ is computed as

$$\begin{aligned} L(x; w, \theta) = &\frac{1}{\xi_0(\alpha \times w)}\Big[N\Big(-d_2(0, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha)\Big)k_\alpha e^{-rT} - N\Big(-d_1(0, x, r, \frac{\kappa}{\gamma+\theta}, k_\alpha)\Big)x\Big] \\ &+ (W_{k_\alpha} - k_\alpha)E\Big[\frac{\xi_T}{\xi_0}\boldsymbol{1}_{\{X_T \le k_\alpha\}}\Big]\Big/(\alpha \times w), \end{aligned}$$

where initial wealth $w = \phi(0, x)$.

Proof. See Appendix 6.3. **Q.E.D.**

Basak and Shapiro (2001) have a negative point of view of using VaR-based risk management, because it leads to severe losses during economic downturns. However, this is not necessarily true when one considers the ambiguity-averse fund manager. The loss amount $L(\cdot)$ is dependent on ambiguity aversion and we can infer from Figure 1 that the largest losses may not be severe when the fund manager exploits a robust portfolio management strategy. Now we turn to investigate the effect of ambiguity aversion on the loss amount $L(\cdot)$. $L(\cdot)$ is sensitive to changes of ambiguity aversion θ and initial wealth x without the VaR constraint (Table 1). $L(\cdot)$ decreases significantly as ambiguity aversion θ increases. For instance, when

$x = 0.5$, the loss amount is up to 126.1% in the absence of ambiguity aversion ($\theta = 0$), whereas it has substantially lower losses of 66.8% in the presence of high ambiguity aversion ($\theta = 5$). Hence, our robust portfolio management with the VaR constraint can address the shortcoming that the VaR constraint can cause severe losses when happen, and we assert that in that case the VaR constraint can be an effective tool in risk management. The loss amount decreases as initial wealth x without the VaR constraint increases, because in that case large losses are not frequent.

[Insert Table 1 here.]

Optimal wealth process W_t^* has a one to one relation with the wealth process X_t without the VaR constraint (Figure 2). When X_t is less than k_α, W_t^* reveals a linear relationship with X_t; this linearity occurs because the fund manager with the VaR constraint does not protect against large losses. W_t^* exceeds X_t where it lies between k_α and $\underline{W}$. At the optimum, the manager tries very hard to protect against small losses, so she increase her wealth to be higher than $\underline{W}$. The difference between W_t^* and X_t increases as ambiguity aversion θ increases and a moderate value of θ may even lead to a discontinuous shape of optimal wealth, which is observed at maturity. In the higher values of X_t, W_t^* is again linear with respect to X_t.

[Insert Figure 2 here.]

The optimal risky investment π_t^* is not a constant function of wealth W_t (Figure 3). To maintain the manager's wealth to be above $\underline{W}$, she increases her risky investment when W_t lies between W_{k_α} and $\underline{W}$. The manager attempts to have her wealth reach the level $\underline{W}$ by investing aggressively in the stock. The manager with ambiguity aversion invests a significantly lower fraction of her wealth in the stock than the manager without ambiguity aversion does.[10] Basak and Shapiro (2001) demonstrate that VaR-based portfolio management allows the fund manager to have a larger risky investment than the fund manager without VaR constraint does, so that the VaR-based management strategy incurs severe losses during economic downturns. However, our robust portfolio management suggests that if the fund manager formulates her investment strategy by considering the VaR constraint together with

[10] We support the results suggested by Maenhout (2004) in that the stock investment has a negative relationship with ambiguity aversion.

her ambiguity aversion, then her risky investment is substantially lower compared to that without the ambiguity aversion, and consequently mitigates the large losses.

[Insert Figure 3 here.]

4.2 Optimal Strategies with ES constraint

With the ES constraint optimal terminal wealth W_T^* is a piecewise linear function of X_T (Figure 4). The optimal terminal wealth is continuous, and consequently does not suffer from large losses compared to the VaR constraint case. As the fund manager with the VaR constraint does, the fund manager with the ES constraint tends to insure against small losses when X_T lies between k_α and $\underline{W}$. She allows the shortfall when X_T is below k_α with probability α, which induces large losses for the case of the VaR constraint, but due to the continuity of optimal terminal wealth the severe losses are not always followed. The ambiguity aversion of fund manager allows her to have smaller risk tolerance, or equivalently, larger k_α, when compared with the one in the absence of the ambiguity aversion (Figure 4 (B)). This implies that the ambiguity-averse fund manager with the ES constraint may be less exposed to losses than the fund manager without ambiguity aversion.

[Insert Figure 4 here.]

ES-based risk management has been regarded as an alternative risk measure to address the shortcomings of VaR-based risk management. Indeed, a fund manager who uses robust portfolio management with the ES constraint experiences a substantial loss reduction compared to that given by VaR-based portfolio management. Moreover, our robust portfolio management with the ES constraint has a superiority in that the ambiguity aversion of fund manager reduces losses significantly, compared to the ES-based risk management formulated by Basak and Shapiro (2001).

