Liquidity Hedging with Futures and Forward Contracts

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Abstract

We present a model for developing hedging strategies using both futures and forward contracts. Although financially constrained firms suffer from liquidity problems, they can recover much of the lost value by hedging with futures contracts. However, firms with a limited cash balance must raise risky debt to remain operational for the long term, and then hedge their liquidity using futures and forward contracts. Adding forward contracts into hedging strategies raises the firm value higher than that when hedging with futures contracts alone. We numerically show that a financially constrained firm can increase its firm value close to that of a financially unconstrained firm by issuing minimal risky debt and hedging with both futures and forward contracts.

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1. Introduction

Hedging liquidity from adverse cash flows has been a central issue for corporations.[3] When firms cannot provide the cash flows required to continue its operation, it has to liquidate and incur deadweight costs. Even when faced with a shortage in cash flows, firms might be reluctant to be sold at prevailing fire-sale prices,[4] aggravating the financial crisis. Among financial derivatives, firms can manage their liquidity risk using futures contracts, which are the most liquid and convenient for risk management. However, when firms continuously lose cash flows during long-term operations, hedging with futures contracts alone is not enough to remain solvent because futures contracts also drain cash flows in some states when the firm needs cash flows in most states. Furthermore, firms cannot delay cash settlements for futures contracts because of marking-to-market requirements.

However, forward contracts do not require interim cash settlements before their maturity. When a hedge using forward contracts loses money, a firm accumulates a liability and carries it until maturity. By using forward contracts the firm appears to obtain a loan from a line of credit that matures at the same time because the liability is naturally added to the firm's existing debt. Although the nature of an implicit line of credit built in forward contracts provides hedging opportunities to long-term operations, it also creates complications with respect to the value of risky debt. That is, the firm should consider value changes in their existing risky debt when hedging liquidity using forward contracts.

We consider a financially constrained firm operating in a long horizon and allow it to issue risky debt and trade both futures and forward contracts for hedging purposes. Recognizing that minimizing the variance of the liquidity value of cash balances is the optimal hedging strategy, as in Mello and Parsons (2000), we develop a model to show how the firm can issue minimal risky debt and hedge liquidity using both futures and forward contracts for long-term operations. Hence, hedging with both futures and forward contracts enables the firm to improve liquidity and increase its value to a level higher than by hedging with futures contracts alone.

When the firm experiences cash shortages and needs to stay open for a longer period, it can no longer generate the necessary cash flows from hedging with futures contracts alone. Persistent losses can make hedging infeasible because the firm needs cash in most states. It must raise risky debt to continue operating for the long term. We allow the firm to issue risky debt just once to

[3] See Graham and Harvey (2001) and Lins et al. (2010).
[4] See Diamond and Rajan (2009).

avoid delaying liquidation from additional debt issuances ex-post. Then, the question becomes the amount of risky debt appropriate for the firm. With a minimum amount of risky debt required to cover cash shortages just to continue to the next period, the firm faces cash shortages again in the subsequent period. We set the maximum risky debt amount as that required to cover cash shortages up to the period before the firm is sold for its full value.

Within the range between the minimum and maximum amount of risky debt, we explore an optimal risky debt amount and hedging with futures and/or forward contracts. Using only futures or only forward contracts, the firm cannot significantly reduce its risky debt without liquidating. However, hedging with both futures and forward contracts allows the firm to issue a little more risky debt than the minimum amount. In addition, we are able to construct the optimal risky debt amount to avoid wealth transfer between shareholders and bondholders resulting from hedging with both contracts. Using numerical examples, we illustrate the benefits from hedging using both contracts for long-term operations.

After Meulbroek (2001, 2002) addressed the notion of integrating different risks in corporations and of evaluating various methods to control net exposure, integrated risk management has been examined using two forms of interactions. On the one hand, Allayannis et al. (2001), Bartram (2008), Hankins (2011), and Pantzalis et al. (2001) find a negative interaction between hedging and operating flexibility. Mello et al. (1995) present a model that shows how hedging and operating flexibility interact as alternative risk management.

On the other hand, the interaction between liquidity and hedging has been studied. Geczy et al. (1997) include liquidity as a regression variable and find evidence that firms not using derivatives have higher short-term liquidity than those using derivatives. Allayannis and Schill (2010) analyze the relationship between payout and leverage policies and hedging and liquidity, and observe a positive relationship between firm value and conservative policies.

Mello and Parsons (2000, MP) incorporate hedging and liquidity to present a dynamic model in which a financially constrained firm receives benefits from hedging with futures contracts and increases its value to greater than that without hedging. Their model emphasizes the benefits of the improved liquidity from hedging and shows that the optimal hedge is to minimize the variance of the liquidity value of cash balances rather than the variance of firm value or cash flows. They use a numerical example to show that the firm can increase its value by reducing deadweight costs through hedging. However, they confine the firm to issuing only riskless debt and hedging liquidity problems with futures contracts alone. Their numerical example works well for short-term operations. We build on MP and allow the firm to issue both riskless and risky debt and to hedge using both futures and forward contracts for long-term operations.

Disatnik et al. (2012) examine the interaction between liquidity and a hedging policy to address cash flow risks and find that the benefit of cash flow hedging to firm value is significantly positive. Bolton et al. (2011) consider a dynamic model that combines liquidity and hedging and allows investments to be partially reversible. They focus on the relationship between marginal q[5] and investment for the existence of financing frictions that can be relieved through cash reserves, lines of credit, or financial derivatives.

More recently, Gamba and Triantis (2013) study corporate risk management that incorporates liquidity management, hedging with derivatives, and flexible operating policies. They show that liquidity is crucial to justifying high cash reserves and that the marginal value of hedging with derivatives is low. They also find that operating leverage affects complex substitution effects in a non-monotonic way. Although they combined liquidity and hedging, cash balance was a control variable, which differs in this study.

This study contributes to the literature on risk management in three ways. First, we present a dynamic model that combines liquidity and hedging for firms operating with a long horizon. Second, we highlight the benefits of hedging with both futures and forward contracts. Both contracts imply potentially redundant substitution. However, we show that both contracts can complement each other for firms with a long horizon. To our knowledge, this study is the first on hedging liquidity using both futures and forward contracts. Third, we illustrate that hedging with both contracts can significantly reduce the amount of required risky debt issuance.

This article is structured as follows. Section 2 presents the dynamic model that allows a firm to issue risky debt when it exhausts risky debt. Section 3 develops a model that incorporates liquidity and hedging with both futures and forward contracts. Section 4 presents a numerical example that illustrates liquidity problems from hedging with futures contracts alone and value increases from hedging with both contracts. Section 5 discusses hedging with alternative financial derivatives. We conclude in Section 6.

2. The Model

[5] While Tobin (1969) suggests “Q” as the ratio of the firm’s market value over its replacement cost, Hayashi (1982) define the asset price determining investment as marginal q.

Consider a firm that produces a commodity. Following MP, we start with a dynamic model using a flexible operating policy subject to stochastic factors: the input cost and the output price. The stochastic processes for the input cost and the output price are given by

$$dc_t = \nu c_t dt + \sigma_c c_t dz_1(t), \text{ and } dp_t = \mu p_t dt + \sigma_p p_t dz_2(t), \tag{1}$$

where c_t is the input cost and p_t is the output price per unit, v and μ are their drifts, σ_c and σ_p are their standard deviations, and $dz_1(t)$ and $dz_2(t)$ are Gauss-Wiener processes with correlation ρ. The firm chooses to either abandon production or produce q units to realize cash flow $q(p_t - c_t)$ at any given time. Assuming a risk-free interest rate r and the net convenience yields κ_c and κ_p, for simplicity we fix the convenience yields as constant proportions of c and p: $\kappa_c(c_t) = \kappa_c c_t$ and $\kappa_p(p_t) = \kappa_p p_t$.

2.1 A Financially Unconstrained Firm

When a firm is financially unconstrained with respect to its cash balance, it can choose the first-best operating policy $\varphi^u(c)$ that determines the time to abandon operations and that depends only on the dominating input cost and output price. For any given input cost, the operating policy provides the critical price below which the firm abandons operations. That is, the optimal operating policy $\varphi^u(c)$ is to abandon production when the output price p falls below $\varphi^u(c)$. MP apply Ito's lemma and present a model describing the instantaneous behavior of the value of the firm as

$$dV = V_p dp + V_c dc + \frac{1}{2}\left[V_{pp}(dp)^2 + 2V_{pc}(dp)(dc) + V_{cc}(dc)^2\right]. \tag{2}$$

Adopting appropriate boundary conditions, MP obtained the explicit solution to the first-best firm value depending on the input cost, the output price, and its operating policy: $V(c, p| \varphi^u)$. Since this solution is also our benchmark case, we solve a corresponding partial differential equation (PDE) in Appendix 1 to confirm the firm value, but only present the results here. The explicit solution to the first-best firm value is slightly different from that presented in MP:

$$V^U(c,p) = \frac{qc}{\kappa_c}\frac{1}{(1-\gamma)}\left(\frac{(\gamma-1)}{\gamma}\frac{p}{c}\right)^{\gamma} + q\left(\frac{p}{\kappa_p} - \frac{c}{\kappa_c}\right), \tag{3}$$

where

$$\gamma = \theta - \left(\theta^2 + \frac{2\kappa_c}{\sigma^2}\right)^{1/2}; \quad \theta = \frac{1}{2} - \frac{\kappa_c - \kappa_p}{\sigma^2}; \quad \sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_c + \sigma_c^2. \tag{4}$$

with the optimal operating policy as

$$\phi^u(c) = \frac{\gamma}{\gamma - 1}c. \tag{5}$$

2.2 A Financially Constrained Firm

When a firm is financially constrained, it has limited cash reserves to fund negative cash flows from operation. If its cash reserves are exhausted, it must liquidate and incur deadweight costs. MP compute the firm value as $V^{fc}(c,p,W)$ by adding the firm's accumulated cash balance and a new operating policy, ϕ^{fc}, to the model. When they extend the model to a financially constrained firm, they also show that the firm value is lower than that of a financially unconstrained firm because the constrained firm can be forced to liquidate at a price higher than the optimal operating policy for the unconstrained firm. The extension to hedging considers only riskless debt when hedging liquidity risk using futures contracts. Hedging provides cash flows when the constrained firm desperately needs cash to stay open and increase its firm value to higher than that of an unhedged firm.

In contrast, hedging with futures contracts alone is good enough in the numerical example of MP because the firm needs to operate in a short horizon by being sold at its full value only two periods later. When the firm continues to lose cash flows from operations and cannot be sold soon at its full value, it will be liquidated at a value less than its full value because hedging with futures contracts alone cannot provide cash flows required for long-term operations. Since the firm continues to lose cash flows from operations, it requires cash inflows from hedging in almost all states. However, hedging with futures contracts also reduces cash in some states and generates cash in other states in the future. When the firm exhausts its riskless debt capacity and does not receive cash flows from hedging positions, it must issue risky debt to keep its operations open.

2.3 Cash Balance with Risky Debt

A financially constrained firm starts operations with a cash balance W_0. When the firm exhausts its cash balance from operations, we allow the firm to issue risky debt just once to remove moral hazards, such as delaying liquidation using additional risky debt issues ex post. Let $\tau > 0$ be the time when the firm issues risky debt upon exhausting its cash balance and riskless debt capacity.

