Realized Higher-Order Comoments

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Abstract

We define a realized third comoment of arithmetic returns, which can be obtained from subperiod returns and option prices. In a similar manner, we also propose realized fourth (joint) cumulants, which are standardized (co)moments, and show that there are no realized fourth (co)moments. These realized estimators help to access the ex-post moments of a return for a specific period because they are obtained from the data only within the period. Moreover, unlike realized estimators suggested by previous studies, our estimators can reflect stochastic volatility as well as jump components. Furthermore, we show that neither realized fourth moments nor third comoments of log returns exist under the similar condition. Lastly, empirical results about our realized estimators are consistent with the literature.

JEL Classification: G11, G12, G13

Keywords: Realized Cumulant, Realized Joint Cumulant, Realized Coskewness, Realized Kurtosis, Realized Cokurtosis, Aggregation Property.

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I. Introduction

If periodical returns of an asset are i.i.d., sample moments of a total return for a period become accurate as sample size increases. However, distribution of security returns appears to be time varying, and it is prominent when there is a shock in the market as a series of financial crises show (Engle 1982; Ang and Timmermann 2012; Baur 2012). Therefore, sample moments can deviate from true moments of a total return for a specific period. Accordingly, alternative methods have been developed for the moment estimation. One of them is a forward-looking measure (or implied moments), which is obtained from option prices. Because option prices reflect perspectives on their underlying assets, they can provide moments of a return on their underlying asset for the next period (e.g., Bakshi, Kapadia and Madan (2003)).

Although implied moments provide information for the future, it is also needed to access what happened in a past specific period. For example, when we test whether an implied variance of a monthly return really forecasts realization, we use an ex-post variance estimation as a reference. As aforementioned, sample variance of monthly returns cannot be used for the reference because it is contaminated by returns from the other months.[1] In this respect, the following two characteristics are required for the reference estimator: Data only within the specific period (a month, in the example) are enough to yield the estimator, and horizon of the estimator coincides with coverage of data (a month, in the example). Realized variance in the literature satisfies these two characteristics (e.g., Andersen et al. (2003)).[2] Accordingly, realized variance helps to understand the specific period clearly and is obtainable even for newly issued securities, which have limited data period. Because of these merits, Neuberger (2012) develops realized third moment. However, none of realized fourth moment, realized third comoments, and realized fourth comoments are known although many theoretical and empirical studies show that the third and the fourth (co)moments are related to returns of securities (e.g., Kraus and Litzenberger 1976; Harvey and Siddique 2000; Dittmar 2002; Ang et al. 2006; Conrad, Dittmar and Ghysels 2013).

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[1] Alternatively, one may use sample variance of daily returns within the specific month but it is a variance of daily returns rather than a variance of the monthly return.

[2] Accordingly, Jiang and Tian (2005) test whether the implied volatility really forecasts the realized volatility of the future period, and Bollerslev, Gibson and Zhou (2011) estimate volatility risk premium from the implied volatility and realized volatility.

In this paper, by extending Neuberger (2012), we provide the realized third comoment of arithmetic returns with an assumption that price of each asset is a martingale. As a realized comoment, it helps to access the ex-post third comoment of returns for a specific period of our concern because it does not require data from any extra periods, which may have different statistical characteristics from the specific period. In the case of the fourth order, we show that there are neither realized fourth moment nor realized fourth comoments. Instead, we propose definitions of realized fourth (joint) cumulants, which are standardized (co)moments. Moreover, we show that neither realized third comoments nor fourth moments of log returns exist under the similar condition.

According to our data set from January 1996 to August 2014, sample moments are closer to average of realized moments than average of implied moments. In addition, empirical result with the realized cumulants is consistent with the literature although there are some differences in the significances. For example, the empirical result shows that portfolio with high beta, low gamma, low skewness, or high kurtosis is linked with low return like the result of Harvey and Siddique (2000), Conrad, Dittmar and Ghysels (2013), Frazzini and Pedersen (2014), and Amaya et al. (2015).

