A Simple Closed-form Approximation for Constant Elasticity of Variance Spread Options

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Abstract

By applying the Lie-Trotter operator splitting method and the idea of the WKB method, we have developed a simple, accurate and efficient analytical approximation for pricing the constant elasticity of variance (CEV) spread options. The derived option price formula bears a striking resemblance to Kirk's formula of the Black-Scholes spread options. Illustrative numerical examples show that the proposed approximation is not only extremely fast and robust, but also it is remarkably accurate for typical volatilities and maturities of up to two years.

Keywords : Constant elasticity of variance model, Black-Scholes equation, Spread options, Kirk's approximation, Lie-Trotter operator splitting method, WKB method

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1. Introduction

Spread options whose payoff is contingent upon the price difference (or the spread) of two lognormal underlying assets are very popular in a wide range of financial markets nowadays, *e.g.* interest rate markets, currency and foreign exchange markets, commodity markets, energy markets, etc. (Carmona and Durrleman, 2003) In spite of their popularity, pricing spread options is a very challenging task and receives much attention in the literature. This is mainly due to the lack of knowledge about the distribution of the spread which can assume negative values. The simplest approach is to obtain the joint probability distribution of the two correlated lognormal underlyings and evaluate the expectation of the final payoff by means of numerical integration. However, spread option traders often prefer to use analytical approximations rather than numerical methods because of their computational ease. Among various analytical approximations Kirk's approximation seems to be the most widely used, especially in the energy markets (Kirk, 1995), for it is not only remarkably accurate but also it is extremely efficient and robust.

A great deal of empirical evidence indicates that implied risk-neutral probability densities tend to be heavily skewed to the left and highly leptokurtic relative to the lognormal assumption made in the Black-Scholes model. To overcome these biases, researchers consider a wide range of stochastic term structures for the volatilities of the underlying processes. Among various possibilities the constant elasticity of variance (CEV) process:

$$dS = \mu S dt + \sigma S^{\beta/2} dZ \quad \text{for} \quad 0 \le \beta < 2 \quad , \tag{1}$$

which is both capable of reproducing the volatility smile observed in the empirical data and consistent with the so-called leverage effect (*i.e.* the existence of a negative correlation between stock returns and realized stock volatility), seems to be a good candidate according to the empirical studies (Cox and Ross, 1976; Emanuel and MacBeth, 1982; Chen and Lee, 1993; Cox, 1996; Chen et al., 2009). Nevertheless, it is unfortunate that the joint probability distribution of two correlated CEV processes is not available. Thus, the approach of numerical integration no longer applies in pricing spread options with two correlated CEV underlying assets and alternative numerical techniques like

Monte Carlo simulation or finite-difference method are needed. It is well known that the implementation cost of these alternative numerical methods is very high. Even for the special case of two uncorrelated CEV underlying assets the valuation via numerical integration is still rather slow because it requires the computation of the noncentral chi-square distribution. Hence, accurate and efficient analytical approximations for pricing these CEV spread options are highly desirable.

This paper considers the valuation of a spread option with two correlated CEV underlying assets. Our aim is to propose a simple, accurate and efficient analytical approximation for pricing the CEV spread option. By applying both the Lie-Trotter operator splitting method (Trotter, 1959) and the idea of the WKB method (Morse and Feshbach, 1953), we derive an approximate closed-form price formula which bears a great resemblance to Kirk's formula. In fact, the proposed approximation is reduced to Kirk's approximation when it is applied to the Black-Scholes spread options. Numerical investigations demonstrate that our approximate price formula possesses the same advantages as Kirk's formula.