As we defined in the previous section, we introduce the following loss amount and compute it:

Theorem 4.2 *We define a loss amount as*

$$L(x; w, \theta) = E\Big[\frac{\xi_T}{\xi_0}\Big(\underline{W} - g(X_T)\Big)\boldsymbol{1}_{\{g(X_T)<k_\alpha\}}\Big]\Big/(\alpha \times w).$$

Then $L(\cdot)$ is computed as

$$L(x; w, \theta) = \frac{\underline{W}}{k_\alpha \xi_0(\alpha \times w)} \Big[N\Big(- d_2(0, x, r, \frac{\kappa}{\gamma + \theta}, k_\alpha)\Big) k_\alpha e^{-rT} - N\Big(- d_1(0, x, r, \frac{\kappa}{\gamma + \theta}, k_\alpha)\Big) x \Big].$$

Proof. See Appendix 6.4. **Q.E.D.**

In this case, $L(\cdot)$ is relatively insensitive to changes of ambiguity aversion θ and initial wealth x without the ES constraint (Table 2). As we expected by the implications derived from the optimal terminal wealth, the fund manager with the ES constraint achieves significant loss reduction compared to the loss induced by the VaR-based portfolio management. In particular, the loss amount is significantly less sensitive to changes of wealth x. This implies that the fund manager who adopts the ES constraint is more reluctant to allow the shortfall from the lower bound $\underline{W}$ than the fund manager with the VaR constraint is. Furthermore, when we consider the ambiguity aversion of a fund manager, the loss amount decreases as the ambiguity aversion increases. Apparently, the effect of ambiguity aversion on the loss reduction is somewhat small, but our robust portfolio management is certainly effective for reducing losses, especially it can substantially decrease the severe losses caused by the VaR constraint. (Table 1).

[Insert Table 2 here.]

Optimal wealth process W_t^* is a one to one correspondence with X_t (Figure 5). In contrast to the VaR constraint case (Figure 2), the optimal wealth W_t^* has no discontinuity when X_t is between k_α and $\underline{W}$. This is the case when we consider the ambiguity aversion of a fund manager; it implies that the ambiguity-averse fund manager is not willing to allow large losses, which would not happen when the manager is constrained by the VaR constraint.

[Insert Figure 5 here.]

Optimal risky investment strategy with the ES constraint is not a constant function of X_t (Figure 6). The fund manager formulates her investment strategy to reduce the stock investment in the intermediate region of W_t. This is opposite to the strategy of the manager with the VaR constraint, and consistent with the result suggested by Basak and Shapiro (2001), in which the fund manager never increases her stock investment to satisfy the ES constraint. Furthermore, the investment strategy of the ambiguity-averse fund manager with the ES constraint tends to become increasingly similar to the one without the constraint as ambiguity aversion increases; this implies that at the optimum the highly-ambiguity averse fund manager may not care about her risk limits. In this sense, the ambiguity aversion may be a more significant factor in portfolio management than are risk limits such as the VaR and the ES constraint.

[Insert Figure 6 here.]

5 Conclusion

We present robust portfolio management with risk limits. We formulate an optimal portfolio strategy in the presence of both portfolio constraints and ambiguity aversion. We jointly consider risk limits represented by Value-at-Risk (VaR) constraint or expected shortfall (ES) constraint and ambiguity, and show quantitatively that these limits are significant. Contrary to traditional investment rules, optimal portfolio management takes different forms in which it is significantly affected by the presence of risk limits and ambiguity aversion.

With reasonably calibrated market parameters, we find that the loss amount due to our robust portfolio management with the VaR constraint significantly decreases as ambiguity aversion increases. Hence, in that case the VaR constraint can be an effective tool in risk management. Our numerical experiments confirm the intuition of Basak and Shapiro (2001) that the fund manager constrained by the ES constraint is more reluctant to allow the shortfall from the lower bound $\underline{W}$ than the fund manager with the VaR constraint is, and show that when we consider the ambiguity aversion of the fund manager, the loss amount with the ES constraint decreases as the ambiguity aversion increases.

Interestingly, our robust portfolio management suggests that if the fund manager formulates her investment strategy by considering the VaR constraint together with her ambiguity aversion, then her risky investment is substantially lower compared to that of Basak and Shapiro (2001). Relative to the ES constraint, the robust portfolio management amplifies the effect of the constraint, and, thus, the investment strategy of the ambiguity-averse fund manager tends to become increasingly similar to the one without the ES constraint as the ambiguity aversion increases. In this sense, the ambiguity aversion may be a more significant factor in portfolio management than are risk limits such as the VaR and the ES constraint.

6 Appendix

6.1 The Proof of Theorem 3.1

Utilizing $g(x)$ given in (13) gives

$$\begin{aligned}\phi(t,x) &= E^Q_{t,x}[e^{-r(T-t)}\{X_T + (\underline{W} - X_T)\mathbf{1}_{\{X_T<\underline{W}\}} - (k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}} - (\underline{W} - k_\alpha)\mathbf{1}_{\{X_T<k_\alpha\}}\}] \\ &= x + N\big(-d_2(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W})\big)\underline{W}e^{-r(T-t)} - N\big(-d_1(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W})\big)x \\ &\quad - \{N\big(-d_2(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\big)k_\alpha e^{-r(T-t)} - N\big(-d_1(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\big)x\} \\ &\quad - (\underline{W} - k_\alpha)e^{-r(T-t)}P^Q_{t,x}\{X_T < k_\alpha\} \\ &= x + Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W}\Big) - Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big) - (\underline{W} - k_\alpha)e^{-r(T-t)}P^Q_{t,x}\{X_T < k_\alpha\}.\end{aligned}$$