Among these studies, the work of Amaya et al. (2015) is closely related to ours because the authors also investigate realized third and fourth moments and discover the relation between the moments and subsequent returns. However, unlike their interesting result, their realized moments have a limitation in that these estimators cannot capture volatility of volatility contribution to the moments as they address. As a result, their measures are biased from total cubic and total quartic variations.[3] This problem arises because they define a realized *k*th order moment as a sum of the *k*th powers of sub-period returns, which is a natural extension of a realized variance, a sum of squares of sub-period returns, defined from the following relation

[3] Albeit the limitation, their realized moments are developed to describe moments of total return of a specific period. However, they revise their realized moments when they get their skewness and kurtosis. This revision makes their skewness and kurtosis to be sample measures of sub-periodical returns rather than realized measures of the total specific period because those are represented as $\left(\frac{1}{N}\sum_{i=1}^{N} r_{t_i}^{k}\right) / \left(\frac{1}{N}\sum_{i=1}^{N} r_{t_i}^{2}\right)^{k/2}$ for k=3 and 4, respectively, where each r represents a sub-periodical return.

$$E_0\left[(F_T - F_0)^k\right] = E_0\left[\sum_{i=1}^{N}(F_{t_i} - F_{t_{i-1}})^k\right], 0 = t_0 < t_1 < \cdots < t_N = T \tag{1}$$

with k=2 and a martingale property of F.

However, for $k \geq 3$, Equation (1) does not generally hold, and diffusive contribution in the right hand side diminishes as the partition $\{t_0, \cdots, t_N\}$ become finer. Therefore, a sum of the third or fourth powers of sub-period returns cannot capture whole characteristics of total return. To resolve this problem, Neuberger (2012) generalizes Equation (1) through Aggregation Property and confirms that realized moments over the second order are not obtainable when the provided information is limited to the price process F. Moreover, the author shows that we can additionally obtain the third moment but no higher order moments when the information set is extended to include variance process additionally. The addition of only the variance process is reasonable because of its importance and accessibility; some derivatives are quoted in volatility of their underlying assets. However, as mentioned above, the third and fourth moments are also important, and they are also obtainable as Bakshi, Kapadia and Madan (2003) show. Therefore, we extend our information set to include the implied third moment.[4] As a result, we show that this extension contributes to get the realized fourth moment. In addition, different from Amaya et al. (2015), our realized fourth moment reflects characteristics of volatility of volatility as well as jump contributions. More specifically, volatility clustering increases the fourth moment, and a negative correlation between skewness and returns reduces the fourth moment.

Meanwhile, previous studies show that covariations between securities' returns are also important in asset pricing. For example, traditional CAPM addresses the role of covariance in the asset pricing, and Harvey and Siddique (2000) addresses that of coskewness. In this respect, Neuberger (2011) investigates coskewness although it is omitted in the published version. The author provides a new perspective with a new definition of coskewness, which is a sensitivity of expected realized skewness with respect to the investments. A point that it is defined without covariance process improves its accessibility and usefulness in the future studies. However, another point that it is not developed based on the traditional definition causes a weak link between the new coskewness and the other studies from the traditional coskewness. Therefore,

[4] We do not include implied fourth moment because we guess that realized moment for an order requires lower order implied moments.

we investigate the realized third and fourth comoments in accordance with the traditional definition. Like the case of the fourth moment, we show that lower order implied moments and comoments contribute to yield realized third and fourth comoments. In addition, we show that contagion effect, which represents high correlation between returns during downturns, reduces the realized third comoment.

Despite the usefulness of the realized higher order comoments, estimating them has a practical obstacle. That is, while we require lower order implied moments and comoments to obtain a realized comoment, implied comoments are hardly accessible because they require exotic options like basket options or spread options. We partially overcome this issue by adopting Kempf, Korn and Saßning (2015).[5]

The finance literature mostly uses log returns instead of arithmetic returns because short term returns are easily transformed to long term returns and vice versa because of additivity of log returns. However, we do not require the transformation when the sample period coincides to the time horizon as we get the realized moments. In addition, upon occasion such as asset allocation, arithmetic returns are more adequate than log returns. Therefore, this paper concentrates on the moments of arithmetic returns although we also investigate existence of the realized moments of log returns.

The rest of the paper is organized as follows. Section II reviews about the Aggregation Property of Neuberger (2012) and investigates some properties about higher-order moments and comoments. Section III discusses estimation methods of implied moments and comoments in advance of estimations of realized moments. Section IV presents empirical results about our realized moments. Section V concludes this study.

II. The Aggregation Property Given Comoment Processes

As mentioned above, in general, Equation (1) does not hold unless $k \in \{1,2\}$. Therefore, a sum of the kth orders of sub-period returns is biased from the true kth order moment. For

[5] Kempf, Korn and Saßning (2015) obtain implied covariance just with index and individual options through an assumption of index model and an additional assumption about idiosyncratic risk. As a result, they show that this covariance is effective in asset allocation.

generalizing Equation (1) to higher orders, Neuberger (2012) provides a new framework, Aggregation Property. The first part of this section reviews this framework.