2. Valuation of a CEV spread option

The price of a European call spread option with two correlated CEV underlying assets S_1 and S_2 obeys the two-dimensional CEV pricing equation

$$\begin{aligned} 0 &= \left\{ \frac{1}{2}\sigma_1^2 S_1^\beta \frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2\sqrt{S_1^\beta S_2^\beta}\frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^\beta \frac{\partial^2}{\partial S_2^2} + \right. \\ & \left. rS_1\frac{\partial}{\partial S_1} + rS_2\frac{\partial}{\partial S_2} - r - \frac{\partial}{\partial \tau} \right\} P\left(S_1, S_2, \tau\right) \end{aligned} \tag{2}$$

with the final payoff condition

$$P\left(S_1, S_2, 0\right) = \max\left(S_1 - S_2 - K, 0\right) \ , \tag{3}$$

where σ_i is the volatility of the underlying asset i, ρ is the correlation between the two underlying assets, K is the strike price, r is the risk-free interest rate, and τ denotes the time-to-maturity. Defining

$$\begin{aligned} P\left(S_1, S_2, \tau\right) &= e^{-r\tau} \exp\left\{ \tau \left(rS_1\frac{\partial}{\partial S_1} + rS_2\frac{\partial}{\partial S_2} \right) \right\} \tilde{P}\left(S_1, S_2, \tau\right) \\ &= e^{-r\tau}\tilde{P}\left(S_1 e^{r\tau}, S_2 e^{r\tau}, \tau\right) \ , \end{aligned} \tag{4}$$

we can show that $\tilde{P}(S_1, S_2, \tau)$ obeys the partial differential equation:

$$\begin{aligned} 0 &= \left\{\frac{1}{2}\sigma_1^2 S_1^\beta \frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2\sqrt{S_1^\beta S_2^\beta}\frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^\beta \frac{\partial^2}{\partial S_2^2} - \right. \\ &\qquad \left. e^{-(2-\beta)r\tau}\frac{\partial}{\partial \tau}\right\} \tilde{P}(S_1, S_2, \tau) \end{aligned} \tag{5}$$

where $\tilde{P}(S_1, S_2, 0) = \max(S_1 - S_2 - K, 0)$. In terms of the new variables:

$$R_1 = \frac{S_1}{S_2 + K} \tag{6}$$

$$R_2 = S_2 + K \tag{7}$$

$$\eta = \frac{1}{(2-\beta)r}\left\{e^{(2-\beta)r\tau} - 1\right\} , \tag{8}$$

we can express Eq.(5) as

$$\left\{\hat{L}_0 + \hat{L}_R - \frac{\partial}{\partial \eta}\right\} \tilde{P}(R_1, R_2, \eta) = 0 , \tag{9}$$

where

$$\hat{L}_0 = \frac{1}{2}\tilde{\sigma}^2 R_1^2 \frac{\partial^2}{\partial R_1^2} \tag{10}$$

$$\begin{aligned} \hat{L}_R &= \frac{1}{2}\sigma_2^2 \frac{(R_2 - K)^\beta}{R_2^2}\left\{R_2^2 \frac{\partial^2}{\partial R_2^2} + 2R_1\frac{\partial}{\partial R_1} - 2R_1 R_2 \frac{\partial^2}{\partial R_1 \partial R_2}\right\} \\ &\quad + \rho\sigma_1\sigma_2 \frac{\sqrt{[R_1 R_2 (R_2 - K)]^\beta}}{R_2^2}\left\{R_2 \frac{\partial^2}{\partial R_1 \partial R_2} - \frac{\partial}{\partial R_1}\right\} \end{aligned} \tag{11}$$

$$\begin{aligned} \tilde{\sigma}^2 &= \sigma_1^2 S_1^{\beta-2} - 2\rho\sigma_1\sigma_2 S_1^{(\beta-2)/2}\frac{S_2^{\beta/2}}{S_2 + K} + \sigma_2^2 \frac{S_2^\beta}{(S_2 + K)^2} \\ &= \sigma_1^2 (R_1 R_2)^{\beta-2} - 2\rho\sigma_1\sigma_2 (R_1 R_2)^{(\beta-2)/2}\frac{(R_2 - K)^{\beta/2}}{R_2} + \sigma_2^2 \frac{(R_2 - K)^\beta}{R_2^2} . \end{aligned} \tag{12}$$