We then calculate $U(t,x)$ as

$$\begin{aligned}U(t,x) &= E_{t,x}[\tilde{u}(g(X_T))] \\ &= E_{t,x}\Big[\frac{1}{1-\gamma}\Big\{X_T + (\underline{W} - X_T)\mathbf{1}_{\{X_T<\underline{W}\}} - (k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}} - (\underline{W} - k_\alpha)\mathbf{1}_{\{X_T<k_\alpha\}}\Big\}^{1-\gamma} - \lambda_\alpha\mathbf{1}_{\{x<k_\alpha\}}\Big] \\ &= \frac{e^{\tilde{r}(T-t)}}{1-\gamma}\Big[x^{1-\gamma} + N\big(-d_2(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},\underline{W}^{1-\gamma})\big)\underline{W}^{1-\gamma}e^{-\tilde{r}(T-t)} \\ &\quad - N\big(-d_1\big(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},\underline{W}^{1-\gamma}\big)\big)x^{1-\gamma} - \{N\big(-d_2(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},k_\alpha^{1-\gamma})\big)k_\alpha^{1-\gamma}e^{-\tilde{r}(T-t)} \\ &\quad - N\big(-d_1\big(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},k_\alpha^{1-\gamma}\big)\big)x^{1-\gamma}\}\Big] \\ &\quad - \frac{e^{\tilde{r}(T-t)}}{1-\gamma}(\underline{W}^{1-\gamma} - k_\alpha^{1-\gamma} + (1-\gamma)\lambda_\alpha)e^{-\tilde{r}(T-t)}P_{t,x}\{X_T < k_\alpha\}, \\ &= \frac{e^{\tilde{r}(T-t)}}{1-\gamma}\Big[x^{1-\gamma} + Put\Big(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},\underline{W}^{1-\gamma}\Big) - Put\Big(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},k_\alpha^{1-\gamma}\Big)\Big] \\ &\quad - \frac{e^{\tilde{r}(T-t)}}{1-\gamma}(\underline{W}^{1-\gamma} - k_\alpha^{1-\gamma} + (1-\gamma)\lambda_\alpha)e^{-\tilde{r}(T-t)}P_{t,x}\{X_T < k_\alpha\}.\end{aligned}$$

Now we verify that the value function $V(t,w)$ formulated by (9) is characterized by $U(t,x)$. We know from the terminal condition that

$$U(T,x) = \tilde{u}\big(g(x)\big) = \tilde{u}\big(\phi(T,x)\big) = V(T,w).$$

One can show that $U(t,x)$ satisfies

$$U_t(t,x) = -\Big(r + \frac{\kappa\kappa_\theta}{\gamma+\theta}\Big)xU_x(t,x) - \frac{1}{2}\Big(\frac{\kappa}{\gamma+\theta}\Big)^2 x^2 U_{xx}(t,x),$$
$$U(T,x) = \tilde{u}\big(g(x)\big).$$

If we denote V^* by $V^*(t,\phi(t,x)) = U(t,x)$, then $V^*(t,\phi(t,x))$ is the value function by solving HJB (5) when V is replaced by V^*. Furthermore, if we assume that

$$-\frac{xU_{xx}}{U_x} = -\frac{x\phi_{xx}}{\phi_x} + \gamma,$$

then optimal risky investment π_t^* is given by

$$\pi_t^* = \frac{\mu - r}{\sigma^2}\frac{\phi_x}{\phi}\frac{1}{\Big[\frac{\gamma}{x} + \frac{\theta}{(1-\gamma)U(t,x)}U_x\Big]}.$$

6.2 The Proof of Theorem 3.2

A straightforward calculation gives the optimal wealth process $\phi(t,x)$ and the value function $U(t,x,\xi)$:

$$\begin{aligned}
\phi(t,x) &= E_{t,x}^Q\Big[e^{-r(T-t)}\Big\{X_T + (\underline{W} - X_T)\mathbf{1}_{\{X_T<\underline{W}\}} - \frac{\underline{W}}{k_\alpha}(k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}}\Big\}\Big] \\
&= E_{t,x}^Q\Big[e^{-r(T-t)}\Big\{-\frac{\underline{W}}{k_\alpha}(k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}} + (\underline{W} - X_T)\mathbf{1}_{\{X_T<\underline{W}\}} + X_T\Big\}\Big] \\
&= x + N\big(-d_2(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W})\big)\underline{W}e^{-r(T-t)} - N\big(-d_1(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W})\big)x \\
&\quad - \frac{\underline{W}}{k_\alpha}\Big[N\big(-d_2(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\big)k_\alpha e^{-r(T-t)} - N\big(-d_1(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\big)x\Big] \\
&= x + Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},\underline{W}\Big) - \frac{\underline{W}}{k_\alpha}Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big).
\end{aligned}$$