II.1. Review of the Aggregation Property

We use a notation $X=(X_t, 0\le t\le T)$ for an adapted vector valued stochastic process defined on a filtration. Then, the Aggregation Property is defined as follows.

Definition 1. The Aggregation Property (Neuberger 2012)

A function *g* on a vector valued process *X* has the *Aggregation Property* if and only if

$$E_s[g(X_u-X_s)]=E_s[g(X_u-X_t)]+E_s[g(X_t-X_s)],\ \ \forall(s,t,u)\ \text{ s.t. }\ 0\le s\le t<u\le T. \quad (2)$$

When the above definition is combined with a law of iteration, we have

$$E_0[g(X_T-X_0)]=E_0\left[\sum_{j=1}^{N}g(\Delta X_j)\right] \quad (3)$$

for any partition $0=t_0\le t_1\le\cdots\le t_N=T$.[6] Therefore,

$$\sum_{j=1}^{N}g(\Delta X_j) \quad (4)$$

can be called a realized measure of

$$E_0[g(X_T-X_0)] \quad (5)$$

because the expression (4) is an ex-post estimator of (5). For example, given a martingale process *S*,

$$\sum_{j=1}^{N}(\Delta S_j)^2 \quad (6)$$

[6] Hereafter, when we describe a process *X* only at t_j's with j=0,…,N, t_j is denoted by *j*. In addition, $X_j - X_{j\text{-}1}$ and $X_{i,j} - X_{i,j\text{-}1}$ are denoted by ΔX_j and $\Delta X_{i,j}$, respectively, for an (possibly vector valued) index *i*.

is an unbiased estimator of

$$E_0\left[(S_T - S_0)^2\right] \tag{7}$$

because $g(x) = x^2$ satisfies the Aggregation Property. More interestingly, Neuberger (2012) show that

$$\sum_{j=1}^{N}\left((\Delta S_j)^3 + 3\Delta S_j \Delta V_j\right) \tag{8}$$

satisfies the Aggregation Property. As a result, we have[7]

$$\begin{aligned} E_0\left[(S_T - S_0)^3\right] &= E_0\left[(S_T - S_0)^3 + 3(V_T - V_0)(S_T - S_0)\right] \\ &= E_0\left[\sum_{j=1}^{N}\left((\Delta S_j)^3 + 3\Delta V_j \Delta S_j\right)\right]. \end{aligned} \tag{9}$$

Accordingly, he names (8) as a realized third moment. Furthermore, he shows that there are no additional higher order moments when the information process X is defined as (S,V), and he leaves an open question of whether extended information sets can produce realized higher order moments.

II.2. Realized higher order comoments

In the rest of this section, we investigate the existence of the realized fourth moment and realized comoments up to the fourth order by extending the information set to include all the implied moments and comoments up to the third order. Therefore, let X be a vector valued process $\{(S_{1,t}, S_{2,t}, M_t) : 0 \le t \le T\}$, where S_1 and S_2 are martingale processes and M_t represents:

$$M_t = (M_{2,0,t}, M_{1,1,t}, M_{0,2,t}, M_{3,0,t}, M_{2,1,t}, M_{1,2,t}, M_{0,3,t}) \tag{10}$$

with

$$M_{k,l,t} \equiv E_t\left[(S_{1,T} - S_{1,t})^k (S_{2,T} - S_{2,t})^l\right]. \tag{11}$$

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[7] The first equality is from martingale property of the process S, and the second equality is from the Aggregation Property of a function $g(\Delta S, \Delta V) = (\Delta S)^3 + 3\Delta S \Delta V$.

Then, Proposition 1 provides a general form of functions which have the Aggregation Property on X .

Proposition 1. When S_1 and S_2 are martingale processes, a two dimensional analytic function *g* has the Aggregation Property on the vector valued process *X* if and only if *g* can be represented as follows:

$$
\begin{aligned}
g(\Delta S_1, \Delta S_2, \Delta M) = {} & h_1 \Delta S_1 + h_2 \Delta S_2 + h_3 (\Delta S_1)^2 + h_4 \Delta M_{2,0} + h_5 \Delta S_1 \Delta S_2 \\
& + h_6 \Delta M_{1,1} + h_7 (\Delta S_2)^2 + h_8 \Delta M_{0,2} + h_9 ((\Delta S_1)^3 + 3\Delta S_1 \Delta M_{2,0}) \\
& + h_{10} \Delta M_{3,0} + h_{11} ((\Delta S_1)^2 \Delta S_2 + 2\Delta S_1 \Delta M_{1,1} + \Delta S_2 \Delta M_{2,0}) + h_{12} \Delta M_{2,1} \\
& + h_{13} (\Delta S_1 (\Delta S_2)^2 + 2\Delta S_2 \Delta M_{1,1} + \Delta S_1 \Delta M_{0,2}) + h_{14} \Delta M_{1,2} \\
& + h_{15} ((\Delta S_2)^3 + 3\Delta S_2 \Delta M_{0,2}) + h_{16} \Delta M_{0,3} \\
& + h_{17} ((\Delta S_1)^4 + 6(\Delta S_1)^2 \Delta M_{2,0} + 4\Delta S_1 \Delta M_{3,0} + 3(\Delta M_{2,0})^2) \\
& + h_{18} \begin{pmatrix} (\Delta S_1)^3 \Delta S_2 + \Delta S_2 \Delta M_{3,0} + 3(\Delta M_{1,1} (\Delta S_1)^2 \\ + \Delta M_{2,1} \Delta S_1 + \Delta M_{2,0} \Delta S_1 \Delta S_2 + \Delta M_{2,0} \Delta M_{1,1}) \end{pmatrix} \\
& + h_{19} \begin{pmatrix} ((\Delta S_1)^2 + \Delta M_{2,0})((\Delta S_2)^2 + \Delta M_{0,2}) + 2(\Delta M_{1,1})^2 \\ + 4\Delta M_{1,1} \Delta S_1 \Delta S_2 + 2\Delta M_{1,2} \Delta S_1 + 2\Delta M_{2,1} \Delta S_2 \end{pmatrix} \\
& + h_{20} \begin{pmatrix} \Delta S_1 (\Delta S_2)^3 + \Delta S_1 \Delta M_{0,3} + 3(\Delta M_{1,1} (\Delta S_2)^2 \\ + \Delta M_{1,2} \Delta S_2 + \Delta M_{0,2} \Delta S_1 \Delta S_2 + \Delta M_{0,2} \Delta M_{1,1}) \end{pmatrix} \\
& + h_{21} ((\Delta S_2)^4 + 6(\Delta S_2)^2 \Delta M_{0,2} + 4\Delta S_2 \Delta M_{0,3} + 3(\Delta M_{0,2})^2) \qquad (12)
\end{aligned}
$$

for some constants $h_1, \cdots, h_{21}$.

Proof is in the Appendix.

By symmetry, it is enough to investigate the terms related to $S_1^k S_2^l$ with $k \geq l$. In this regard, a function with the Aggregation Property is represented with a sum of 12 individual terms as follows:

$$
\begin{aligned}
g(\Delta S_1, \Delta S_2, \Delta M) = {} & h_1 \Delta S_1 + h_2 (\Delta S_1)^2 + h_3 \Delta M_{2,0} + h_4 \Delta S_1 \Delta S_2 + h_5 \Delta M_{1,1} \\
& + h_6 ((\Delta S_1)^3 + 3\Delta S_1 \Delta M_{2,0}) + h_7 \Delta M_{3,0} \\
& + h_8 ((\Delta S_1)^2 \Delta S_2 + 2\Delta S_1 \Delta M_{1,1} + \Delta S_2 \Delta M_{2,0}) + h_9 \Delta M_{2,1} \\
& + h_{10} ((\Delta S_1)^4 + 6(\Delta S_1)^2 \Delta M_{2,0} + 4\Delta S_1 \Delta M_{3,0} + 3(\Delta M_{2,0})^2) \\
& + h_{11} \begin{pmatrix} (\Delta S_1)^3 \Delta S_2 + \Delta S_2 \Delta M_{3,0} + 3(\Delta M_{1,1} (\Delta S_1)^2 \\ + \Delta M_{2,1} \Delta S_1 + \Delta M_{2,0} \Delta S_1 \Delta S_2 + \Delta M_{2,0} \Delta M_{1,1}) \end{pmatrix} \\
& + h_{12} \begin{pmatrix} ((\Delta S_1)^2 + \Delta M_{2,0})((\Delta S_2)^2 + \Delta M_{0,2}) + 2(\Delta M_{1,1})^2 \\ + 4\Delta M_{1,1} \Delta S_1 \Delta S_2 + 2\Delta M_{1,2} \Delta S_1 + 2\Delta M_{2,1} \Delta S_2 \end{pmatrix}
\end{aligned}
\tag{13}
$$

In Equation (13), the Aggregation Property of each $\Delta M_{k,l}$ is obvious and those of ΔS_1, $(\Delta S_1)^2$, and $(\Delta S_1)^3 + 3\Delta S_1 \Delta M_{2,0}$ are shown by Neuberger (2012). Among remainder terms, while $\Delta S_1 \Delta S_2$ is well known as an estimator of covariance, the 8$^{\text{th}}$, 10$^{\text{th}}$, 11$^{\text{th}}$, and 12$^{\text{th}}$ terms have not been discovered from the previous studies.