The final payoff condition now becomes

$$\tilde{P}(R_1, R_2, 0) = R_2 \max(R_1 - 1, 0) . \tag{13}$$

Accordingly, the formal solution of Eq.(9) is given by

$$\tilde{P}(R_1, R_2, \eta) = \exp\left\{\eta\left(\hat{L}_0 + \hat{L}_R\right)\right\} R_2 \max(R_1 - 1, 0) . \tag{14}$$

Since the exponential operator $\exp\left\{\eta\left(\hat{L}_0+\hat{L}_R\right)\right\}$ is difficult to evaluate, the Lie-Trotter operator splitting method (Trotter, 1959) can be applied to approximate the operator by (see the Appendix)

$$\hat{O}^{LT}=\exp\left\{\eta\hat{L}_0\right\}\exp\left\{\eta\hat{L}_R\right\} \tag{15}$$

and obtain an approximation to the formal solution $\tilde{P}\left(R_1,R_2,\eta\right)$, namely

$$\begin{aligned} P^{LT}\left(R_1,R_2,\eta\right) &= \hat{O}^{LT}R_2\max\left(R_1-1,0\right) \\ &= R_2\exp\left\{\eta\hat{L}_0\right\}\max\left(R_1-1,0\right) \\ &\equiv R_2C\left(R_1,\eta\right)\ , \end{aligned} \tag{16}$$

for

$$\begin{aligned} \hat{L}_RR_2\max\left(R_1-1,0\right) &= 0 \\ \Rightarrow \exp\left\{\eta\hat{L}_R\right\}R_2\max\left(R_1-1,0\right) &= R_2\max\left(R_1-1,0\right)\ . \end{aligned} \tag{17}$$

In accordance with Eq.(16), one could easily recognise that $C\left(R_1,\eta\right)$ satisfies the partial differential equation

$$\left\{\frac{1}{2}\tilde{\sigma}^2R_1^2\frac{\partial^2}{\partial R_1^2}-\frac{\partial}{\partial\eta}\right\}C\left(R_1,\eta\right)=0 \tag{18}$$

with the initial condition: $C\left(R_1,0\right)=\max\left(R_1-1,0\right)$. Obviously, if $\tilde{\sigma}$ is replaced by a term σ_0 which is independent of R_1, then Eq.(18) is the Black-Scholes equation for a lognormal underlying asset price R_1 with null interest rate and the solution is simply given by

$$C_0\left(R_1,\eta\right)=R_1N\left(\xi_1\right)-N\left(\xi_2\right) \tag{19}$$

where $N\left(\cdot\right)$ denotes the cumulative normal distribution function, and

$$\xi_1 = \frac{\ln\left(R_1\right)}{\sigma_0\sqrt{\eta}}+\frac{1}{2}\sigma_0\sqrt{\eta} \tag{20}$$

$$\xi_2 = \xi_1-\sigma_0\sqrt{\eta}\ . \tag{21}$$

As $\tilde{\sigma}$ is a function of both R_1 and R_2, the solution in Eq.(19) does not satisfy Eq.(18). Nevertheless, if $\tilde{\sigma}$ is a slowly-varying function of R_1, then, based upon the solution in Eq.(19), we can apply the idea of WKB method [Footnote 1, Lo (2013)], which is a powerful tool for obtaining a global approximation to the solution of a linear ordinary differential equation, to derive an accurate approximate solution to Eq.(18). The closed-form approximate solution turns out to closely resemble Kirk's formula. Furthermore, it should be noted that for the Lie-Trotter operator splitting approximation to be valid, $\tilde{\sigma}^2\tau$ needs to be sufficiently small, namely $\tilde{\sigma}^2\tau \ll 1$.