Suppose that the firm is sold at its full value at time T. In addition, the firm is allowed to issue risky debt with a principal amount B and an annual coupon rate b only if its cash balance falls to $-aV^u$ for the first time.[6] With a coupon paying the risk-free interest rate, the firm should pay the principal amount if its value is greater than or equal to the principal value at time T. With an initial cash balance $W_0 > 0$, the firm's accumulated cash balance at time $t < T$ is given as

$$W_t = W_0 e^{rt} + \int_0^t e^{r(t-s)} q\left(p_s - c_s\right) ds + I\left(\tau < t\right)\left(D_\tau - b_\tau B_\tau\right) e^{r(t-\tau)}, \tag{6}$$

where D_τ is the cash raised from issuing risky debt at time $\tau > 0$. The instantaneous change in the cash balance is $q(p_t - c_t) + rW_t$. A positive change increases the cash balance and a negative change reduces it. The firm with risky debt outstanding continues to operate even with temporary negative cash flows as long as its cash balance is higher than its liquidation value at $-aV^u$. The firm is assumed to liquidate at $\alpha V^U(c,p)$, $0 \le \alpha < 1$, indicating that the constrained firm value is smaller than its first-best value.

When the firm is sold, it is sold at the first-best firm value, which is added to any remaining cash balance to first pay off risky debt. Since bondholders receive lower than the face value of their bond at some nodes, the bonds are risky. By discounting payoffs to bondholders at the risk-free interest rate and adjusting for risk-neutral probabilities, we compute the raised cash D equal to the present value of risky debt.

2.4 Valuation of the Financially Constrained Firm

The operating policy and the value of the constrained firm become a function not only of c, p, and W but also of D. Let V^C and V^R be the firm values before and after it issues risky debt, respectively. Applying Ito's lemma, the firm value process before issuing risky debt described by

$$dV = V_p dp + V_c dc + V_W dW + \frac{1}{2}\left[V_{pp}\left(dp\right)^2 + 2V_{pc}\left(dp\right)\left(dc\right) + V_{cc}\left(dc\right)^2\right], \tag{7}$$

satisfies PDE

[6] Mello and Parsons (2000) recognized that the firm must liquidate at its value –αVu to make its debt riskless.

$$
\begin{aligned}
&\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp}^C + 2\rho\sigma_p\sigma_c V_{pc}^C + \sigma_c^2 c^2 V_{cc}^C\right] + p\left(r-\kappa_p\right)V_p^C \\
&\quad + \left\{q\left[p\left(1+r-\kappa_p\right) - c\left(1+r-\kappa_c\right)\right] + rW\right\}V_W^C + c\left(r-\kappa_c\right)V_c^C - rV^C = 0,
\end{aligned}
\tag{8}
$$

before it issues risky debt and

$$
\begin{aligned}
&\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp}^R + 2\rho\sigma_p\sigma_c V_{pc}^R + \sigma_c^2 c^2 V_{cc}^R\right] + p\left(r-\kappa_p\right)V_p^R \\
&\quad + \left\{q\left[p\left(1+r-\kappa_p\right) - c\left(1+r-\kappa_c\right)\right] + rW\right\}V_W^R + c\left(r-\kappa_c\right)V_c^R - rV^R = 0,
\end{aligned}
\tag{9}
$$

after it issues risky debt. Both PDEs can be reduced into PDEs with time t and a variable $X = p/c$ to facilitate numerical implementation, as shown in Appendix 2.

Since the firm value must be zero when it is abandoned whether or not it issued risky debt, one boundary condition is obtained:

$$
V^C\left(c,p,W\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,B)} = V^R\left(c,p,W,D\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,B)} = 0, \tag{10}
$$

where φ^C is the operating policy given that the constrained firm can issue risky debt once. Another boundary condition results from the fact that the ratio of value to price must be finite as the ratio of price to cost increases:

$$
\lim_{(p/c)\to\infty} \frac{V^C\left(c,p,W\,|\,\varphi^C\right)}{p} < \infty, \quad \text{and} \quad \lim_{(p/c)\to\infty} \frac{V^R\left(c,p,W,D\,|\,\varphi^C\right)}{p} < \infty. \tag{11}
$$

If the firm value in operation falls below that in liquidation, liquidating the firm is preferred even if it has adequate riskless debt capacity to cover the negative cash balance. Hence, we have the following boundary condition

$$
V^C\left(c,p,W\,|\,\varphi^C\right) \geq \alpha V^U\left(c,p\right). \tag{12}
$$

This condition is not recognized in MP because they considered a numerical example for short-term operations, where the firm's operational value is always greater than the liquidation value before it is sold at its full value in period 2.

Similarly, even after the firm issues risky debt, its operational value should remain higher than its liquidation value net of the current value of the risky debt because it can always liquidate whenever its value falls below the net liquidation value; this condition is shown as

$$V^R\left(c,p,W,D\,|\,\varphi^C\right)\geq\alpha V^U\left(c,p\right)-D. \tag{12}$$

The smooth pasting condition on the firm values around issuing risky debt is as follows:

$$V^C\left(c,p,W\,|\,\varphi^C\right)|_{W=-\alpha V^U(c,p)}=V^R\left(c,p,W,D\,|\,\varphi^C\right)|_{W=-\alpha V^U(c,p)+D}. \tag{13}$$

That is, when the firm exhausts its riskless debt capacity as $W = -\alpha V^U$ with a value V^C, it issues risky debt D once and continues operating with a value of V^R. Once the firm issues risky debt, it should liquidate when it reaches the same cash balance as just before it issued the risky debt, with the liquidation point noted as:

$$V^R\left(c,p,W,D\,|\,\varphi^C\right)|_{W=-\alpha V^U(c,p)}=\alpha V^U\left(c,p\right). \tag{14}$$

As the cash balance increases, the firm value should converge to the first-best value regardless of whether it issued risky debt:

$$\lim_{W\to\infty}V^C\left(c,p,W\,|\,\varphi^C\right)=\lim_{W\to\infty}V^R\left(c,p,W,D\,|\,\varphi^C\right)=V^U\left(c,p\right). \tag{15}$$

Furthermore, the marginal value of the firm in each variable must be continuous under the optimal operating policy:

$$\begin{aligned}
V_c^C\left(c,p,W\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,D)}&=V_c^R\left(c,p,W,D\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,D)}=0,\\
V_p^C\left(c,p,W\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,D)}&=V_p^R\left(c,p,W,D\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,D)}=0,\\
V_W^C\left(c,p,W\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,D)}&=V_W^R\left(c,p,W,D\,|\,\varphi^C\right)|_{p=\varphi^C(c,W,D)}=0.
\end{aligned} \tag{16}$$

We must simultaneously solve for φ^C, D, V^C, and V^R. Since the firm value is path-dependent of whether or not it issues risky debt, the explicit solutions to V^C and V^R are not available and we must rely on numerical methods.

However, it is obvious that the value of the firm with riskless debt alone in MP, V^{fc}, is smaller than that with risky debt, V^C, because the firm with risky debt increases the probability of being sold at its full value. Furthermore, it is clear that the firm value, V^R, with risky debt outstanding is smaller than the firm value, V^C, with risky debt yet to be issued. That is, the value of the firm with risky debt capacity is greater than that with riskless debt capacity alone: $V^{fc} < V^C$. The difference is the advantage of issuing risky debt over riskless debt alone. Since the firm with risky debt capacity has access to additional cash flows, it can continue to operate at a lower price $p = \phi^{C} < \phi^{fc}$, whereas the firm with riskless debt capacity alone must be liquidated at $p = \phi^{fc}$.

Figure 1 illustrates the value of the firm with risky debt capacity relative to that with only riskless debt at given cash balances and a liquidation value of $\alpha = 0.6$. The increasing cash balances allow the firm value to approach the first-best value regardless of the riskiness of the debt. In contrast, as the cash balance decreases, the deadweight cost to the firm increases and the value of the firm precipitates to the liquidation value. Furthermore, when the firm's liquidation costs are small and α is high, its bankruptcy costs are minimized. Low liquidation costs with high α increase the riskless debt capacity for the relaxed financial constraint, which in turn boosts the value of the firm given any cash balance amount. Hence, the firm shows low bankruptcy costs.

When the firm is allowed to issue risky debt, its value becomes higher than that with only riskless debt capacity because it can issue risky debt after exhausting riskless debt capacity and continues operating. Once the firm issues risky debt, its value appears lower than that with only riskless debt. However, the comparison is incorrect because when the firm value reaches its liquidation value, its value becomes zero with only riskless debt.

Even after issuing risky debt, the firm cannot eliminate financial constraints and its value falls short of that for a financially unconstrained firm in the amount of the deadweight costs given as $V^{U}(c, p) - V^{C}(c, p, W)$. Although the required cash reserves to cover cash shortages from operations can be enormous, the firm's risky debt capacity is limited to its maximum firm value. Given the limited capacity of risky debt issuance, the firm can further improve its financial constraints and reduce the deadweight costs by hedging with futures and forward contracts.

3. Hedging with Futures and Forward Contracts

While hedging does not add any value to a financially unconstrained firm, the financially constrained firm could be benefited from hedging because a dollar value inside the firm can be higher than that outside the firm. A fairly priced hedge on the market moves cash from states with low shadow value of liquidity to states with high shadow value of liquidity. Hedging reduces the expected financing costs and overcomes the financial constraints to the firm, increasing its value.

MP analyzed the benefits of hedging with futures contracts to a financially constrained firm and evaluated different hedging strategies. They showed that the optimal hedging strategy is to minimize the variance in the marginal value of cash balances as opposed to minimizing the variance in the cash balances or firm value. Since our analysis of hedging also examines liquidity improvements, the optimal hedging strategy remains the same as in MP.

While MP's analysis focused on the benefits from hedging with only futures contracts, we allow the firm to use both futures contracts and forward contracts for hedging. To extend the operation for a long horizon, we consider the firm taking positions in futures contracts, which are dynamically rebalanced and instantaneously matured, and in forward contracts, which are not marked-to-market and have times to maturity longer than futures contracts. When implementing hedging, we consider hedging only the output price to avoid a perfect hedge, which is possible by also hedging the input cost.

Given the previously assumed convenience yield, the futures price maturing in *s* periods is given as

$$f\left(p_t, s\right) = p_t e^{\left(r-\kappa_p\right)s}. \tag{17}$$

A hedging position in futures contracts, h_s, generates instantaneous cash flows $h_s df(p_s)$, where

$$df\left(p_s\right) = f_p dp + \frac{1}{2} f_{pp}\sigma_p^2 p^2 dt - f_s dt, \tag{18}$$

whereas that in forward contracts, g_s, provides cash flows $g_s df(p_s)$ only at maturity because forward contracts do not have interim cash settlements. Thus, the firm can dynamically hedge using futures and forward contracts held at any given time subject to the output and input prices and the cash balance. Since the firm is assumed to start hedging only when it issues risky debt, its cash balance from hedging positions at any time *t* becomes

$$\begin{aligned} W_t = {} & W_0 e^{rt} + \int_0^t e^{r(t-s)} q\left(p_s - c_s\right) ds + I\left(\tau < t\right)\left(D_s - b_s B\right) e^{r(t-\tau)} \\ & + \int_\tau^t e^{r(t-s)}\left(h_s + g_s^*\right) df\left(p_s\right) ds, \end{aligned} \tag{19}$$

where $g_s^* = g_s$ if the forward contract is matured and $g_s^* = 0$ otherwise.