To analyze the 8$^{\text{th}}$ term among them, let us define a sum $cTM_{1,1,2}^{real}$ as[8]

$$
cTM_{1,1,2}^{real} \equiv \sum_{j=1}^{N} \left((\Delta S_{1,j})^2 \Delta S_{2,j} + \Delta S_{2,j} \Delta M_{2,0,j} + 2\Delta S_{1,j} \Delta M_{1,1,j} \right). \tag{14}
$$

Then we have[9]

$$
\begin{aligned}
E_0 \left[cTM_{1,1,2}^{real} \right] &= E_0 \left[(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0}) - (S_{2,T} - S_{2,0}) M_{2,0,0} - 2(S_{1,T} - S_{1,0}) M_{1,1,0} \right] \\
&= E_0 \left[(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0}) \right].
\end{aligned}
\tag{15}
$$

Because the last line of Equation (15) is the third comoment, $cTM_{1,1,2}^{real}$ is an unbiased estimator of the third comoment. Accordingly, we name it a realized third comoment. These discussion

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[8] At the terminology in Equation (14), prefix *c* and *TM* are from *co*(moment) and *third moment*, respectively. Additionally, a subscript *a*,*b*,*c* implies that the term is about $E_0[(S_{a,T}-S_{a,0})(S_{b,T}-S_{b,0})(S_{c,T}-S_{c,0})]$ and the superscript 'real' represents that the estimator is a realized moment. Afterward, *TM* is replaced by *FM* for the fourth cumulant, which is linked to the *fourth moment*. Then, FM_a denotes an estimator which is related to the fourth moment of $(S_{a,T} - S_{a,0})$, and $cFM_{a,b,c,d}$ denotes an estimator which is related to the *fourth comoment*, $E_0[(S_{a,T}-S_{a,0})(S_{b,T}-S_{b,0})(S_{c,T}-S_{c,0})(S_{d,T}-S_{d,0})]$. In the strict sense, *FM* and *cFM* represent the fourth cumulant and joint cumulant, respectively, and more detailed description of *FM* and *cFM* will be given later.

[9] The first equality is from the Aggregation Property and identities $M_{k,l,N}=0$ for any *k* and *l*. Then, the second equality is from the martingale property for each S_1 and S_2.

implies that an estimator $\sum_{j=1}^{N}(\Delta S_{1,j})^2 \Delta S_{2,j}$ cannot capture whole the third comoment. However, addition of $\Delta S_{2,j}\Delta M_{2,0,j}$ and $2\Delta S_{1,j}\Delta M_{1,1,j}$ makes the estimator complete. Among the additional terms, inclusion of the former term into the realized third comoment implies that, if S_1 becomes more volatile while S_2 decreases, third comoment is reduced. Similarly, inclusion of the latter term implies that, if the covariance between returns increases while S_2 decreases, third comoment is reduced. From a series of financial crises, we observe that covariance of returns increases in economic downturns, which is known as contagion effect and interdependence (e.g., Allen and Gale 2000; Forbes and Rigobon 2002; Cespa and Foucault 2014). This empirical evidence implies that $\sum_{j=1}^{N}(\Delta S_{1,j})^2 \Delta S_{2,j}$ may be biased upward from the actual third comoment.

Now let us investigate the fourth moment through the 10th term in Equation (13). To analyze this term, let us define a sum FM_1^{real} as

$$FM_1^{real} \equiv \sum_{j=1}^{N}\left((\Delta S_{1,j})^4 + 6(\Delta S_{1,j})^2 \Delta M_{2,0,j} + 4\Delta S_{1,j}\Delta M_{3,0,j} + 3(\Delta M_{2,0,j})^2\right). \tag{16}$$

Then we have

$$\begin{aligned} E_0\left[FM_1^{real}\right] &= E_0\left[(S_{1,T}-S_{1,0})^4 - 6(S_{1,T}-S_{1,0})^2 M_{2,0,0} - 4(S_{1,T}-S_{1,0})M_{3,0,0} + 3(M_{2,0,0})^2\right] \\ &= E_0\left[(S_{1,T}-S_{1,0})^4\right] - 3\left(E_0[(S_{1,T}-S_{1,0})^2]\right)^2 \end{aligned} \tag{17}$$

due to the Aggregation Property, martingale property, and identities $M_{k,l,N} = 0$. Although the last line of Equation (17) is not the fourth moment, it is also an important value as the numerator of kurtosis and is called the fourth cumulant. Accordingly, we name FM_1^{real} a realized fourth cumulant because it is an unbiased estimator of the fourth cumulant as Equation (17) shows.