Proposition 1 :

If $\tilde{\sigma}$ is a s*lowly-varying function* of R_1, *i.e.*

$$\frac{R_1}{\tilde{\sigma}^2}\left|\frac{\partial\tilde{\sigma}^2}{\partial R_1}\right| \ll 1 \ , \tag{22}$$

then the solution $C\left(R_1,\eta\right)$ of Eq.(18) can be approximated by

$$C_{\text{eff}}\left(R_1,\eta\right) = R_1 N\left(d_1\right) - N\left(d_2\right) \tag{23}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(R_1\right)}{\tilde{\sigma}\sqrt{\eta}} + \frac{1}{2}\tilde{\sigma}\sqrt{\eta} && (24)\\ d_2 &= d_1 - \tilde{\sigma}\sqrt{\eta}\ , && (25)\end{aligned}$$

which has the same form as the solution $C_0\left(R_1,\eta\right)$ in Eq.(19).

Proof :

First of all, it is not difficult to show that

$$\begin{aligned} \frac{R_1}{\tilde{\sigma}^2}\left|\frac{\partial\tilde{\sigma}^2}{\partial R_1}\right| &= \frac{2-\beta}{\tilde{\sigma}^2}\left|\sigma_1^2\left(R_1R_2\right)^{\beta-2} - \rho\sigma_1\sigma_2\frac{\left[R_1R_2\left(R_2-K\right)\right]^{\beta/2}}{R_1R_2^2}\right| \\ &= \frac{\left(2-\beta\right)f\left(R_1,R_2\right)\left|f\left(R_1,R_2\right)-\rho\right|}{1-2\rho f\left(R_1,R_2\right)+f\left(R_1,R_2\right)^2} \end{aligned} \tag{26}$$

where

$$\begin{aligned} f\left(R_1,R_2\right) &= \frac{\sigma_1}{\sigma_2}\left(\frac{1}{R_1}\right)\left(\frac{R_1R_2}{R_2-K}\right)^{\beta/2} \\ &= \frac{\sigma_1}{\sigma_2}\left(\frac{S_2+K}{S_2}\right)\left(\frac{S_2}{S_1}\right)^{(2-\beta)/2} \ . \end{aligned} \tag{27}$$

Then, provided that $(2-\beta)\,f\,(R_1, R_2) \ll 1$, we attain the condition given in Eq.(22). It is obvious that for $\beta \to 2$ the requirement in Eq.(22) is automatically satisfied; otherwise the requirement can be achieved if $K \ll S_2 \ll S_1$.

Next, substituting $C_{\text{eff}}\,(R_1, \eta)$ into the left-hand side $(L.H.S.)$ of Eq.(18), we obtain, after simplification,

$$\begin{aligned} L.H.S. \quad = \quad & \frac{1}{2}\tilde{\sigma}^2 R_2 \left\{ \frac{1}{4}\tilde{\sigma}\sqrt{\eta}\,(d_1 d_2 - 1)\,\Phi\,(d_2) \left(\frac{R_1}{\tilde{\sigma}^2}\frac{\partial\tilde{\sigma}^2}{\partial R_1} \right)^2 - \right. \\ & \left. d_2 \Phi\,(d_2) \left(\frac{R_1}{\tilde{\sigma}^2}\frac{\partial\tilde{\sigma}^2}{\partial R_1} \right) + \frac{1}{2}\tilde{\sigma}\sqrt{\eta}\Phi\,(d_2) \left(\frac{R_1^2}{\tilde{\sigma}^2}\frac{\partial^2\tilde{\sigma}^2}{\partial R_1^2} \right) \right\} \end{aligned} \tag{28}$$

where $\Phi\,(\cdot)$ denotes the normal distribution function. Hence, if $\tilde{\sigma}$ is a slowly-varying function of R_1 as shown in Eq.(22), it can be inferred that $L.H.S. \approx 0$ in Eq.(29) and $C_{\text{eff}}\,(R_1, \eta)$ can be a good approximate solution of Eq.(18). (Q.E.D.)