$$\begin{aligned}
U(t,x,\xi) &= E_{t,x}\big[\tilde{u}\big(g\big((X_T)\big),\xi_T\big)\big] \\
&= E_{t,x}\Big[\frac{1}{1-\gamma}\Big\{X_T + (\underline{W} - X_T)\mathbf{1}_{\{X_T<\underline{W}\}} - \frac{\underline{W}}{k_\alpha}(k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}}\Big\}^{1-\gamma} \\
&\quad - \lambda_\alpha\xi_T\Big(\underline{W} - \frac{\underline{W}}{k_\alpha}X_T\Big)\mathbf{1}_{\{X_T\le\underline{W}\}}\Big] \\
&= E_{t,x}\Big[\frac{1}{1-\gamma}\Big\{X_T^{1-\gamma} + (\underline{W}^{1-\gamma} - X_T^{1-\gamma})\mathbf{1}_{\{X_T^{1-\gamma}<\underline{W}^{1-\gamma}\}} - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{1-\gamma}(k_\alpha^{1-\gamma} - X_T^{1-\gamma})\mathbf{1}_{\{X_T^{1-\gamma}<k_\alpha^{1-\gamma}\}}\Big\}\Big] \\
&\quad - \lambda_\alpha\frac{\underline{W}}{k_\alpha}E_{t,x}\Big[\xi_T(k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}}\Big] \\
&= \frac{e^{\tilde{r}(T-t)}}{1-\gamma}\Big[x^{1-\gamma} + N\big(-d_2(t,x^{1-\gamma},\tilde{r},\frac{(1-\gamma)\kappa}{\gamma+\theta},\underline{W}^{1-\gamma})\big)\underline{W}^{1-\gamma}e^{-\tilde{r}(T-t)}
\end{aligned}$$

$$
\begin{aligned}
&- N\big(-d_1(t,x^{1-\gamma},\tilde r,\frac{(1-\gamma)\kappa}{\gamma+\theta},\underline{W}^{1-\gamma})\big)x^{1-\gamma} - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{1-\gamma}\Big\{N\big(-d_2(t,x^{1-\gamma},\tilde r,\frac{(1-\gamma)\kappa}{\gamma+\theta},k_\alpha^{1-\gamma})\big)k_\alpha^{1-\gamma}\\
&\times e^{-\tilde r(T-t)} - N\big(-d_1(t,x^{1-\gamma},\tilde r,\frac{(1-\gamma)\kappa}{\gamma+\theta},k_\alpha^{1-\gamma})\big)x^{1-\gamma}\Big\}\Big] - \lambda_\alpha\frac{\underline{W}}{k_\alpha}\xi\Big[N\big(-d_2(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\big)\\
&\times k_\alpha e^{-r(T-t)} - N\big(-d_1(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\big)x\Big]\\
&= \frac{e^{\tilde r(T-t)}}{1-\gamma}\Big[x^{1-\gamma} + Put\Big(t,x^{1-\gamma},\tilde r,\frac{(1-\gamma)\kappa}{\gamma+\theta},\underline{W}^{1-\gamma}\Big) - \Big(\frac{\underline{W}}{k_\alpha}\Big)^{1-\gamma}Put\Big(t,x^{1-\gamma},\tilde r,\frac{(1-\gamma)\kappa}{\gamma+\theta},k_\alpha^{1-\gamma}\Big)\Big]\\
&\qquad - \lambda_\alpha\frac{\underline{W}}{k_\alpha}\xi Put\Big(t,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha\Big).
\end{aligned}
$$

The remainder of the proof is similar to that of Theorem (3.1).

The remaining task is to verify that $U(t,x,\xi)$ is indeed the value function $V(t,w)$ given by (11). From the terminal condition,

$$
U(T,x,\xi) = \tilde u\big(g(x),\xi\big) = \tilde u\big(\phi(T,x),\xi\big) = V(T,w),
$$

one can calculate that U satisfies the following differential equation:

$$
\begin{aligned}
U_t(t,x,\xi) = &-\Big(r + \frac{\kappa\kappa_\theta}{\gamma+\theta}\Big)xU_x(t,x,\xi) - \frac{1}{2}\Big(\frac{\kappa}{\gamma+\theta}\Big)^2 x^2U_{xx}(t,x) + r\xi U_\xi(t,x,\xi)\\
&- \frac{1}{2}\kappa_\theta^2 U_{\xi\xi}(t,x,\xi) + \frac{\kappa\kappa_\theta}{\gamma+\theta}U_{\xi x}(t,x,\xi).
\end{aligned}
$$

If we define V^* by $V^*\big(t,\phi(t,x,),\xi\big) = U(t,x,\xi)$, then $V^*\big(t,\phi(t,x,),\xi\big)$ is the value function by solving HJB (1) when V is replaced by V^*. Moreover, if we assume that

$$
-\frac{xU_{xx}}{U_x - U_{\xi x}\xi} = -\frac{x\phi_{xx}}{\phi_x} + \gamma,
$$

then optimal risky investment π_t^* follows

$$
\pi_t^* = \frac{\mu-r}{\sigma^2}\frac{\phi_x}{\phi}\frac{1}{\Big[\frac{\gamma}{x} + \frac{\theta}{(1-\gamma)}U_x\Big]}.
$$