We would like to decompose the terms in Equation (16) into three parts: $(\Delta S_{1,j})^4$, $6(\Delta S_{1,j})^2 \Delta M_{2,0,j} + 3(\Delta M_{2,0,j})^2$, and $4\Delta S_{1,j}\Delta M_{3,0,j}$. Like the third comoment case, the first part $(\Delta S_{1,j})^4$ does not entirely capture the fourth cumulant of total return even though it is related to the fourth cumulant of total return. More specifically, Amaya et al. (2015) address that the first part captures only jump contribution and cannot capture the volatility of volatility contribution.

However, this problem can be resolved by the second part $6(\Delta S_{1,j})^2\Delta M_{2,0,j} + 3(\Delta M_{2,0,j})^2$. To investigate its property, let us take an approximation:[10]

$$6(\Delta S_{1,j})^2\Delta M_{2,0,j} + 3(\Delta M_{2,0,j})^2 \approx 3(\Delta S_{1,j})^2\Delta M_{2,0,j}. \tag{18}$$

This approximation shows that the second part is related to autocorrelation of variance because $\Delta M_{2,0,j}$ and $(\Delta S_{1,j})^2$ represent variance (until the maturity) innovation and instantaneous variance, respectively. The literature such as Ghose and Kroner (1995) reports that autocorrelation of variance is positive, which is known as volatility clustering, and such literature shows that this is related to fat-tail of distribution. Therefore, Equations (16), (17), and (18) do not only show the necessity of $6(\Delta S_{1,j})^2\Delta M_{2,0,j} + 3(\Delta M_{2,0,j})^2$ for the estimation of the fourth cumulant, but also confirm the relation between volatility clustering and fat-tail, consistent with the literature. Additionally, the other part in Equation (16), $4\Delta S_{1,j}\Delta M_{3,0,j}$, implies that fourth cumulant is related to the correlation between skewness and return. Unlike the second part, this part seems to reduce fourth cumlant because empirical evidence shows that there is a negative relation between skewness and return (e.g., Boyer, Mitton and Vorkink (2010)).

Like the realized fourth cumulant, we can also define realized fourth order joint cumulants based on the 11th and 12th terms of Equation (13) as follows:[11]

$$\begin{aligned} cFM_{1,1,1,2}^{real} \equiv & \sum_{j=1}^{N}\left((\Delta S_{1,j})^3\Delta S_{2,j} + \Delta S_{2,j}\Delta M_{3,0,j}\right) \\ & + 3\sum_{j=1}^{N}(\Delta M_{1,1,j}(\Delta S_{1,j})^2 + \Delta M_{2,1,j}\Delta S_{1,j} + \Delta M_{2,0,j}\Delta S_{1,j}\Delta S_{2,j} + \Delta M_{2,0,j}\Delta M_{1,1,j}) \end{aligned} \tag{19}$$

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[10] The approximation is from $E_{j-1}[\Delta M_{2,0,j}+(\Delta S_{1,j})^2]=0$.

[11] The third order joint cumulant of (X_1, X_2, X_3) and the fourth order joint cumulant of (X_1, X_2, X_3, X_4) are $E[X_1X_2X_3]$ and $E[X_1X_2X_3X_4] - E[X_1X_2]E[X_3X_4] - E[X_1X_3]E[X_2X_4] - E[X_1X_4]E[X_2X_3]$, respectively, when expectation of each X_i is zero. Therefore, the last line of Equation (15) is the third order joint cumulant as well as the third comoment because each S_i is martingale.

and

$$cFM_{1,1,2,2}^{real} \equiv \sum_{j=1}^{N}((\Delta S_{1,j})^2 + \Delta M_{2,0,j})((\Delta S_{2,j})^2 + \Delta M_{0,2,j}) \\ + \sum_{j=1}^{N}(2(\Delta M_{1,1,j})^2 + 4\Delta M_{1,1,j}\Delta S_{1,j}\Delta S_{2,j} + 2\Delta M_{1,2,j}\Delta S_{1,j} + 2\Delta M_{2,1,j}\Delta S_{2,j}). \tag{20}$$

From the realized third comoment and the realized fourth (joint) cumulants, we can define the third and the fourth (co)moment swaps described in Table 1. Then, these swaps can be hedged with some securities as Proposition 2 describes.