The hedged firm value is now obtained using the new cash balance:

$$dV = V_p dp + V_c dc + V_W dW + \frac{1}{2}\left[V_{pp}\left(dp\right)^2 + 2V_{pc}\left(dp\right)\left(dc\right) + V_{cc}\left(dc\right)^2\right]. \tag{20}$$

Since we have $df\left(p_s\right)=f_p p\left[\left(\mu-r\right)+\kappa_p\right]dt+f_p p\sigma\, dz_2$ and $\mu=r-\kappa_p$, the value of the hedged firm satisfies PDE

$$\begin{aligned}&\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp}^H+2\rho\sigma_p\sigma_c V_{pc}^H+\sigma_c^2 c^2 V_{cc}^H\right]+p\left(r-\kappa_p\right)V_p^H\\&+\left\{q\left[p\left(1+r-\kappa_p\right)-c\left(1+r-\kappa_c\right)\right]+rW\right\}V_W^H+c\left(r-\kappa_c\right)V_c^H-rV^H=0,\end{aligned}\tag{21}$$

before it issues risky debt and

$$\begin{aligned}&\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp}^{HR}+2\rho\sigma_p\sigma_c V_{pc}^{HR}+\sigma_c^2 c^2 V_{cc}^{HR}\right]+p\left(r-\kappa_p\right)V_p^{HR}\\&+\left\{q\left[p\left(1+r-\kappa_p\right)-c\left(1+r-\kappa_c\right)\right]+rW\right\}V_W^{HR}+c\left(r-\kappa_c\right)V_c^{HR}-rV^{HR}=0,\end{aligned}\tag{22}$$

after it issues risky debt with boundary conditions similar to those in Section 2.4:

$$\begin{aligned}&V^H\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,B)}=V^{HR}\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,B)}=0,\\&V^H\left(c,p,W,D\,|\,\varphi^H\right)|_{W=-\alpha V^U(c,p)}=V^{HR}\left(c,p,W,D\,|\,\varphi^H\right)|_{W=-\alpha V^U(c,p)+D},\\&V^{HR}\left(c,p,W,D\,|\,\varphi^H\right)|_{W=-\alpha V^U(c,p)}=\alpha V^U\left(c,p\right),\end{aligned}\tag{23}$$

$$\lim_{(p/c)\to\infty}\frac{V^H\left(c,p,W,D\,|\,\varphi^H\right)}{p}<\infty;\quad \lim_{(p/c)\to\infty}\frac{V^{HR}\left(c,p,W,D\,|\,\varphi^H\right)}{p}<\infty\tag{24}$$

$$\begin{aligned}&\lim_{W\to\infty}V^H\left(c,p,W,D\,|\,\varphi^H\right)=\lim_{W\to\infty}V^{HR}\left(c,p,W,D\,|\,\varphi^H\right)=V^U\left(c,p\right),\\&V_c^H\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,D)}=V_c^{HR}\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,D)}=0,\\&V_p^H\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,D)}=V_p^{HR}\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,D)}=0,\\&V_W^H\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,D)}=V_W^{HR}\left(c,p,W,D\,|\,\varphi^H\right)|_{p=\varphi^H(c,W,D)}=0.\end{aligned}\tag{25}$$

where φ^H is the optimal operating policy of the implemented hedging strategy. The boundary conditions in Eqs. (23)–(25) show that when the firm uses up its riskless debt capacity, it issues risky debt once and continues to operate with a value of V^{HR}.

Figure 2 illustrates the effect of hedging on firm values as a function of the cash balance. The solid curves are the values of the hedged firm before and after the issuance of risky debt and the dashed curves represent the unhedged firm. As cash balances increases, the financial constraint is less likely to be bound and the firm value approaches the first-best value, as in Eq. (3).

At any cash balance amount, hedging with futures and forward contracts improves liquidity by adding the cash balance when the firm needs it most and by lowering the probability that its cash balance will reach the liquidation point. Thus, the value of the hedged firm becomes higher than that of the unhedged firm both before and after the issuance of risky debt. The difference is the value of hedging:

$$V^{HC}(c, p, W, D) - V^{C}(c, p, W, D). \tag{26}$$

Hence, a hedge increases the value of the firm by reducing its deadweight cost incurred from constrained liquidity.

Hedging with futures and forward contracts is jointly determined by the amount of risky debt issued. This joint determination becomes optimal by maximizing Eq. (26). The maximization is consistent with maximizing the value of the hedged firm, V^H. We define $h^*(c, p, W)$ in futures contracts and $g^*(c, p, W)$ in forward contracts as optimal hedges and D^* as the optimal risky debt that maximize its expected firm value V^* and need to solve the hedged value function V^* for h^*, g^*, and D^*. MP showed that the optimal hedging strategy also minimizes the variance in the shadow value of the cash balance by presenting a hedging strategy using futures contracts.

Since we consider a firm in a long horizon and hedge its liquidity using futures and forward contracts, an optimal hedging strategy is consistent with that in MP. That is, the optimal hedge should maximize the value of a hedged firm or minimize the variance in the marginal value of the cash balance. However, the optimal hedge with risky debt as presented improves the hedge with only riskless debt in MP in the sense that firms are flexible in issuing risky debt and in hedging with an additional financial derivative: forward contracts. Hahnenstein and Röder (2004) also consider the case in which a firm's optimal hedging is jointly determined using its capital structure policies.

4. A Numerical Example

A discrete example provides insight into the benefits of hedging by reducing liquidity costs for alternative hedging instruments.

4.1 The Financially Unconstrained Firm

To illustrate the example in long-term operations, the firm is assumed to be sold at its full value in period 4, as opposed to in period 2 in MP. We take the same parameter values as in MP except for the convenience yields, which are smaller than those in MP and are set to 2.5%: $\kappa_p = \kappa_c = 2.5\%$. The smaller convenience yield allows the firm to stay open at nodes without abandoning operations until period 3, as opposed to the example in MP in which the firm should shut down at a node in period 1. The other parameter values are taken as $q = 1$, $p = c = \$10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, and $\rho = 0$. Using these parameter values, the firm begins operations in period 0 with an output price of $10 per unit and its annual variance of 10%, and it faces an input cost of $10 per unit and an annual variance of 40%. That is, the output price moves up to $13.72 or down to $7.29 and the input cost increases to $18.81 or declines to $5.31 in period 1.

Using the optimal operating policy found in Eq. (5), we compute this policy as $\gamma = p_t/c_t = 0.0839$. The firm is assumed to have unlimited cash reserves and continues to operate as long as its current price–cost ratio remains higher than 0.0839. That is, the firm produces output even if it loses cash flows from operations with $p_t < c_t$ because it can later recover the losses from profits with $p_t > c_t$. However, when the price–cost ratio falls to less than 0.0839, abandoning operations permanently is optimal.

Table 1 shows 16 possible states during periods 0–2. The node (0, 0, 0) in period 0 can move to one of the four nodes (1, 0, k), k = 0, 1, 2, 3 in period 1. If the node falls to (1, 0, 1) in period 1, it can develop into another 16 possible states during periods 1–3, as shown in Table 2. Similarly, Table 3 shows that the 64 possible states are available in period 4 out of the node (1, 0, 1) in period 1. That is, during periods 0–4, both price and cost change separately through their own variances, resulting in 256 different states in period 4.[7] We show the 64 possible states in Table 3 because the liquidity problem in these states are most serious compared with all of the other states in period 3 if the firm is financially constrained, as subsequently examined.

Using the solution to Eq. (3) and the optimal operating policy in Eq. (5), we compute the full value of the firm in each of the 256 states in period 4. The firm values in period 4 are discounted at the risk-free interest rate using the risk-neutral probability and are added to the

---

[7] Although we only show the 64 possible states for brevity, all of the 256 possible states are available on request.

operating cash flows $(p_t - c_t)$ in period 3 to determine its value in that period. For example, the firm values in nodes (4, 0, 0, 0) – (4, 0, 0, 3) in Table 3 are discounted using risk-neutral probabilities and are added to the operating cash flow of –$1.44 to compute the firm value in node (3, 0, 0), which is $102.04 in the last column of Table 2. Similar calculations generate firm values in nodes (3, 0, 1) – (3, 0, 3) out of (4, 0, 1, 0) – (4, 0, 3, 3) as $36.53, $229.91, and $137.65, respectively. The four firm values are discounted and added to the corresponding cash flow of –$4.69 to compute the firm value of node (2, 1, 0) as $121.52. The repeated calculations at nodes (2, 1, 1) – (2, 1, 3) provide a firm value of $135.77 for node (1, 0, 1) in Table 2 and Table 1.

We observe that out of node (1, 0, 1) in period 1, abandoning operations in three nodes (3, 1, 1), (3, 1, 3), (3, 3, 1) in period 3 is optimal, as shown in Table 2. Abandoning operations at node (3, 1, 1) is clear because the price–cost ratio (3.87/66.68 = 0.0580) at this node is less than γ = 0.0839. Although the price–cost ratio (7.29/66.68 = 0.1093) at nodes (3, 1, 3) and (3, 3, 1) is higher than γ = 0.0839, abandoning operations is still better because the firm value may be negative if operations are continued through the subsequent period. From node (3, 1, 3), the firm can realize its first-best firm values as $26.36, $0, $144.55, and $0 at nodes (4, 1, 3, 0) – (4, 1, 3, 3), respectively. By discounting using the risk-free rate and applying risk-neutral probabilities, we compute the first-best firm value as $48.83 at node (3, 1, 3). However, subtracting operating cash flow of –$59.39 at the same node as in Table 2 would result in a firm value of less than zero. Hence, abandoning operation at nodes (3, 1, 3) and (3, 3, 1) is also optimal.

The firm should shut down in 11 out of the 64 nodes in period 4, as shown in Table 4. Although not shown for brevity, we also find that abandoning operations in period 3 is never optimal, which emanates from nodes (1, 0, 0), (1, 0, 2), and (1, 0, 3) in period 1. In contrast, closing down operations in some nodes in period 4 is optimal.

By applying this backward calculation to all 256 possible states, we compute the firm value in the first-best case as $292.84, as shown in Table 1. Comparing the first-best value of $292.02 by using Eq.(3) with that of $292.84 from backward calculations confirms that the value of the discrete solution is very close to that of the exact solution. We continue analyzing the discrete solution to illustrate liquidity concerns of a financially constrained firm and the benefits from alternative hedging.

4.2 The Financially Constrained Firm

The financially constrained firm has limited cash reserves and must liquidate when it cannot fund negative cash flows from operations using its accumulated cash balance and riskless debt capacity. We consider the firm with an initial cash balance of $10 in Table 4 for comparison with the case in MP. Since the operating cash flow in period 0 is $0, the ending cash balance remains the same at $10 and the firm can continue operating into period 1 with an increased cash balance of $10.50 at the risk-free interest rate. The firm loses $11.53 in operating cash flows at node (1, 0, 1) and faces a liquidity problem with an available cash balance of only $10.50.