6.3 The Proof of Theorem 4.1

One can show that

$$
\begin{aligned}
L(x;w,\theta) &= E\Big[\frac{\xi_T}{\xi_0}\Big(W_{k_\alpha} - g(X_T)\Big)\mathbf{1}_{\{g(X_T)<k_\alpha\}}\Big]\Big/(\alpha\times w)\\
&= E\Big[\frac{\xi_T}{\xi_0}\Big(W_{k_\alpha} - g(X_T)\Big)\mathbf{1}_{\{X_T<k_\alpha\}}\Big]\Big/(\alpha\times w)\\
&= E\Big[\frac{\xi_T}{\xi_0}(W_{k_\alpha} - X_T)\mathbf{1}_{\{X_T<k_\alpha\}}\Big]\Big/(\alpha\times w)
\end{aligned}
$$

$$
\begin{aligned}
&= E\Big[\frac{\xi_T}{\xi_0}(k_\alpha - X_T)\mathbf{1}_{\{X_T \le k_\alpha\}}\Big]\Big/(\alpha \times w) + E\Big[\frac{\xi_T}{\xi_0}(W_{k_\alpha} - k_\alpha)\mathbf{1}_{\{X_T \le k_\alpha\}}\Big]\Big/(\alpha \times w)\\
&= \frac{1}{\xi_0(\alpha \times w)}\Big[N\Big(-d_2(0,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\Big)k_\alpha e^{-rT} - N\Big(-d_1(0,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\Big)x\Big]\\
&+ (W_{k_\alpha} - k_\alpha)E\Big[\frac{\xi_T}{\xi_0}\mathbf{1}_{\{X_T \le k_\alpha\}}\Big]\Big/(\alpha \times w).
\end{aligned}
$$

6.4 The Proof of Theorem 4.2

A straightforward calculation yields

$$
\begin{aligned}
L(x;w,\theta) &= E\Big[\frac{\xi_T}{\xi_0}\Big(\underline{W} - g(X_T)\Big)\mathbf{1}_{\{g(X_T)<k_\alpha\}}\Big]\Big/(\alpha \times w)\\
&= E\Big[\frac{\xi_T}{\xi_0}\Big(\underline{W} - g(X_T)\Big)\mathbf{1}_{\{X_T<k_\alpha\}}\Big]\Big/(\alpha \times w)\\
&= E\Big[\frac{\xi_T}{\xi_0}\Big(\underline{W} - \frac{\underline{W}}{k_\alpha}X_T\Big)\mathbf{1}_{\{X_T<k_\alpha\}}\Big]\Big/(\alpha \times w)\\
&= \frac{\underline{W}}{k_\alpha\xi_0(\alpha \times w)}E\Big[\xi_T(k_\alpha - X_T)\mathbf{1}_{\{X_T<k_\alpha\}}\Big]\\
&= \frac{\underline{W}}{k_\alpha\xi_0(\alpha \times w)}\Big[N\Big(-d_2(0,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\Big)k_\alpha e^{-rT} - N\Big(-d_1(0,x,r,\frac{\kappa}{\gamma+\theta},k_\alpha)\Big)x\Big].
\end{aligned}
$$

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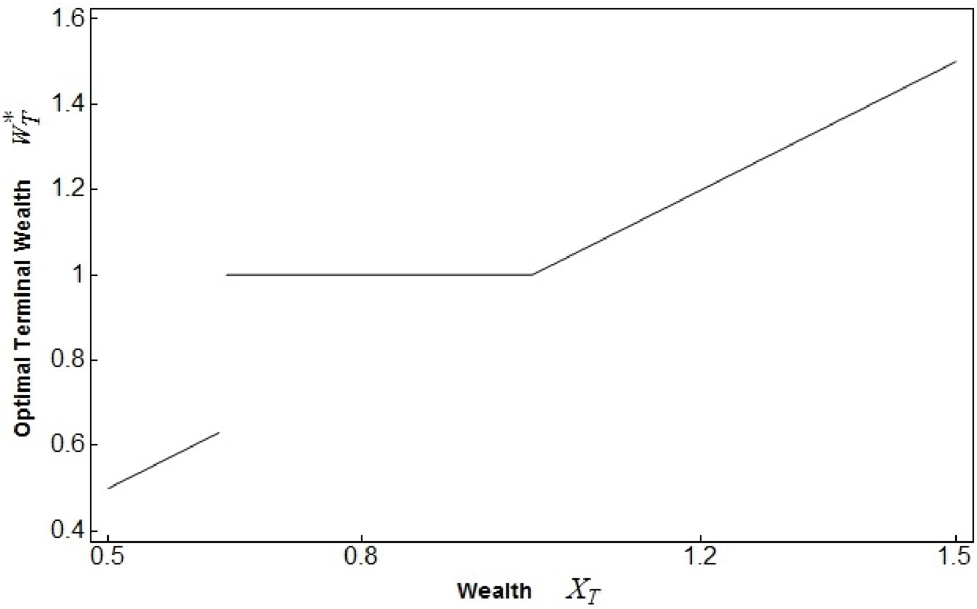
$\theta \setminus x$	0.5	0.7	0.9	1.1	1.3
0	126.1	96.5	64.7	35.6	17.9
1	109.6	79.1	47.5	18.8	5.6
2	89.6	61.7	33.3	9.1	2.5
3	78.3	52.1	25.5	4.8	1.9
4	71.4	46.2	20.7	2.9	1.8
5	66.8	42.2	17.5	2.0	1.6