[Table 1 about here]

Proposition 2. Higher order (co)moment swaps

When we define higher order (co)moment swaps as Table 1 describes, each swap can be replicated with risk free asset and the securities that pay $S_{1,T}$, $S_{2,T}$, $S_{1,T}^2$, $S_{1,T}S_{2,T}$, $S_{1,T}^3$, $S_{1,T}^2S_{2,T}$, or $S_{1,T}S_{2,T}^2$ at time T.

Proof is in the appendix.

One may wonder about the Aggregation Property for log prices because of log returns' merits like an additivity over time. However, in the case of realized moment, we do not require the transformation. In addition, upon occasion such as asset allocation, arithmetic returns are more adequate than log returns. So we want to focus on the Aggregation Property with arithmetic returns, but we also investigate a generalized function with the Aggregation Property for log price series through the next propositions. Because the Aggregation Property is meaningful when we can find realized moments, we define generalized comoments and realized comoments as follows, to explore realized moments in a general sense:

Definition 2. A generalized (k,l)-comoment function

We call $f^{k,l}$ a *generalized* (*k*,*l*)-*comoment function* if and only if $f^{k,l}$ is a two dimensional analytic function such that $\frac{f^{k,l}(s_1,s_2)}{s_1^k s_2^l} \to 1$ as $(s_1,s_2) \to (0,0)$. In addition, we call both a *generalized* (*k*,0)-*comoment function* and a *generalized* (0,*k*)-*comoment function* as a *generalized k-moment function* and denote them with f^k.

Definition 3. A realized (*k*,*l*)-comoment element

Let $x = (s_1, s_2, m)$ be a partitioned vector process where m_t consists of $m_{k,l,t}$ such that $m_{k,l,t} = E_t[f^{k,l}(s_{1,T} - s_{1,t}, s_{2,T} - s_{2,t})]$ for a generalized (*k*,*l*)-comoment function $f^{k,l}$. Then, a function *g* with the Aggregation Property is called a *realized* (*k*,*l*)-*comoment element* if and only if it is decomposed as follows

$$g(x_\tau - x_t) = \eta(x_\tau - x_t) + g_r^{k,l}(s_{1,\tau} - s_{1,t}, s_{2,\tau} - s_{2,t}) \tag{21}$$

where η is a function that satisfies $E_t[\eta(x_\tau - x_t)] = 0$ for $t < \tau$ and $g_r^{k,l}$ is a function that satisfies the condition of a generalized (*k*,*l*)-comoment function. In addition, we call both a *realized* (*k,0*)-*comoment element* and a *realized* (*0*,*k*)-*comoment element* as a *realized k-moment element*.

As Neuberger shows, there is a realized (3,0)-comoment element with the following decomposition:

$$\begin{aligned} g(\Delta x) &= -12(e^{\Delta s_1} - 1) + 6\Delta s_1 - 3\Delta v_1 + 3e^{\Delta s_1}(\Delta v_1 + 2\Delta s_1) \\ &= 3\Delta v_1(e^{\Delta s_1} - 1) + 6(\Delta s_1 e^{\Delta s_1} - 2e^{\Delta s_1} + \Delta s_1 + 2) \end{aligned} \tag{22}$$

However, as shown in the next propositions and corollaries, we cannot get other realized moments under our condition; more specifically, both realized (2,1)-comoment element and realized 4-moment element do not exist. We finish this section with presenting propositions and corollaries about the Aggregation Property of log prices and the existence of realized moments, respectively. In the propositions, $s_{i,t}$ denotes $\ln(S_{i,t})$ for a martingale process S_i, and

$m_{k,l,t}$ denotes $E_t[f^{k,l}(s_{1,T}-s_{1,t}, s_{2,T}-s_{s,t})]$ with a generalized (*k*,*l*)-comoment function $f^{k,l}$ (or f^k when *l*=0).