Although the firm needs to borrow $1.03 to continue operating, it has no riskless debt capacity. Its riskless debt capacity is computed using the guaranteed repayment available at nodes (2, 1, 0) – (2, 1, 3) in the subsequent period. We assume that the firm liquidates at α = 60% of the first-best value. The ending cash flow of –$1.03 without riskless debt capacity forces the firm to liquidate at $81.46 out of its full value at $135.77 in Table 2, and the firm incurs a deadweight loss of $54.31 at node (1, 0, 1), as shown in the last column in Table 4.

The deadweight costs at the other nodes in period 2 are also computed as $16.03, $76.76, and $9.72 at nodes (2, 0, 1), (2, 3, 1), and (2, 3, 3), respectively, in the last column of Table 4. Since these losses occur in period 3, their liquidation values are computed in period 3 and not shown in period 2. However, the losses are carried backward to period 2 as shown at the three corresponding nodes. The three losses are again carried backward to period 1 as $3.01 at node (1, 0, 0) and $15.97 at node (1, 0, 3) in Table 4.

Although the firm has positive ending cash balances at nodes (1, 0, 0) and (1, 0, 3), it still suffers from deadweight costs of $3.01 and $15.97, respectively, because of the deadweight costs at period 3 nodes emanating from nodes (2, 0, 1), (2, 3, 1), and (2, 3, 3), which again emanate from nodes (1, 0, 0) and (1, 0, 3). The losses of $3.01, $54.31, and $15.97 from liquidation are discounted at the risk-free interest rate using the corresponding risk-neutral probabilities in Table 1 and are realized as the deadweight cost of $13.74, resulting in the firm value of $279.10 in Table 4.

While the constrained value of $89.72 at node (2, 3, 1) is computed from its corresponding values in period 4, the firm is better off by liquidating operations at this node because it can realize a higher value of $99.89 from liquidation, as in Table 5.[8] This type of liquidation is not forced by liquidity concerns but by a boundary condition in Eq. (12). Hence, the deadweight

[8] The liquidation value of $99.89 is computed by multiplying its first-best firm value of $166.48 at node (2, 3, 1) in Table 1 by α = 60%.

loss is reduced from $13.74 in Table 4 to $13.43 in Table 5 and the firm value increases from $279.10 to $279.41.

4.3 Issuing Risky Debt

Although the firm incurs liquidation costs in period 3 nodes emanating from period 2 nodes (2, 0, 1), (2, 3, 1), and (2, 3, 3) in Table 4, the cash shortage in period 3 can be readily managed using futures contracts, as in MP. Since the firm needs additional cash balances from hedging positions using futures contracts in period 3, it can trade futures contracts for hedging in period 2 and obtain its needed cash balance in period 3. Since the firm is assumed to be sold at its full value one period later in period 4, the situation appears similar to the short-term operations in MP.

However, the firm’s liquidity problems at node (1, 0, 1) are not readily managed using futures contracts because the cash shortage at this node can persist in future periods; thus, the firm should resolve its liquidity concerns in long-term operations from period 0 to period 4. Liquidity issues in long-term operations attract attention for hedging with futures and forward contracts. Hence, we focus our analysis on liquidity issues at the nodes emanating from node (1, 0, 1) in period 1.

To relieve the cash constraint with exhausted riskless debt capacity at node (1, 0, 1), the firm can attempt to hedge liquidity using just futures contracts. Although a hedge is available for period 2, the firm cannot hedge liquidity for that period, as shown in Table 2. Hedging with futures contracts on output prices can bring in cash flows for two out of the four nodes in period 2, but doing so erodes cash flows at the other two nodes. Since the firm experiences negative cash flows at the three nodes in period 2 of (2, 0, 1), (2, 3, 1), and (2, 3, 3) in Table 4, it remains short of cash flows at least at one node by hedging with futures contracts and cannot continue operations beyond period 2.

The only way to continue operations up to period 4 is to issue risky debt in period 1. We allow the firm to issue risky debt exactly once when its cash balance and riskless debt capacity is exhausted, indicating that the firm cannot alter the capital structure ex post to delay liquidation without a cost. We also assume that the risky debt matures in period 4 and is repaid when the firm is sold at the first-best value. A replenished cash balance increased from issuing risky debt in period 1 provides a cushion for a negative cash balance in period 2 and allows the firm to continue operations up to period 4. The firm begins to issue a small amount of risky debt, barely enough to cover its cash shortages in period 2.

The risky debt issued by the firm should be fairly priced in the sense that both the issuing firm and bond investors neither gain nor lose. When the firm issues risky debt, it promises to pay back a principal amount *B* at maturity with an annual coupon *b* equal to the riskless interest rate. We compute the market price *D* of risky debt by discounting payoffs of risky debt at maturity with risk-neutral probabilities. The firm receives its market value, which replenishes its cash balance.

When the firm issues risky debt of $31.55, it safely resolves its liquidity problem in period 2. However, the issue of risky debt of $31.55 cannot prevent liquidation at those nodes and the firm still suffers from deadweight losses of $12.19 in period 1, as shown in Table 6. The source of the loss at $10.55 is liquidation at nodes (3, 1, 0), (3, 1, 2), (3, 3, 0), (3, 3, 2), and (3, 3, 3) in period 3. At these nodes, the firm must be liquidated because the ending cash balances are negative. Hence, liquidating the firm at nodes (2, 1, 1) and (2, 1, 3) is better for realizing the higher liquidation values of $12.07 and $84.50, respectively, than the operational value of $0.02 as bounded by Eq. (14).[9] Recall that the ending cash balances at nodes (3, 3, 1), (3, 1, 3) and (3, 3, 1) are irrelevant because even the firm abandons operation at these nodes.

In contrast, the firm has to issue enough risky debt to finance a negative cash balance up to period 3 to restore its first-best firm value. Since the ending cash balance at node (3, 3, 3) is –$50.28 in Table 6, we need additional risky debt to make the ending cash balance zero. When the firm issues risky debt of $81.29 as in Table 7, the ending cash balance at node (3, 3, 3) becomes zero and the firm restores the first-best firm value of $135.77 at node (1, 0, 1) without any hedge.

As the firm issues more risky debt, it increases the cash balance and restores its first-best firm value. A natural question arises regarding the amount of the optimal risky debt. How much risky debt should the firm issue? Table 8 shows the value of the firm related to the issuance of risky debt. When the firm initially issues a little risky debt, it can significantly increase its firm value. For example, when the firm issues risky debt of $1.03, its value increases by $19.04, from $81.29 to $100.50. The rate of increase in the value the firm decreases as the amount of risky debt increases, as shown in the figure accompanying Table 8.[10]

[9] To compute the operational value of $0.18, we discount the constrained values at nodes (3, 1, 0) – (3, 1, 3) using the corresponding risk-neutral probabilities and subtract the operating cash flow at node (2, 1, 1) in Table 1.

[10] We show the firm valuations with risky debt issues of $31.55 and $81.29 in Table 6 and Table 7, respectively; its valuations at other levels of risky debt issues are available on request.

Although in general, the values of the firm take a concave form with respect to the amount of risky debt issuance, as shown in the graph accompanying Table 8, the graph has two kinks. One occurs when the firm issues risky debt of $45.49 and the other occurs when it issues risky debt of $26.78 and $31.55. The two kinks are observed because of the boundary condition in Eq. (14), which forces the firm to liquidate rather than continue operations at some nodes. That is, issuing risky debt up to these two kinks does not contribute much to the value of the firm.

4.4 Optimal Risky Debt and Optimal Hedges

Optimal risky debt is not a balance between tax benefits and bankruptcy costs because we do not consider the two factors in the model. Rather, we consider optimal risky debt as that minimally required to restore the first-best firm value when the amount of debt is jointly determined using optimal hedges. Since a hedge is available and alters cash flows if implemented, the firm must simultaneously consider optimal debt and optimal hedging, as in Hahnenstein and Röder (2004).[11]

Hedging with only futures contracts or forward contracts does not significantly decrease the required amount of risky debt issuance. For example, when we attempt to hedge liquidity using only futures contracts, as in MP, the firm should issue a risky debt of $58.85 at a minimum to restore its first-best firm value. Table 9 shows the required risky debt amount and hedging with only futures contracts. When the firm issues risky debt of $58.85, it can make ending cash balances non-negative in period 3 for all of the nodes except for two nodes: (3, 3, 0) and (3, 3, 3). When the firm sells 4.52 futures contracts at node (2, 1, 1) and buys 6.44 futures contracts at node (2, 1, 3) in period 2, its hedging positions provide adequate cash flows in period 3 to make zero cash balances at nodes emanating from the two nodes without making the cash balances at other nodes negative. The value of the hedge is $1.29.

Since the firm can hedge using both futures and forward contracts, its liquidity management becomes more flexible when using both hedges than when using one hedge. That is, the firm can tailor hedges to its cash needs in separate nodes. Table 10 shows the optimal risky debt computed as $39.90 and the optimal hedges. When the firm reaches node (1, 0, 1) in period 1, it exhausts cash balances and riskless debt capacity, issues risky debt of $39.90, and hedges its remaining liquidity using futures and forward contracts.

[11] Hahnenstein and Röder (2004) examine the interaction between leverage decision and hedging with a forward contract. However, their setup is different from ours in the sense that their optimal capital structure trades off the bankruptcy costs and corporate taxes, which are not considered in this study.

In period 1, the firm buys $7.37 of forward contracts on the output price that mature two periods later in period 3. The cash flows from the forward contracts are –$15.93 and $18.63 in period 2 and are carried forward to period 3 with the balance of risky debt outstanding and without immediate cash settlements. The forward contracts are finally settled in period 3. For example, cash flows of –$27.94 and –$2.76 are settled at nodes (3, 0, 0) and (3, 0, 2), respectively.

In contrast, the firm should sell futures contracts of $6.08 at (2, 1, 0), $5.69 at (2, 1, 1), and $0.93 at (2, 1, 3) in period 2. The short position of $6.08 in futures contracts at (2, 1, 0) generates cash flows of $9.58 or –$11.20 at nodes (3, 0, 0) – (3, 0, 3) in period 3. Similarly, the short positions in futures contacts at the other nodes generate corresponding cash flows in period 3 as well. Given the optimal risky debt and the optimal hedges, the firm restores its first-best firm value of $135.77 at node (1, 0, 1) in Table 10 and $292.84 at node (0, 0, 0) in Table 1. Since the firm issues risky debt of $39.90, which is smaller than $55.18 using futures hedging, the value of the hedge becomes $4.96, which is larger than the $1.29 using futures hedging.

The minimal risky debt of $39.90 is considered optimal in the sense that it is the least amount to be issued to restore the first-best firm value using the jointly determined hedging positions in futures and forward contracts. In addition, the amount is a little larger than the $31.55 required to cover the cash shortage just one period in the future, as in Table 6. Hedging positions in futures and forward contracts are also considered optimal because they reduce the required risky debt issuance to restore the first-best value from $81.29 in Table 7 to $39.90 in Table 10.

When Mello and Parsons (2000) consider hedging only futures contracts, they recognize the benefits of hedging to improve just liquidity. When considering the issuance of risky debt and hedging with futures and forward contracts, another benefit in addition to enhanced liquidity is recognized. Since hedging redistributes cash flows when they are needed most, it also reduces the riskiness of risky debt and interest payments. For example, when the firm issues risky debt of $39.90 without any hedging, the face value of risky debt becomes $48.21 and the coupon payment becomes $2.41. However, when the firm issues the same amount of risky debt and hedges using both contracts, the face value is reduced to $45.05 and the coupon payment becomes $2.25.