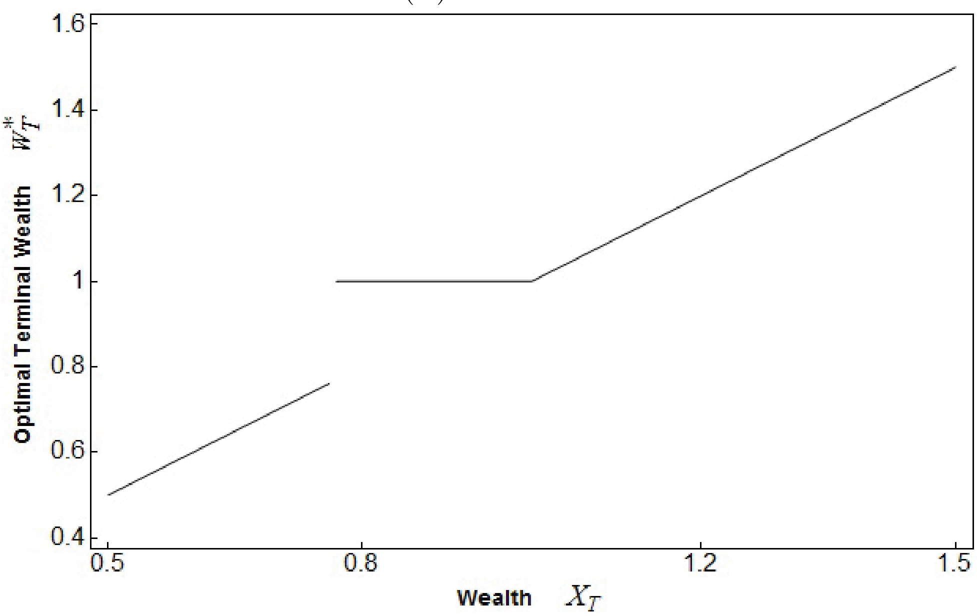
Table 1: **The sensitivity of the (%) loss amount $L(x; w, \theta)$ to changes of ambiguity aversion θ and initial wealth x without the VaR constraint.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, $T = 1$, and $\xi_0 = 1$.

$\theta \setminus x$	0.5	0.7	0.9	1.1	1.3
0	19.3	19.2	18.3	16.3	14.1
1	9.2	9.2	9.0	7.9	6.7
2	5.8	5.8	5.7	5.0	4.3
3	4.2	4.2	4.1	3.6	3.1
4	3.3	3.3	3.2	2.8	2.4
5	2.7	2.7	2.6	2.3	1.9

Table 2: **The sensitivity of the (%) loss amount $L(x; w, \theta)$ to changes of ambiguity aversion θ and initial wealth x without the ES constraint.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, $T = 1$, and $\xi_0 = 1$.



(A) $\theta = 0$



(B) $\theta = 2$

Figure 1: **Optimal terminal wealth W_T^* as a function of X_T.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $x = 0.9$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, and $T = 1$.

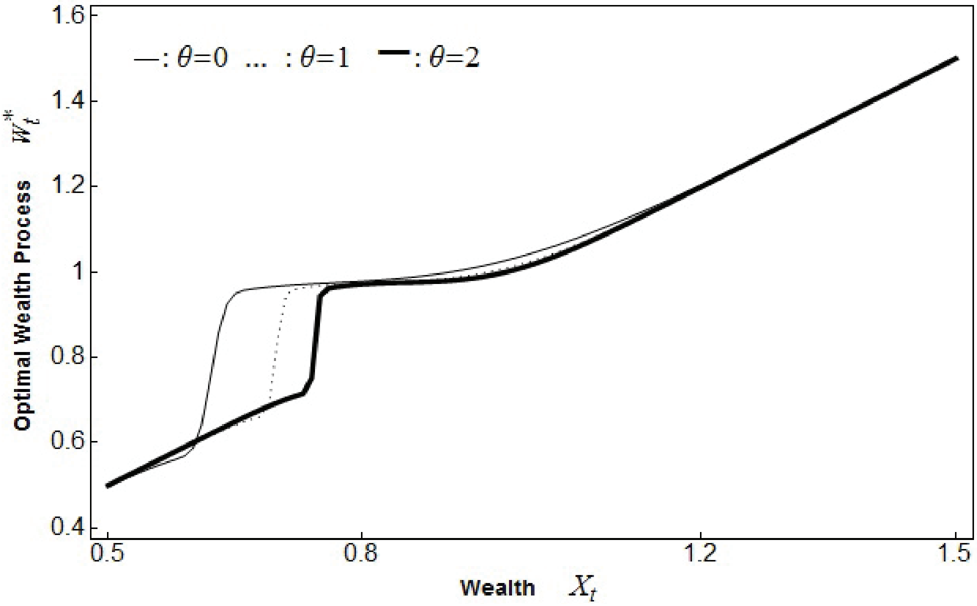


Figure 2: **Optimal wealth process W_t^* as a function of X_t.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $x = 0.9$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, $T = 1$, and $t = 0.5$.