As a result, we have succeeded in deriving an approximate closed-form price formula for the CEV call spread option, which is a generalisation of Kirk's formula, and it is given by

$$P\,(S_1, S_2, \tau) \approx S_1 N\,(\theta_1) - \left(S_2 + K e^{-r\tau} \right) N\,(\theta_2) \tag{29}$$

where

$$\theta_1 \quad = \quad \frac{\ln(S_1) - \ln(S_2 + K e^{-r\tau})}{\sigma_{eff}\sqrt{\eta}} + \frac{1}{2}\sigma_{eff}\sqrt{\eta} \tag{30}$$

$$\theta_2 \quad = \quad \theta_1 - \sigma_{eff}\sqrt{\eta} \tag{31}$$

$$\sigma_{eff}^2 \quad = \quad e^{(2-\beta)r\tau} \left\{ \sigma_1^2 S_1^{\beta-2} - 2\rho\sigma_1\sigma_2 S_1^{(\beta-2)/2} \frac{S_2^{\beta/2}}{S_2 + K e^{-r\tau}} + \sigma_2^2 \frac{S_2^{\beta}}{(S_2 + K e^{-r\tau})^2} \right\} \tag{32}$$

and η is defined in Eq.(8). It should be noted that by taking the limit of $\beta \to 2$ we recover the well-known result, *i.e.* Kirk's formula.

3. Illustrative numerical results

In this section illustrative numerical examples are presented to demonstrate the accuracy of the proposed closed-form approximation for the CEV call spread option. Table 1-3 tabulate the approximate option prices for different values of the strike price K and time-to-maturity T with $\beta = 1.5$. Other input model parameters are set as follows: $r = 0.05$, $\rho = 0.5$, $S_1 = 60$, $S_2 = 50$ and

$$\begin{cases} \sigma_1 = \sigma_2 = 0.3 & \text{(Table 1)} \\ \sigma_1 = 0.3 \;\; \& \;\; \sigma_2 = 0.6 & \text{(Table 2)} \\ \sigma_1 = \sigma_2 = 0.6 & \text{(Table 3)} \end{cases} . \tag{33}$$

Monte Carlo estimates and the corresponding standard deviations are also presented for comparison.[2] It is observed that the computed errors of the approximate option prices for in-the-money (ITM) and at-the-money (ATM) cases are capped at 1% (in magnitude). In fact, most of them are less than 0.2% (in magnitude). For out-to-money (OTM) cases the errors are comparatively larger due to the smallness of the option prices but they do not exceed 2.3% (in magnitude). Moreover, the approximation generally works better for options with smaller volatilities and shorter maturities.

In Table 4-6 the effect of varying the parameter β upon the approximate estimation of the option prices is investigated. It is found that the accuracy of the proposed approximation remains more or less the same for different β in both ITM and ATM cases. However, for the OTM cases the errors exhibit more prominant fluctuations due to the smallness of the option prices. Table 7 then examines how the correlation parameter ρ affects the performance of the proposed approximation for options with $T = 0.25$. Obviously, the accuracy of the proposed approximation is not very sensitive to the changes in ρ for both ITM and ATM cases, but for the OTM cases the errors of the approximate estimates increase significantly with ρ.

In conclusion, all the numerical results show that the proposed closed-form approximation for the CEV spread option is not only extremely fast and robust, but also it is very accurate for typical volatilities and maturities of up to two years.

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[2]As discussed in Kahl and Jäckel (2006), Andersen (2008) and Lord *et al.* (2010), simulating a CEV process using a naive Euler discretization scheme may produce some error bias, especially in the presence of a stochastic volatility. In order to overcome the bias, we employ the special discretization scheme of Lord *et al.* (2010) with 60 time-steps a year to simulate the asset prices. According to Andersen (2008), our choice of time-steps should be able to make the bias statistically insignificant for there is no stochastic volatility in our case.

4. Conclusion

By applying both the Lie-Trotter operator splitting method and the idea of the WKB method, we have presented a simple approach to rigorously derive a closed-form approximation for the CEV spread options. The derived option price formula turns out to bear a striking resemblance to Kirk's formula of the Black-Scholes spread options. In fact, the proposed approximation is reduced to Kirk's approximation when it is applied to the Black-Scholes spread options. As demonstrated by illustrative numerical examples, our proposed approximation has the advantage that it is not only extremely fast and robust, but also it is remarkably accurate for typical volatilities and maturities of up to two years Moreover, we believe that this new approach can be generalised to price the multi-asset CEV spread options in a straightforward manner[3], and a detailed report of such a study will be presented elsewhere.