Although forward contracts are similar to the implicit line of credit to a counterpart that is losing cash flows from the contracts, they should be protected from default. In our example, the two-period forward contracts initially traded in period 1 are settled in period 3 and the firm is sold at the first-best value in period 4. Hence, forward contracts are well protected. In contrast, when the firm loses cash flows from the forward contracts at nodes (2, 1, 0) and (2, 1, 1) in Table 10, it does not need an immediate cash settlement but simply carries the losses from the forward contracts along with its risky debt outstanding until the contracts mature in period 3.

5. Discussion on Hedging using Alternative Instruments

Other alternative contracts may be used to implement liquidity hedging. In addition to futures and forward contracts, firms can hedge liquidity using swaps and options. Swaps share common characteristics with forward contracts in the sense that both do not require interim cash settlements and have effects on firm value by potentially triggering debt covenants through the accumulated losses from the contracts.

In contrast, futures, forward, and swaps require a firm to fix its operating horizon. For example, a firm expects to operate in a long horizon and hedges liquidity using forward contracts. If the firm is sold before the forward contracts mature, it could suffer losses from adverse cash flows that outstanding forward contracts subsequently incur.

When the firm faces uncertainty in its operating horizon, it should hedge its liquidity using option contracts that mature long enough for stretched operations. Although the firm can be sold at any time at its full value, potential losses on options contracts will be capped at the initial premium paid for the option. However, the firm should be willing to pay out the initial premium to buy options for hedging purposes.

6. Conclusion

We study how hedging with both futures and forward contracts can reduce the deadweight costs to a financially constrained firm more than it does using only futures contracts. We set up a dynamic model for the operating and financial policies of that firm that allow it to issue risky debt just once and show that hedging with both contracts also reduces the level of required risky debt issuance to increase financial flexibility. Without interim cash settlements, forward contracts allow the firm to carry potential losses and to shift cash balances across states and times. Our numerical example illustrates the benefits of hedging with both contracts over that with only futures contracts. Although hedging with only futures contracts is suitable for a short horizon, hedging with both futures and forward contracts works well for a long horizon.

Appendix

1. The financially unconstrained firm in MP

The firm value process with Ito's lemma described by

$$dV = V_p dp + V_c dc + \frac{1}{2}\left[V_{pp}\left(dp\right)^2 + 2V_{pc}\left(dp\right)\left(dc\right) + V_{cc}\left(dc\right)^2\right], \tag{27}$$

satisfies PDE:

$$\begin{aligned}&\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp} + 2\rho\sigma_p\sigma_c V_{pc} + \sigma_c^2 c^2 V_{cc}\right]\\ &\quad + p\left(r-\kappa_p\right)V_p + c\left(r-\kappa_c\right)V_c + q\left(p-c\right) - rV = 0\end{aligned} \tag{28}$$

with boundary conditions:

$$\begin{aligned}&V\left(c, p \mid \varphi^u\right)\big|_{p=\varphi^u(c)} = 0,\\ &\lim_{(p/c)\to\infty} \frac{V\left(c, p \mid \varphi^u\right)}{p} < \infty,\\ &V_c\left(c, p \mid \varphi^u\right)\big|_{p=\varphi^u(c)} = 0,\\ &V_p\left(c, p \mid \varphi^u\right)\big|_{p=\varphi^u(c)} = 0.\end{aligned} \tag{29}$$

Since the explicit solution that we obtain is a little different from that in MP, we provide a brief sketch to derivation using a method in Weinberger (1965, pp117-119). Observing that $V(c,p|\phi^u)$ is homogeneous in p and c, we reduce the PDE in Eq.(28) using substitution

$$X = \frac{p}{c} \Rightarrow V\left(p,c\right) = cU\left(X\right), \tag{30}$$

into Ordinary Differential Equation (ODE) as

$$\frac{1}{2}\sigma^2 X^2 U_{XX} + \left(\kappa_c - \kappa_p\right) XU_X - \kappa_c U = -q\left(X-1\right). \tag{31}$$

with converted boundary conditions reflecting the optimal operating policy as in Eq.(29),

$$
\begin{aligned}
&U\left(X \mid \varphi^{u}\right)|_{X=\varphi^{u}} = 0, \\
&\lim_{X\to\infty} \frac{V\left(X \mid \varphi^{u}\right)}{X} < \infty, \\
&U_X\left(X \mid \varphi^{u}\right)|_{X=\varphi^{u}} = 0.
\end{aligned} \tag{32}
$$

We start solving the homogeneous equation, which has zero on the right-hand side as,

$$
\frac{1}{2}\sigma^2 X^2 U_{XX} + \left(\kappa_c - \kappa_p\right) X U_X - \kappa_c U = 0, \tag{33}
$$

for a general solution $U(X) = c_1U_1(X) + c_2U_2(X)$. A trial solution $U(X) = X^{\delta}$ and its derivatives $U_X = \delta X^{\delta-1}$ and $U_{XX} = \delta(\delta-1)X^{\delta-2}$ are substituted to find the two roots for δ as β and γ,

$$
\beta = \theta + \sqrt{\theta^2 + \frac{2\kappa_c}{\sigma^2};} \quad \gamma = \theta - \sqrt{\theta^2 + \frac{2\kappa_c}{\sigma^2};} \quad \theta = \frac{1}{2} - \frac{\kappa_c - \kappa_p}{\sigma^2}. \tag{34}
$$

Hence, we have a general solution to Eq.(33) as

$$
U\left(X\right) = c_1 U_1\left(X\right) + c_2 U_2\left(X\right) = c_1 X^{\beta} + c_2 X^{\gamma}. \tag{35}
$$

Recognizing the ODE in Eq.(31) as a form $a(X)U_{XX} + b(X)U_X + c(X)U = F(X)$ and multiplying both sides by

$$
p\left(X\right) = \exp\left(\int_{\phi}^{X} \frac{b\left(\xi\right)}{a\left(\xi\right)} d\xi\right), \tag{36}
$$

we rewrite it as

$$
p\left(X\right)U_{XX} + \frac{\left(\kappa_c - \kappa_p\right)}{\frac{1}{2}\sigma^2 X^2} p\left(X\right) X U_X - \frac{\kappa_c}{\frac{1}{2}\sigma^2 X^2} p\left(X\right) = \frac{-q\left(X-1\right)}{\frac{1}{2}\sigma^2 X^2} p\left(X\right), \tag{37}
$$

and define $f(X)$ as

$$
f\left(X\right) = \frac{-q\left(X-1\right)}{\frac{1}{2}\sigma^2 X^2} p\left(X\right). \tag{38}
$$

The explicit solution to the ODE in Eq.(31) is derived as

$$U(X)=\int_{\phi}^{X}R(X,\xi)\,f(\xi)\,d\xi=\int_{\phi}^{X}R(X,\xi)\frac{-q(\xi-1)}{\frac{1}{2}\sigma^2\xi^2}\,p(\xi)\,d\xi, \tag{39}$$

where

$$R(X,\xi)=\frac{U_1(X)U_2(\xi)-U_1(\xi)U_2(X)}{p(X)\left[U_1^{'}(X)U_2(\xi)-U_1(\xi)U_2^{'}(X)\right]}=\frac{X^{\beta}\xi^{\gamma}-\xi^{\beta}X^{\gamma}}{p(X)\left(\beta X^{\beta-1}X^{\gamma}-\gamma X^{\beta}X^{\gamma-1}\right)}. \tag{40}$$

Converting X in $U(X)$ back to the ratio p/c, we derive the explicit solution, which is slightly different from that in Mello and Parsons (2000), as

$$V^{U}(c,p)=\frac{qc}{\kappa_c}\frac{1}{(1-\gamma)}\left(\frac{(\gamma-1)}{\gamma}\frac{p}{c}\right)^{\gamma}+q\left(\frac{p}{\kappa_p}-\frac{c}{\kappa_c}\right), \tag{41}$$

where

$$\gamma=\theta-\left(\theta^2+\frac{2\kappa_c}{\sigma^2}\right)^{1/2};\quad \theta=\frac{1}{2}-\frac{\kappa_c-\kappa_p}{\sigma^2};\quad \sigma^2=\sigma_p^2-2\rho\sigma_p\sigma_c+\sigma_c^2, \tag{42}$$

with the optimal operating policy as

$$\phi^{u}(c)=\frac{\gamma}{\gamma-1}c. \tag{43}$$

2. The Financially Constrained Firm

The 2-variable PDE in Eq.(8) can be reduced to a 1-variable PDE. We recognize that the cash balance W in Eq.(6) depends on time t and rewrite V_W in terms of V_t as

$$\begin{aligned} V_t&=\frac{\partial V}{\partial t}=\frac{\partial V}{\partial W}\frac{\partial W}{\partial t}=V_W\left[rW_t+q\left(p_t-c_t\right)\right]\\ V_W&=\frac{V_t}{rW_t+q\left(p_t-c_t\right)}. \end{aligned} \tag{44}$$

Then, Eq.(8) becomes

$$\frac{1}{2}\left[\sigma_p^2 p^2 V_{pp}^C + 2\rho\sigma_p\sigma_c V_{pc}^C + \sigma_c^2 c^2 V_{cc}^C\right] + p\left(r-\kappa_p\right)V_p^C + c\left(r-\kappa_c\right)V_c^C$$
$$+\left\{q\left[p\left(1+r-\kappa_p\right) - c\left(1+r-\kappa_c\right)\right] + rW\right\}\frac{V_t^C}{rW_t + q\left(p_t - c_t\right)} - rV^C = 0, \qquad (45)$$

which is reduced to

$$ZV_t + \frac{1}{2}\sigma_p^2 p^2 V_{pp}^C + \rho\sigma_p\sigma_c V_{pc}^C + \frac{1}{2}\sigma_c^2 c^2 V_{cc}^C + p\left(r-\kappa_p\right)V_p^C + c\left(r-\kappa_c\right)V_c^C - rV^C = 0, \qquad (46)$$

where Z is defined as $Z = rW_t + q\left(p_t - c_t\right)$. Since the firm value function V is homogenous in p and c, we rewrite it as a function of $X = p/c$ and time t into $V(p, c, t) = cU(X, t)$. The resulted PDE is given as

$$ZU_t + \frac{1}{2}\sigma^2 X^2 U_{XX} + \left(\kappa_c - \kappa_p\right)XU_X - \kappa_c U = 0, \qquad (47)$$

where $\sigma^2 = \sigma_p^2 - 2\rho\sigma_p\sigma_c + \sigma_c^2$.