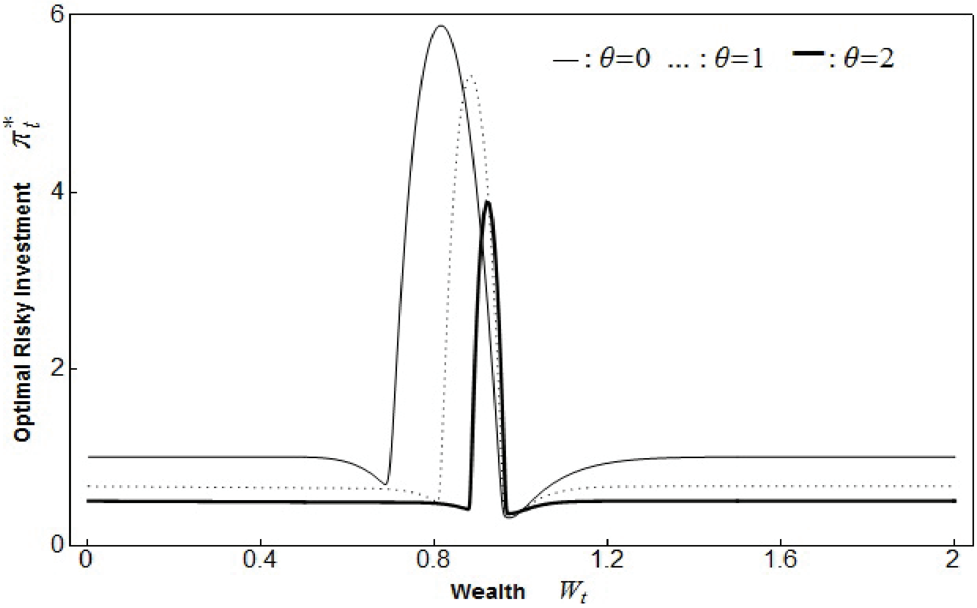
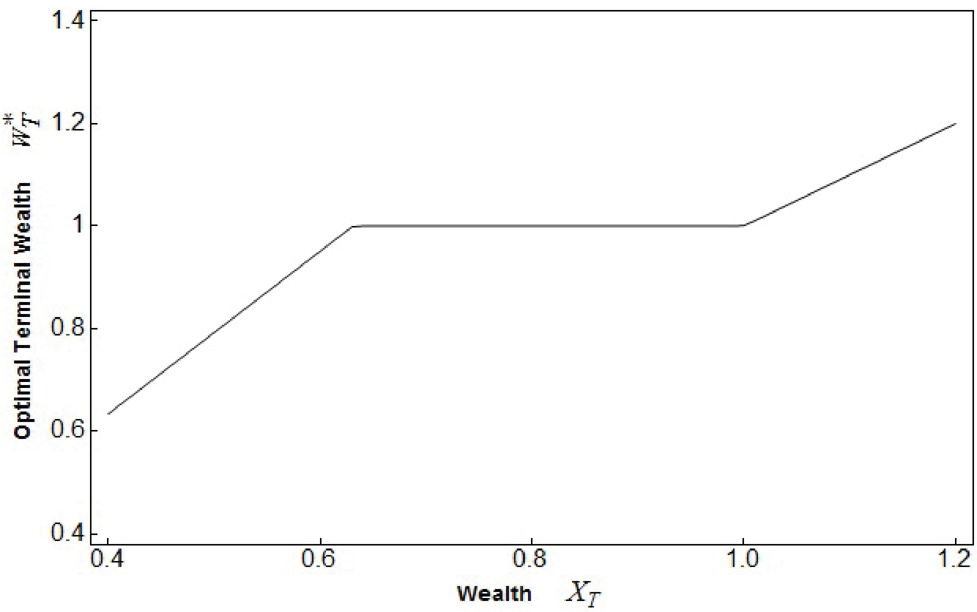
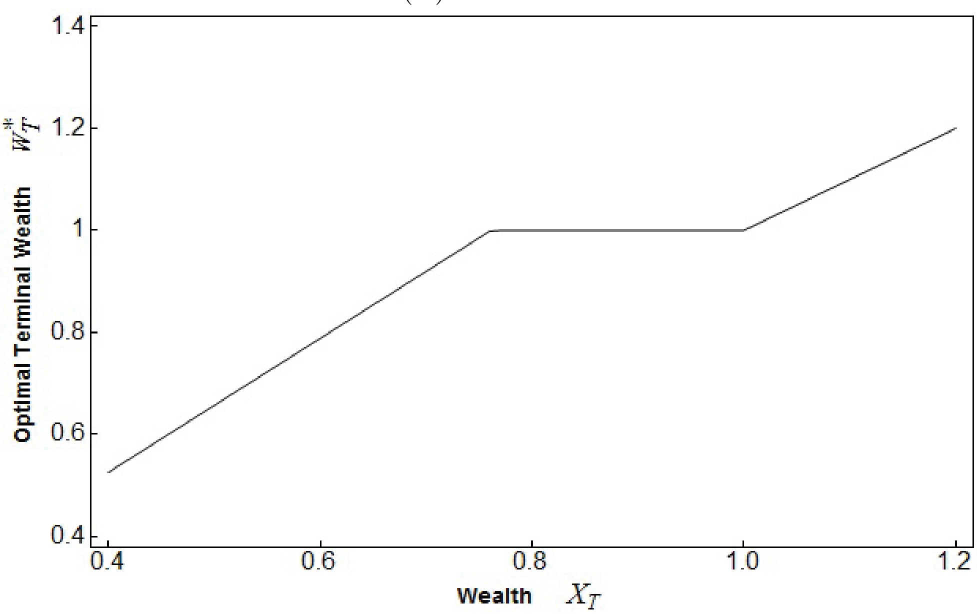


Figure 3: **Optimal risky investment π_t^* as a function of W_t.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $x = 1.1$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, $T = 1$, and $t = 0.5$.