[3]The Lie-Trotter operator splitting method has already been applied by Lo (2014a, 2014b, 2014c) to derive a generalization of Kirk's approximation for the case of multi-asset Black-Scholes spread option.

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Appendix: *Lie-Trotter operator splitting method*

Suppose that one needs to exponentiate an operator $\hat{C}$ which can be split into two different parts, namely $\hat{A}$ and $\hat{B}$. For simplicity, let us assume that $\hat{C} = \hat{A} + \hat{B}$, where the exponential operator $\exp\left(\hat{C}\right)$ is difficult to evaluate but $\exp\left(\hat{A}\right)$ and $\exp\left(\hat{B}\right)$ are either solvable or easy to deal with. Under such circumstances the exponential operator $\exp\left(\varepsilon\hat{C}\right)$, with ε being a small parameter, can be approximated by the Lie-Trotter operator splitting formula:

$$\exp\left(\varepsilon\hat{C}\right) = \exp\left(\varepsilon\hat{A}\right)\exp\left(\varepsilon\hat{B}\right) + \mathcal{O}\left(\varepsilon^2\right) \quad . \tag{I.1}$$

This can be seen as the approximation to the solution at $t = \varepsilon$ of the equation $d\hat{Y}/dt = \left(\hat{A} + \hat{B}\right)\hat{Y}$ by a composition of the exact solutions of the equations $d\hat{Y}/dt = \hat{A}\hat{Y}$ and $d\hat{Y}/dt = \hat{B}\hat{Y}$ at time $t = \varepsilon$. Details of the Lie-Trotter splitting approximation can be found in Trotter (1958 & 1959), Suzuki (1985), Drozdov and Brey (1998), Hatano and Suzuki (2005), and Blanes et al. (2012). The Lie-Trotter splitting approximation is particularly useful for studying the short-time behaviour of the solutions of evolutionary partial differential equations of parabolic type because for this class of problems it is sensible to split the spatial differential operator into several parts each of which corresponds to a different physical contribution (e.g., reaction and diffusion).

$K\backslash T$	0.25	0.5	1	2	
1	9.0136	9.0510	9.2225	9.7006	LTWKB
	0.01%	−0.01%	0.01%	0.04%	error
	9.0128 ± 0.0019	9.0515 ± 0.0039	9.2214 ± 0.0028	9.6970 ± 0.0067	MC
5	5.1206	5.3617	5.8779	6.7894	LTWKB
	−0.02%	−0.02%	−0.05%	0.01%	error
	5.1214 ± 0.0023	5.3626 ± 0.0031	5.8810 ± 0.0034	6.7887 ± 0.0040	MC
10	1.2768	1.8394	2.6651	3.8843	LTWKB
	−0.02%	−0.01%	0.05%	0.04%	error
	1.2770 ± 0.0011	1.8395 ± 0.0016	2.6639 ± 0.0029	3.8829 ± 0.0033	MC
15	0.0786	0.3206	0.8804	1.9325	LTWKB
	2.21%	1.52%	1.07%	0.78%	error
	0.0769 ± 0.0003	0.3158 ± 0.0005	0.8711 ± 0.0015	1.9176 ± 0.0028	MC

Table 1: Prices of a European CEV call spread option for $\beta = 1.5$. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, $\rho = 0.5$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.