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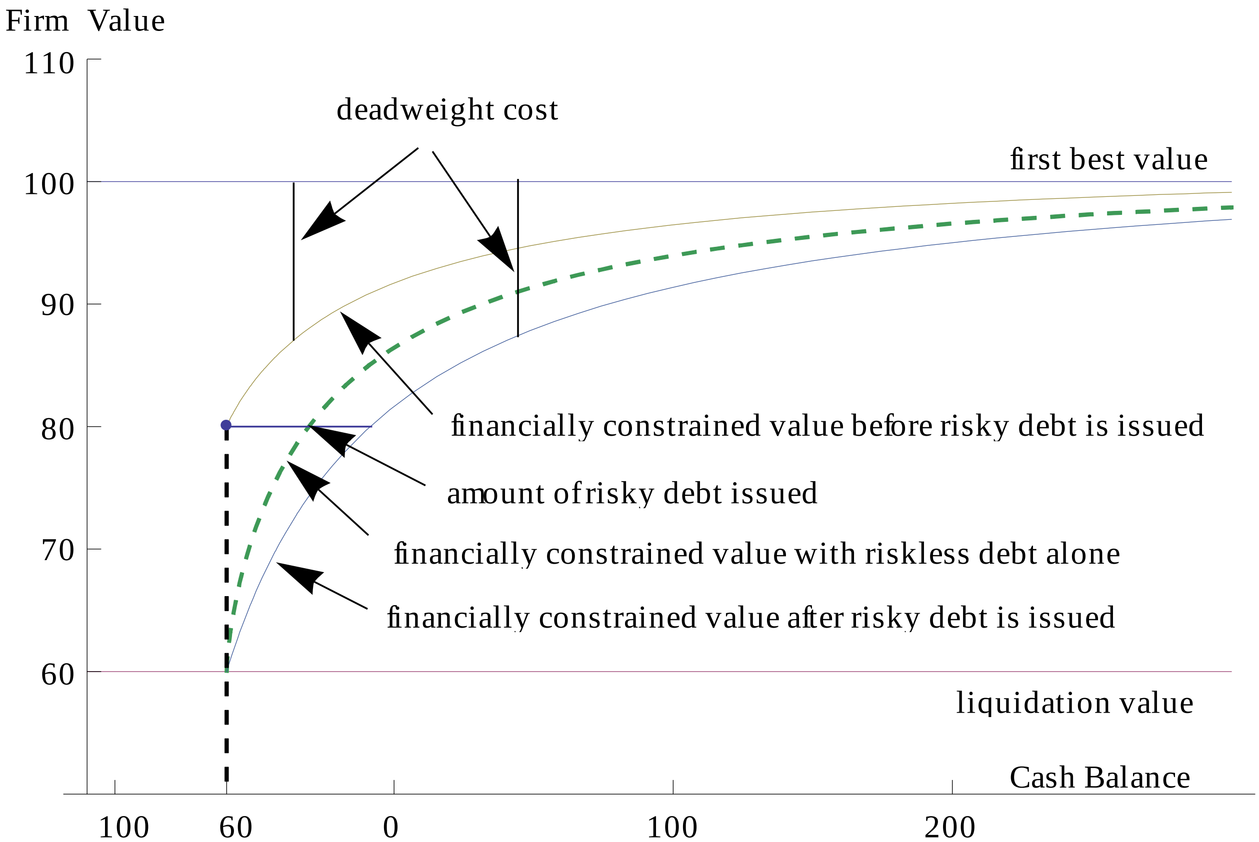


Figure 1

The value of the financially constrained firm with risky debt

The value of the financially constrained firm with risky debt is shown at various levels of cash balances given a liquidation cost: $1 - \alpha = 0.4$. As the cash balance decreases, the deadweight cost to the financially constrained firm increases. On the other hand, the increasing cash balance allows its value to approach close to the first-best value. When the firm is allowed to issue risky debt, its value is higher than that with riskless debt capacity alone because it can issue risky debt upon exhausting riskless debt and continues to operate. However, once it issues risky debt, its value is lower than that with riskless debt alone due to interest payments.

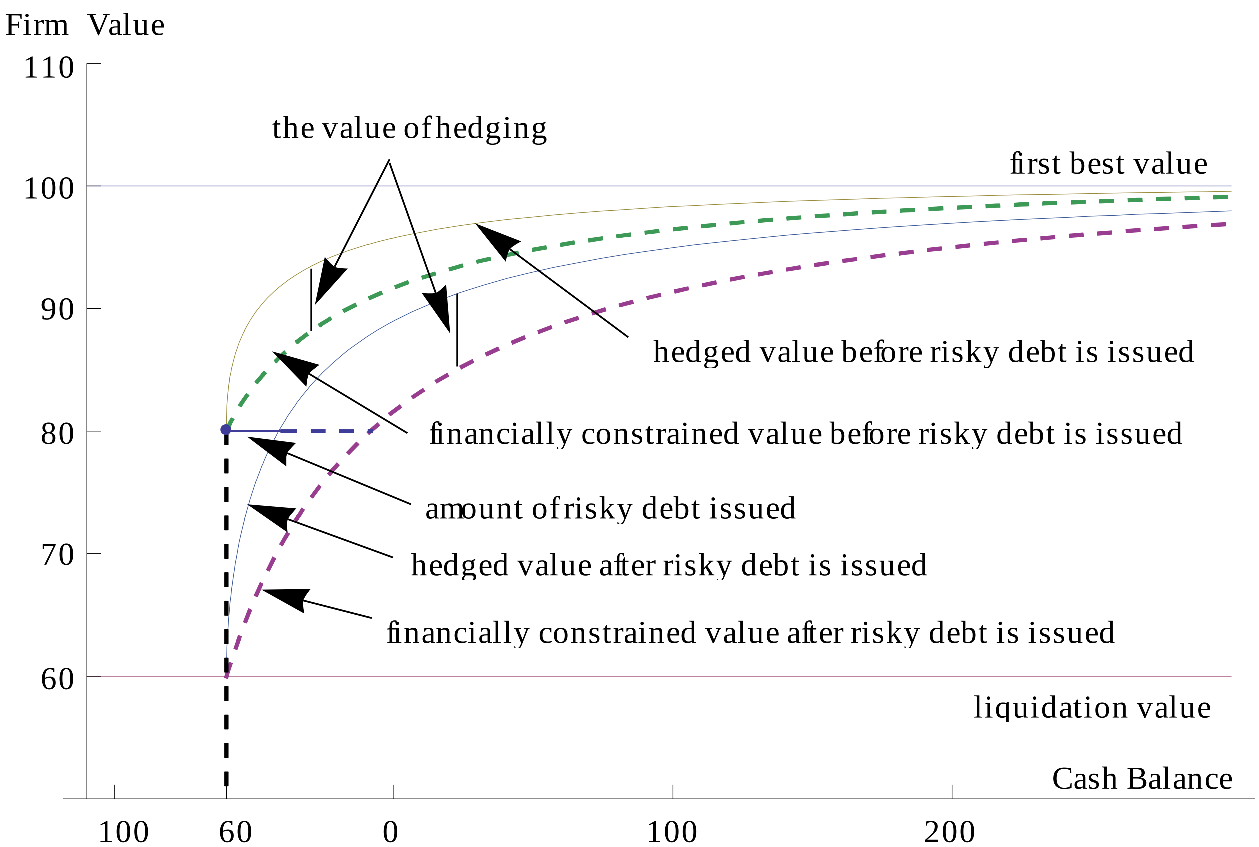


Figure 2

The hedged and unheged values of the financially constrained firm

The hedged and unhedged values of the financially constrained firm with risky debt are shown for various levels of cash balances given a liquidation cost: 1 – α = 0.4. The value of the hedged firm is higher than that of the unhedged firm for both prior to and following the issuance of risky debt given the level of cash balances because the probability that the future cash balances decline to the liquidation point for the hedged firm is smaller than that for the unhedged firm. As the cash balance increases, the financial constraints are less likely to be bound and the values of the hedged and the unhedged firm increase close to the first-best value.

Table 1. First-best firm valuation

Node i	Node j	Node k	Risk-neutral transition probability	Output price	Input cost	Futures price	Period operating cash flow	First-best firm value
0	0	0		10.00	10.00	10.25	0.00	292.84
1	0	0	0.3419	7.29	5.31	7.47	1.98	230.19
1	0	1	0.1971	7.29	18.82	7.47	-11.53	135.77
1	0	2	0.2924	13.72	5.31	14.07	8.41	480.05
1	0	3	0.1686	13.72	18.82	14.07	-5.10	365.73
2	0	0	0.3419	5.31	2.82		2.49	177.72
2	0	1	0.1971	5.31	10.00		-4.69	121.52
2	0	2	0.2924	10.00	2.82		7.18	361.38
2	0	3	0.1686	10.00	10.00		0.00	292.14
2	1	0	0.3419	5.31	10.00		-4.69	121.52
2	1	1	0.1971	5.31	35.43		-30.11	20.11
2	1	2	0.2924	10.00	10.00		0.00	292.14
2	1	3	0.1686	10.00	35.43		-25.43	140.84
2	2	0	0.3419	10.00	2.82		7.18	361.38
2	2	1	0.1971	10.00	10.00		0.00	292.14
2	2	2	0.2924	18.82	2.82		16.00	711.17
2	2	3	0.1686	18.82	10.00		8.82	629.62
2	3	0	0.3419	10.00	10.00		0.00	292.14
2	3	1	0.1971	10.00	35.43		-25.43	166.48
2	3	2	0.2924	18.82	10.00		8.82	629.62
2	3	3	0.1686	18.82	35.43		-16.61	430.51

The table exhibits a lattice representation of the two variable, continuous time problem presented in Section 2. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4 at 256 nodes (omitted for brevity). Using risk-neutral probabilities and the risk-free interest rate of $r = 5\%$, we discount the first-best firm value in period 4 and add it to the operating cash flow in period 3 to obtain the first-best value in period 3. Similarly, the first-best value in period 2 is calculated by backward programming using the period 3 first-best value with the risk-neutral probability reaching each node, and by adding the period 2 operating cash flow. For example, the values at nodes (2, 0, k), k = 0, 1, 2, 3 at the last column are discounted and added to the operating cash flow of 1.98 to compute the value at node (1, 0, 0) as 230.19. Similarly, we calculate the first-best firm value at node (0, 0, 0) as 292.84 from the values at nodes (1, 0, k), k = 0, 1, 2, 3.

Table 2: First-best firm valuation at node (1, 0, 1)

Node I	j	k	Risk-neutral transition probability	Output price	Input cost	Futures price	Two-year forward Price	Period operating cash flow	First-best firm value
1	0	1		7.29	18.82	7.47	7.66	-11.53	135.77
2	1	0	0.3419	5.31	10.00	5.45		-4.69	121.52
2	1	1	0.1971	5.31	35.43	5.45		-30.11	20.11
2	1	2	0.2924	10.00	10.00	10.25		0.00	292.14
2	1	3	0.1686	10.00	35.43	10.25		-25.43	140.84
3	0	0	0.3419	3.87	5.31			-1.44	102.04
3	0	1	0.1971	3.87	18.82			-14.95	36.53
3	0	2	0.2924	7.29	5.31			1.98	229.91
3	0	3	0.1686	7.29	18.82			-11.53	137.65
3	1	0	0.3419	3.87	18.82			-14.95	36.53
3	1	1	0.1971	3.87	66.68			0.00	**0.00**
3	1	2	0.2924	7.29	18.82			-11.53	137.65
3	1	3	0.1686	7.29	66.68			-59.93	**0.00**
3	2	0	0.3419	7.29	5.31			1.98	229.91
3	2	1	0.1971	7.29	18.82			-11.53	137.65
3	2	2	0.2924	13.72	5.31			8.41	479.02
3	2	3	0.1686	13.72	18.82			-5.10	361.50
3	3	0	0.3419	7.29	18.82			-11.53	137.65
3	3	1	0.1971	7.29	66.68			-59.93	**0.00**
3	3	2	0.2924	13.72	18.82			-5.10	361.50
3	3	3	0.1686	13.72	66.68			-52.96	129.41

The table exhibits a lattice representation of the two variable, continuous time problem up to period 3 emanating from node (1, 0, 1) in Table 1. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm values in period 4, which are used to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Using risk-neutral probabilities and the risk-free interest rate of $r = 5\%$,, we discount the first-best firm value in period 3 and add it to the operating cash flow in period 3 to obtain the first-best value in period 2. For example, the values at nodes (3, 1, k), k = 0, 1, 2, 3 at the last column are discounted using risk-neutral probabilities and added to the operating cash flow of -30.11to compute the value at node (2, 1, 1) as 20.11. Similarly, we calculate the first-best firm value at node (1, 0, 1) as 135.77 from the values at nodes (2, 1, k), k = 0, 1, 2, 3 and the operating cash flow of -$11.53.