(A) $\theta = 0$



(B) $\theta = 2$

Figure 4: **Optimal terminal wealth W_T^* as a function of X_T.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $x = 0.9$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, and $T = 1$.

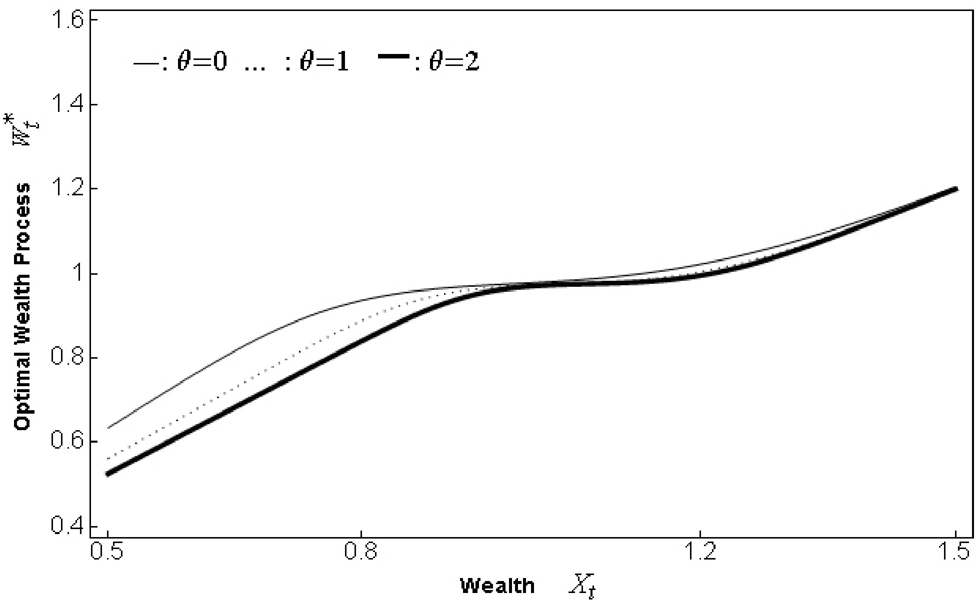


Figure 5: **Optimal wealth process W_t^* as a function of X_t.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $x = 0.9$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, $T = 1$, and $t = 0.5$.

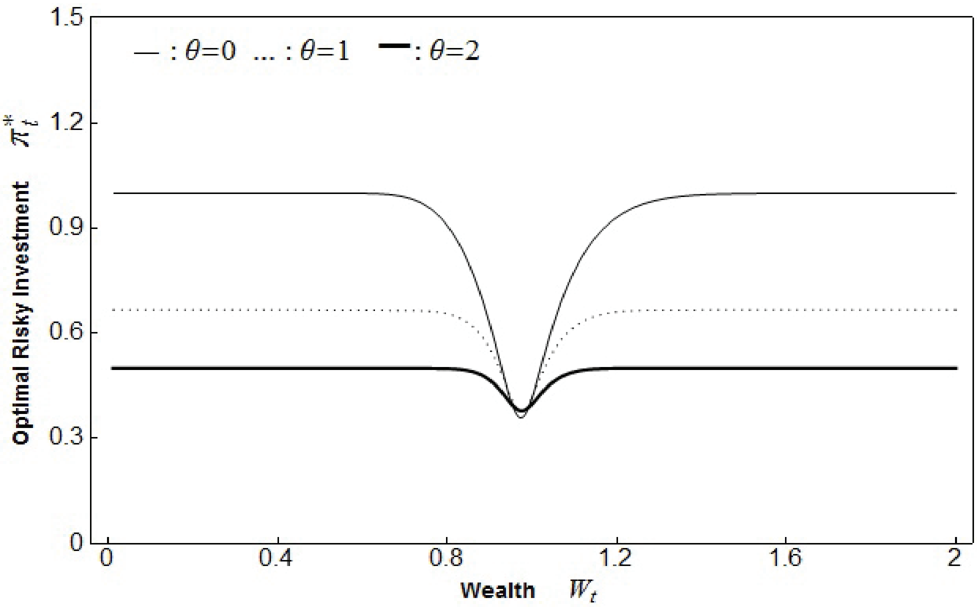


Figure 6: **Optimal risky investment π_t^* as a function of W_t.** Parameter values are set as follows: $\gamma = 2$, $\alpha = 0.01$, $x = 1.1$, $\underline{W} = 1$, $r = 0.05$, $\kappa = 0.4$, $\sigma = 0.2$, $T = 1$, and $t = 0.5$.