$K\backslash T$	0.25	0.5	1	2	
1	9.1088	9.4379	10.1885	11.5296	LTWKB
	0.11%	0.30%	0.58%	0.91%	error
	9.0986 ± 0.0031	9.4101 ± 0.0047	10.1294 ± 0.0049	11.4257 ± 0.0078	MC
5	5.4929	6.1327	7.2026	8.8514	LTWKB
	0.18%	0.32%	0.44%	0.66%	error
	5.4830 ± 0.0029	6.1133 ± 0.0026	7.1712 ± 0.0060	8.7931 ± 0.0048	MC
10	2.0168	2.8861	4.1460	5.9796	LTWKB
	0.00%	0.05%	0.16%	0.31%	error
	2.0168 ± 0.0018	2.8846 ± 0.0035	4.1392 ± 0.0044	5.9612 ± 0.0080	MC
15	0.3827	0.9704	2.0159	3.7233	LTWKB
	−0.16%	0.06%	0.08%	0.24%	error
	0.3833 ± 0.0008	0.9698 ± 0.0014	2.0142 ± 0.0020	3.7143 ± 0.0045	MC

Table 2: Prices of a European CEV call spread option for $\beta = 1.5$. Other input parameters are: $r = 0.05$, $\sigma_1 = 0.3$, $\sigma_2 = 0.6$, $\rho = 0.5$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.

$K\backslash T$	0.25	0.5	1	2	
1	9.1922	9.6578	10.6115	12.2199	LTWKB
	-0.02%	-0.01%	0.03%	0.14%	error
	9.1942 ± 0.0039	9.6584 ± 0.0047	10.6085 ± 0.0037	12.2032 ± 0.0097	MC
5	5.7398	6.5574	7.8590	9.8052	LTWKB
	-0.03%	-0.05%	-0.03%	0.05%	error
	5.7415 ± 0.0042	6.5608 ± 0.0065	7.8615 ± 0.0060	9.8003 ± 0.0106	MC
10	2.4892	3.5488	5.0683	7.2437	LTWKB
	0.06%	0.03%	0.11%	0.19%	error
	2.4877 ± 0.0029	3.5477 ± 0.0028	5.0627 ± 0.0057	7.2301 ± 0.0076	MC
15	0.7670	1.6356	3.0425	5.1984	LTWKB
	1.21%	0.96%	0.77%	0.71%	error
	0.7578 ± 0.0010	1.6200 ± 0.0044	3.0193 ± 0.0045	5.1615 ± 0.0059	MC

Table 3: Prices of a European CEV call spread option for $\beta = 1.5$. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = 0.6$, $\rho = 0.5$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.

$K\backslash T$	0.25	0.5	1	2	
1	9.0124	9.0247	9.0488	9.0952	LTWKB
	0.00%	0.00%	0.00%	0.01%	error
	9.0124 ± 0.0003	9.0245 ± 0.0005	9.0484 ± 0.0007	9.0946 ± 0.0008	MC
5	5.0621	5.1235	5.2439	5.4758	LTWKB
	0.00%	0.00%	0.01%	0.01%	error
	5.0622 ± 0.0002	5.1233 ± 0.0004	5.2436 ± 0.0005	5.4754 ± 0.0008	MC
10	0.2318	0.3727	0.6211	1.0728	LTWKB
	0.00%	0.03%	0.03%	0.02%	error
	0.2318 ± 0.0002	0.3726 ± 0.0003	0.6209 ± 0.0005	1.0726 ± 0.0005	MC
15	0.0000	0.0000	0.0000	0.0002	LTWKB
	NA	NA	NA	6.65%	error
	0.0000 ± 0.0000	0.0000 ± 0.0000	0.0000 ± 0.0000	$(1.98 \pm 0.152) \times 10^{-4}$	MC

Table 4: Prices of a European CEV call spread option for $\beta = 0.5$. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, $\rho = 0.5$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.

$K\backslash T$	0.25	0.5	1	2	
1	9.0124	9.0247	9.0488	9.0965	LTWKB
	0.00%	0.01%	0.00%	0.02%	error
	9.0124 ± 0.0008	9.0242 ± 0.0011	9.0484 ± 0.0016	9.0951 ± 0.0022	MC
5	5.0621	5.1237	5.2500	5.5210	LTWKB
	-0.01%	0.00%	0.00%	0.00%	error
	5.0624 ± 0.0010	5.1239 ± 0.0009	5.2498 ± 0.0017	5.5212 ± 0.0019	MC
10	0.5081	0.7560	1.1431	1.7595	LTWKB
	0.02%	0.03%	-0.01%	0.03%	error
	0.5080 ± 0.0006	0.7558 ± 0.0008	1.1432 ± 0.0008	1.7589 ± 0.0016	MC
15	0.0000	0.0008	0.0232	0.1910	LTWKB
	NA	7.94%	4.04%	1.87%	error
	0.0000 ± 0.0000	$(7.23 \pm 0.0975) \times 10^{-4}$	0.0223 ± 0.0002	0.1875 ± 0.0005	MC

Table 5: Prices of a European CEV call spread option for $\beta = 1$. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, $\rho = 0.5$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.