Table 3: First-best firm values in period 4 out of (1, 0, 1) in period 1

Node						First-best	Node						First-best
i	j	k	l	price	cost	firm value	i	j	k	l	price	cost	firm value
4	0	0	0	2.82	2.82	82.43	4	2	0	0	5.31	2.82	177.40
4	0	0	1	2.82	10.00	40.80	4	2	0	1	5.31	10.00	121.95
4	0	0	2	5.31	2.82	177.40	4	2	0	2	10.00	2.82	360.50
4	0	0	3	5.31	10.00	121.95	4	2	0	3	10.00	10.00	292.02
4	0	1	0	2.82	10.00	40.80	4	2	1	0	5.31	10.00	121.95
4	0	1	1	2.82	35.43	**0.00**	4	2	1	1	5.31	35.43	26.36
4	0	1	2	5.31	10.00	121.95	4	2	1	2	10.00	10.00	292.02
4	0	1	3	5.31	35.43	26.36	4	2	1	3	10.00	35.43	144.55
4	0	2	0	5.31	2.82	177.40	4	2	2	0	10.00	2.82	360.50
4	0	2	1	5.31	10.00	121.95	4	2	2	1	10.00	10.00	292.02
4	0	2	2	10.00	2.82	360.50	4	2	2	2	18.82	2.82	709.26
4	0	2	3	10.00	10.00	292.02	4	2	2	3	18.82	10.00	628.47
4	0	3	0	5.31	10.00	121.95	4	2	3	0	10.00	10.00	292.02
4	0	3	1	5.31	35.43	26.36	4	2	3	1	10.00	35.43	144.55
4	0	3	2	10.00	10.00	292.02	4	2	3	2	18.82	10.00	628.47
4	0	3	3	10.00	35.43	144.55	4	2	3	3	18.82	35.43	432.05
4	1	0	0	2.82	10.00	40.80	4	3	0	0	5.31	10.00	121.95
4	1	0	1	2.82	35.43	**0.00**	4	3	0	1	5.31	35.43	26.36
4	1	0	2	5.31	10.00	121.95	4	3	0	2	10.00	10.00	292.02
4	1	0	3	5.31	35.43	26.36	4	3	0	3	10.00	35.43	144.55
4	1	1	0	2.82	35.43	**0.00**	4	3	1	0	5.31	35.43	**26.36**
4	1	1	1	2.82	125.51	**0.00**	4	3	1	1	5.31	125.51	**0.00**
4	1	1	2	5.31	35.43	**0.00**	4	3	1	2	10.00	35.43	**144.55**
4	1	1	3	5.31	125.51	**0.00**	4	3	1	3	10.00	125.51	**0.00**
4	1	2	0	5.31	10.00	121.95	4	3	2	0	10.00	10.00	292.02
4	1	2	1	5.31	35.43	26.36	4	3	2	1	10.00	35.43	144.55
4	1	2	2	10.00	10.00	292.02	4	3	2	2	18.82	10.00	628.47
4	1	2	3	10.00	35.43	144.55	4	3	2	3	18.82	35.43	432.05
4	1	3	0	5.31	35.43	**26.36**	4	3	3	0	10.00	35.43	144.55
4	1	3	1	5.31	125.51	**0.00**	4	3	3	1	10.00	125.51	**0.00**
4	1	3	2	10.00	35.43	**144.55**	4	3	3	2	18.82	35.43	432.05
4	1	3	3	10.00	125.51	**0.00**	4	3	3	3	18.82	125.51	93.37

The table exhibits the first-best firm values in period 4 emanating from nodes (1, 0, 1) in period 1 Table 2. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4, which is discounted at $r = 5\%$ using risk-neutral probabilities and added to the operating cash flow in period 3 to compute the first-best firm value in that period at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3 in Table 2.

Table 4: The value of unhedged firm with riskless debt alone

Node			Starting	Ending	Riskless Debt	Liquidation	Constrained	Deadweight
i	j	k	Cash	Cash	Capacity	Value	Value	Loss
0	0	0	10	10	87.59		279.10	13.74
1	0	0	10.50	12.48	112.44		227.18	3.01
1	0	1	10.50	-1.03	0.00	81.46	81.46	54.31
1	0	2	10.50	18.91	297.13		480.05	
1	0	3	10.50	5.40	129.19		349.75	15.97
2	0	0	12.57	15.07	97.18		177.72	
2	0	1	12.57	7.89	20.87		105.48	16.03
2	0	2	12.57	19.75	218.96		361.38	
2	0	3	12.57	12.57	131.10		292.14	
2	1	0	-1.09	-10.16	20.87			
2	1	1	-1.09	-35.58	0.00			
2	1	2	-1.09	-5.47	131.10			
2	1	3	-1.09	-30.90	0.00			
2	2	0	19.85	24.25	218.96		361.38	
2	2	1	19.85	17.07	131.10		292.14	
2	2	2	19.85	33.07	456.21		711.17	
2	2	3	19.85	25.89	344.28		629.62	
2	3	0	5.67	2.89	131.10		292.14	
2	3	1	5.67	-22.54	123.25		89.72	76.76
2	3	2	5.67	11.71	344.28		629.62	
2	3	3	5.67	-13.72	123.25		420.79	9.72

The table exhibits the value of the financially constrained firm, which starts operating with the initial cash balance of \$10 and can issue riskless debt alone. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4, which is discounted with risk-neutral probabilities and added to the operating cash flow in period 3 to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. "Starting Cash" is the cash balance carried over from the previous ending cash balance with the interest. "Ending Cash" is obtained from "Starting Cash" added to "Period Operating Cash Flow" in Table 1. Since the ending cash balance at nodes (1, 0, 0), (1, 0, 2), and (1, 0, 3) is positive, no financing is required. However, the negative ending cash at node (1, 0, 1) combined with no riskless debt capacity requires the firm to liquidate with the deadweight cost, which is assumed at 1 – α = 40% of the first-best value. The deadweight losses at nodes (2, 0. 1), (2, 3, 1), and (2, 3, 3) in period 2 result from liquidation at some nodes in period 3.

Table 5: The value of unhedged firm with riskless debt alone by liquidating at node (2, 3, 1)

Node			Starting	Ending	Riskless Debt	Liquidation	Constrained	Deadweight
i	j	k	Cash	Cash	Capacity	Value	Value	Loss
0	0	0	10	10	87.59		279.41	13.43
1	0	0	10.50	12.48	112.44		227.18	3.01
1	0	1	10.50	-1.03	0.00	81.46	81.46	54.31
1	0	2	10.50	18.91	297.13		480.05	
1	0	3	10.50	5.40	129.19		351.66	14.06
2	0	0	12.57	15.07	97.18		177.72	
2	0	1	12.57	7.89	20.87		105.48	16.03
2	0	2	12.57	19.75	218.96		361.38	
2	0	3	12.57	12.57	131.10		292.14	
2	1	0	-1.09	-10.16	20.87			
2	1	1	-1.09	-35.58	0.00			
2	1	2	-1.09	-5.47	131.10			
2	1	3	-1.09	-30.90	0.00			
2	2	0	19.85	24.25	218.96		361.38	
2	2	1	19.85	17.07	131.10		292.14	
2	2	2	19.85	33.07	456.21		711.17	
2	2	3	19.85	25.89	344.28		629.62	
2	3	0	5.67	2.89	131.10		292.14	
2	3	1	5.67	-22.54	123.25		**99.89**	**66.59**
2	3	2	5.67	11.71	344.28		629.62	
2	3	3	5.67	-13.72	123.25		420.79	9.72

The table exhibits the value of the financially constrained firm, which starts operating with the initial cash balance of \$10 and can issue riskless debt alone. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4, which is discounted with risk-neutral probabilities and added to the operating cash flow in period 3 to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. “Starting Cash” is the cash balance carried over from the previous ending cash balance with the interest. “Ending Cash” is obtained from “Starting Cash” added to “Period Operating Cash Flow” in Table 1. Since the ending cash balance at nodes (1, 0, 0), (1, 0, 2), and (1, 0, 3) is positive, no financing is required. However, the negative ending cash at node (1, 0, 1) combined with no riskless debt capacity requires the firm to liquidate with the deadweight cost, which is assumed at 1 – α = 40% of the first-best value. The deadweight losses at nodes (2, 0. 1) and (2, 3, 3) in period 2 result from liquidation at some nodes in period 3. It is better to liquidate at node (2, 3, 1) to realize the firm value of \$99.82 as opposed to that of \$89.72 in Table 4, where the price is \$10 and the cost is \$35.43. The ratio is 10/35.43 = 0.2822.

Table 6: Constrained firm valuation with a small amount of risky debt

Risky debt to cover cash shortages up to period 2 out of node (1, 0, 1)

Node i	j	k	Risky debt	Starting cash bal.	After interest	Ending cash bal.	Liquidation value	Constrained value	Deadweight loss
1	0	1	31.55	42.05		30.51		125.22	10.55
2	1	0	37.58	32.04	30.11	25.43		121.52	
2	1	1	10.14	32.04	30.11	0.00	12.07	12.07	8.04
2	1	2	38.50	32.04	30.11	30.11		292.14	
2	1	3	30.22	32.04	30.11	4.69	84.50	84.50	56.33
3	0	0	38.50	26.70	24.77	23.33		102.04	
3	0	1	33.64	26.70	24.77	9.82		36.53	
3	0	2	38.50	26.70	24.77	26.75		229.91	
3	0	3	38.50	26.70	24.77	13.24		137.65	
3	1	0	22.22	0.00	-1.92	-16.87	21.92	21.92	14.61
3	1	1	2.11	0.00	-1.92	-1.92		0.00	
3	1	2	34.00	0.00	-1.92	-13.46	82.59	82.59	55.06
3	1	3	2.11	0.00	-1.92	-1.92		0.00	
3	2	0	38.50	31.62	29.70	31.67		229.91	
3	2	1	38.50	31.62	29.70	18.16		137.65	
3	2	2	38.50	31.62	29.70	38.10		479.02	
3	2	3	38.50	31.62	29.70	24.59		361.50	
3	3	0	34.97	4.92	3.00	-8.54	82.59	82.59	55.06
3	3	1	7.04	4.92	3.00	3.00		0.00	
3	3	2	38.50	4.92	3.00	-2.11	216.90	216.90	144.60
3	3	3	30.91	4.92	3.00	-49.97	77.64	77.64	51.76