$K\backslash T$	0.25	0.5	1	2	
1	9.5803	10.4777	12.0359	14.4675	LTWKB
	0.01%	0.00%	−0.01%	−0.01%	error
	9.5798 ± 0.0053	10.4775 ± 0.0072	12.0376 ± 0.0084	14.4694 ± 0.0120	MC
5	6.4224	7.6528	9.5146	12.2396	LTWKB
	−0.03%	−0.04%	−0.06%	−0.05%	error
	6.4245 ± 0.0042	7.6557 ± 0.0059	9.5200 ± 0.0078	12.2455 ± 0.0113	MC
10	3.3904	4.8254	6.8794	9.8209	LTWKB
	−0.03%	−0.03%	−0.01%	0.00%	error
	3.3914 ± 0.0046	4.8269 ± 0.0071	6.8798 ± 0.0077	9.8213 ± 0.0118	MC
15	1.5088	2.8177	4.8199	7.7983	LTWKB
	0.19%	0.13%	0.05%	0.00%	error
	1.5060 ± 0.0028	2.8140 ± 0.0048	4.8177 ± 0.0097	7.7981 ± 0.0157	MC

Table 6: Prices of a European CEV call spread option for $\beta = 2$. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, $\rho = 0.5$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.

$\rho\backslash K$	1	5	10	15	
-0.9	9.1726	5.6959	2.4122	0.6932	LTWKB
	0.02%	-0.01%	0.00%	0.64%	error
	9.1704 ± 0.0047	5.6966 ± 0.0035	2.4121 ± 0.0023	0.6888 ± 0.0013	MC
-0.5	9.1025	5.5238	2.1519	0.5144	LTWKB
	0.01%	-0.03%	0.04%	0.88%	error
	9.1013 ± 0.0034	5.5256 ± 0.0029	2.1510 ± 0.0026	0.5099 ± 0.0012	MC
-0.1	9.0491	5.3507	1.8540	0.3320	LTWKB
	0.01%	-0.04%	0.02%	0.97%	error
	9.0483 ± 0.0028	5.3530 ± 0.0031	1.8537 ± 0.0018	0.3288 ± 0.0009	MC
0.1	9.0308	5.2668	1.6846	0.2419	LTWKB
	0.01%	-0.02%	-0.06%	1.26%	error
	9.0299 ± 0.0031	5.2681 ± 0.0031	1.6856 ± 0.0017	0.2389 ± 0.0005	MC
0.5	9.0136	5.1206	1.2768	0.0786	LTWKB
	0.01%	-0.02%	-0.02%	2.21%	error
	9.0128 ± 0.0019	5.1214 ± 0.0023	1.2770 ± 0.0011	0.0769 ± 0.0003	MC
0.9	9.0124	5.0621	0.6262	0.0003	LTWKB
	0.01%	0.01%	0.00%	25.09%	error
	9.0119 ± 0.0013	5.0617 ± 0.0007	0.6262 ± 0.0008	0.0002 ± 0.0000	MC

Table 7: Prices of a European CEV call spread option for $\beta = 1.5$. Other input parameters are: $r = 0.05$, $\sigma_1 = \sigma_2 = 0.3$, $T = 0.25$, $S_1 = 60$ and $S_2 = 50$. Here "LTWKB" refers to our proposed approximation while "MC" denotes the Monte Carlo estimates with $30,000,000$ replications. The relative errors of the "LTWKB" option prices with respect to the "MC" estimates are also presented.