The table exhibits the value of the financially constrained firm, which has an initial cash balance of \$10 and issues risky debt of \$31.61 when is exhausts cash balance from operation. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4, which is discounted and added to the operating cash flow in period 3 to compute the first-best firm value in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. "Starting Cash" is the cash balance carried over from the previous ending cash balance with the interest. "Ending Cash" is obtained from "Starting Cash" plus "Risky debt" added to "Period Operating Cash Flow" in Table 2. Once the firm issues the bare minimum risky debt at 33.59 at node (1, 0, 1) to continue to operate in period 2, it again faces liquidity problems in period 3 at nodes (3, 1, 0), (3, 1, 2), (3, 3, 0), and (3, 3, 3) and suffers from deadweight costs in liquidation at node (2, 1, 1), which reduces its value by 4.96 at node (1, 0, 1)

Table 7: Constrained firm valuation with a large amount of risky debt

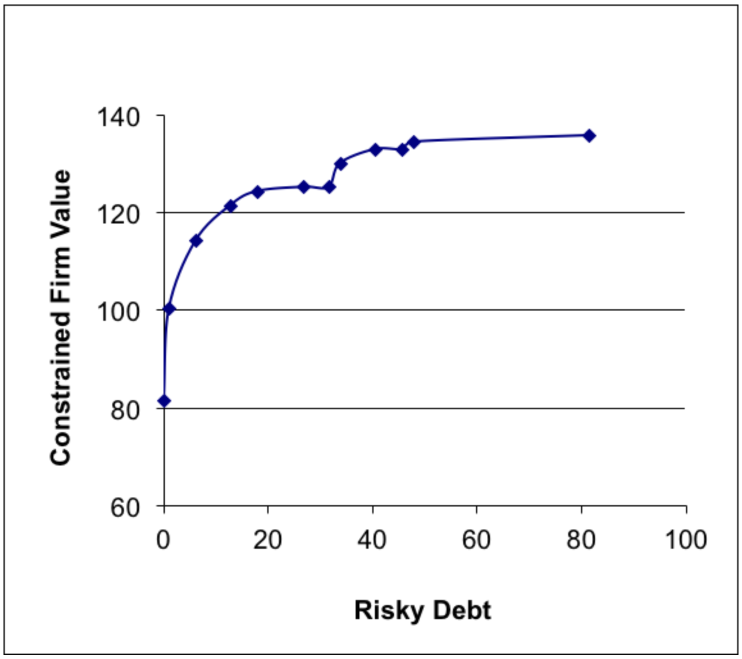
Risky debt to cover cash shortages up to period 3 out of node (1, 0, 1)

Node			Risky	Starting	After	Ending	Liquidation	Constrained	Deadweight
i	j	k	debt	cash bal.	interest	Cash bal.	value	value	loss
1	0	1	**81.29**	91.79		80.26		135.77	**0.00**
2	1	0	85.06	84.27	79.97	75.28		121.52	
2	1	1	68.14	84.27	79.97	49.85		20.11	
2	1	2	86.07	84.27	79.97	79.97		292.14	
2	1	3	79.31	84.27	79.97	54.54		140.84	
3	0	0	86.07	79.05	74.74	73.30		102.04	
3	0	1	80.69	79.05	74.74	59.79		36.53	
3	0	2	86.07	79.05	74.74	76.72		229.91	
3	0	3	86.07	79.05	74.74	63.21		137.65	
3	1	0	66.10	52.35	48.04	33.09		36.53	
3	1	1	48.04	52.35	48.04	48.04		**0.00**	
3	1	2	81.25	52.35	48.04	36.51		137.65	
3	1	3	67.74	52.35	48.04	-11.35		**0.00**	
3	2	0	86.07	83.97	79.66	81.64		229.91	
3	2	1	86.07	83.97	79.66	68.13		137.65	
3	2	2	86.07	83.97	79.66	88.07		479.02	
3	2	3	86.07	83.97	79.66	74.56		361.50	
3	3	0	82.22	57.27	52.96	41.43		137.65	
3	3	1	71.23	57.27	52.96	-6.43		**0.00**	
3	3	2	86.07	57.27	52.96	47.86		361.50	
3	3	3	69.11	57.27	52.96	0.00		129.41	

The table exhibits the value of the financially constrained firm, which has an initial cash balance of \$10 and issues risky debt of \$81.44 when is exhausts cash balance from operation. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4, which is discounted and added to the operating cash flow to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. “Starting Cash” is the cash balance carried over from the previous ending cash balance with the interest. “Ending Cash” is obtained from “Starting Cash” plus “Risky debt” added to “Period Operating Cash Flow” in Table 2. Once the firm issues the maximum risky debt at 81.44 at node (1, 0, 1) to continue to operate in period 2, it does not face liquidity problems in period 3 at all and can be sold at the full value in period 4.

Table 8: Constrained firm value vs. risky debt at node (1, 0, 1)

Risky Debt			Constrained	Deadweight
Market value (D)	Face value (B)	Coupon (b)	firm value	loss
81.29	86.07	4.38	135.77	0
47.75	48.19	2.41	134.44	1.33
45.59	58.21	2.91	132.93	2.84
40.43	50.79	2.54	132.93	2.84
33.73	41.46	2.07	130.05	5.72
31.55	38.50	1.92	125.22	10.55
26.78	32.10	1.60	125.22	10.55
17.83	18.30	0.91	124.33	11.44
12.74	6.24	0.31	121.45	14.32
6.11	4.82	0.24	114.24	21.53
1.03	1.10	0.06	100.50	35.27
0			81.46	54.31



The table exhibits the firm value behaviors given the level of issued risky debt as presented in Section 2. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm values in period 4, which are used to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. The value of the firm significantly increases when it issues a small amount of risky debt and the rate of increase drops as it issues more risky debt. Once the firm issues the maximum risky debt at 81.44 at node (1, 0, 1) to continue operation into period 2, it does not face liquidity problems in period 3 at all.

Table 9: Hedging with futures contracts alone

Node			Risky	Futures	Forward	Starting	After	Ending	Constrained	Value
I	J	k	debt	hedging	hedging	cash	interest	cash	value	of hedge
1	0	1	58.85			69.35		57.81	135.77	1.29
2	1	0	61.66			60.70	57.58	52.89	121.52	
2	1	1	56.28	-4.52		60.70	57.58	27.46	20.11	
2	1	2	62.52			60.70	57.58	57.58	292.14	
2	1	3	48.68	6.44		60.70	57.58	32.15	140.84	
3	0	0	62.52			55.54	52.41	50.97	102.04	
3	0	1	57.95			55.54	52.41	37.46	36.53	
3	0	2	62.52			55.54	52.41	54.39	229.91	
3	0	3	62.52			55.54	52.41	40.88	137.65	
3	1	0	58.39	28.84		57.68	54.55	39.60	36.53	
3	1	1	57.28	28.84		57.68	54.55	54.55	0.00	
3	1	2	60.85	13.40		42.24	39.11	27.58	137.65	
3	1	3	41.07	13.40		42.24	39.11	39.11	0.00	
3	2	0	62.52			60.46	57.33	59.31	229.91	
3	2	1	62.52			60.46	57.33	45.80	137.65	
3	2	2	62.52			60.46	57.33	65.74	479.02	
3	2	3	62.52			60.46	57.33	52.23	361.50	
3	3	0	55.14	-19.10		14.66	11.54	0.00	137.65	
3	3	1	12.12	-19.10		14.66	11.54	11.54	0.00	
3	3	2	62.52	22.33		56.09	52.96	47.86	361.50	
3	3	3	50.20	22.33		56.09	52.96	0.00	129.41	

The table exhibits the value of the financially constrained firm with an optimal risky debt and optimal hedges with futures contracts. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq.(3), we compute the first-best firm value in period 4, which is discounted and added to the operating cash flow in period 3 to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. To restore the first-best firm value as 135.77 at node (1, 0, 1), the firm should issue minimal risky debt as 55.18 at node (1, 0, 1) when it hedges with futures contracts alone. The optimal hedge is shown to be 4.52 short at node (2, 1, 1), and 6.44 long in futures contracts at (2, 1, 3). The value of hedge in this way is computed as 9.49 at node (1, 0, 1).

Table 10: Hedging with both futures and forward contracts

Node i	j	k	Risky debt	Futures hedging	Forward hedging	Starting cash	After interest	Ending cash	Constrained value	Value of hedge
1	0	1	39.90		7.37	50.40		38.86	135.77	4.96
2	1	0	41.98	-6.08	-15.93	40.81	38.55	33.87	121.52	
2	1	1	30.55	-5.69	-15.93	40.81	38.55	8.44	20.11	4.30
2	1	2	45.05		18.63	40.81	38.55	38.55	292.14	
2	1	3	36.15	-0.93	18.63	40.81	38.55	13.13	140.84	25.86
3	0	0	45.05	9.58	-27.94	17.20	14.95	13.51	102.04	
3	0	1	30.69	9.58	-27.94	17.20	14.95	0.00	36.53	
3	0	2	45.05	-11.20	-2.76	21.60	19.35	21.33	229.91	
3	0	3	42.73	-11.20	-2.76	21.60	19.35	7.82	137.65	
3	1	0	30.69	36.28	-27.94	17.20	14.95	0.00	36.53	14.13
3	1	1	15.69	36.28	-27.94	17.20	14.95	14.95	0.00	
3	1	2	43.01	16.85	-2.76	22.96	20.71	9.18	137.65	54.72
3	1	3	21.74	16.85	-2.76	22.96	20.71	20.71	0.00	
3	2	0	45.05		-2.76	37.73	35.48	37.45	229.91	
3	2	1	45.05		-2.76	37.73	35.48	23.94	137.65	
3	2	2	45.05		44.65	85.13	82.88	91.29	479.02	
3	2	3	45.05		44.65	85.13	82.88	77.78	361.50	
3	3	0	41.11	2.75	-2.76	13.78	11.53	0.00	137.65	
3	3	1	12.11	2.75	-2.76	13.78	11.53	11.53	0.00	
3	3	2	45.05	-3.22	44.65	55.21	52.96	47.86	361.50	
3	3	3	36.17	-3.22	44.65	55.21	52.96	0.00	129.41	50.07

The table exhibits the value of the financially constrained firm with an optimal risky debt and optimal hedges with both futures and forward contracts as presented in Section 3. Applying the parameter values: $\kappa_p = \kappa_c = 2.5\%$, $q = 1$, $p = c = 10$, $r = 5\%$, $\sigma_p = 10\%$, $\sigma_c = 40\%$, $\rho = 0$ to Eq. (3), we compute the first-best firm value in period 4, which is discounted and added to the operating cash flow in period 3 to compute the first-best firm values in period 3 at nodes (3, j, k), j = 0, 1, 2, 3 and k = 0, 1, 2, 3. Once the backward computation is done until period 0, we consider the liquidity. When the firm hedges liquidity with both futures and forward contracts, it can reduce risky debt issuance as 39.91 at node (1, 0, 1) to restore its first-best firm value as 135.77 at node (1, 0, 1). The optimal hedge is shown to be 7.37 long in the 2-period forward contracts at node (1, 0, 1), 6.08 short in futures contracts at (2, 1, 0), 5.69 short in futures at (2, 1, 1), and 0.93 short in futures at (2, 1, 3). The value of hedge in this way is computed as 2.10 at node (1, 0, 1).