Proposition 3. An analytic function *g* on a vector valued process $x=(s_1, m_{2,0}, m_{3,0})$ has the Aggregation Property on the vector valued process *x* if and only if *g* is represented as follows:

$$\begin{aligned} g(s_1, m_{2,0}, m_{3,0}) &= h_1(e^{s_1}-1)+h_2 s_1+h_3 m_{2,0}+h_4 m_{3,0} \\ &+h_5(m_{2,0}+am_{3,0}-2s_1)^2+h_6(m_{2,0}+am_{3,0}+2s_1)e^{s_1} \end{aligned} \tag{23}$$

for some constants $h_1,\cdots,h_6$ and *a*, which have one of the following 3 conditions:

i) $h_5=h_6=0$.

ii) $h_6=0$ and $f^2(s)+af^3(s)=2(e^s-s-1)$ for the constant *a*.

iii) $h_5=0$ and $f^2(s)+af^3(s)=2(se^s-e^s+1)$ for the constant *a*.

Proof is in the Appendix B.

Corollary 4. When information set is given by $x=(s_1, m_{2,0}, m_{3,0})$, there is no realized 4-moment element.

Proof is in the Appendix B.

Proposition 5. A multidimensional analytic function *g* has the Aggregation Property on the vector valued process $x=(s_1, s_2, m_{2,0}, m_{0,2}, m_{1,1})$ if and only if *g* is represented as follows:

$$\begin{aligned} g(s_1, s_2, m_{2,0}, m_{0,2}, m_{1,1}) &= h_1(e^{s_1}-1)+h_2 s_1+h_3(e^{s_2}-1)+h_4 s_2+h_5 m_{2,0} \\ &+h_6 m_{0,2}+h_7 m_{1,1}+h_8(m_{2,0}-2s_1)^2+h_9(m_{0,2}-2s_2)^2 \\ &+h_{10}(m_{2,0}-2s_1)(m_{0,2}-2s_2)+h_{11}e^{s_1}\left(2m_{1,1}-m_{0,2}+2s_2\right) \\ &+h_{12}e^{s_2}\left(2m_{1,1}-m_{2,0}+2s_1\right)+h_{13}e^{s_1}(m_{2,0}+2s_1) \\ &+h_{14}e^{s_2}(m_{0,2}+2s_2) \end{aligned} \tag{24}$$

for some constants $h_1, \cdots, h_{14}$ which have one of the following 5 conditions:

i) $h_{12} = h_{13} = h_{14} = 0$, $f^{1,1}(s_1, s_2) = s_2(e^{s_1} - 1)$, and $f^2(s) = 2(e^s - s - 1)$,

ii) $h_{11} = h_{13} = h_{14} = 0$, $f^{1,1}(s_1, s_2) = s_1(e^{s_2} - 1)$, and $f^2(s) = 2(e^s - s - 1)$,

iii) $h_{11} = h_{12} = h_{13} = h_{14} = 0$ and $f^2(s) = 2(e^s - s - 1)$,

iv) $h_8 = h_9 = h_{10} = h_{11} = h_{12} = 0$ and $f^2(s) = 2(se^s - e^s + 1)$,

v) $h_8 = h_9 = h_{10} = h_{11} = h_{12} = h_{13} = h_{14} = 0$.

Proof is in the Appendix B.

Corollary 6. When information set is given by $x = (s_1, s_2, m_{2,0}, m_{0,2}, m_{1,1})$, there is no realized (2,1)-comoment element.

Proof is similar to the proof of Corollary 4.

III. Practical Issues in the Estimation

In this section, we discuss practical issues in the estimation of the realized (joint) cumulants. Since the propositions assume that each security is martingale, we use forward prices for each $S_{i,j}$.[12] Then, without loss of generality, we assume $S_{i,0}$ to be one.[13] Accordingly, $S_{i,T} - S_{i,0}$ implies an arithmetic return between time 0 and *T*.

For the estimation of the realized fourth cumulant, we require the second and the third implied moments. Although we cannot directly observe moments of the price in the market,

[12] Note that prices are derived through the risk neutral measure while they evolve under the real measure. Therefore both of the implied moment and the realized moment in this paper can be understood as proxies. Bias of the estimate due to the different probability measure is the payoff of hedging strategy of each swap in the proof of Proposition 2. Hereafter, expectations in this section are in terms of the risk neutral measure.

[13] In other words, hereafter, $S_{i,j}$ represents $S_{i,j}/S_{i,0}$.

Bakshi, Kapadia and Madan (2003) provide a method to overcome this issue. By adopting their method, we can obtain implied moments of $S_{i,T} - S_{i,j}$ from the following equality:

$$E_j\left[\left(S_{i,T} - S_{i,j}\right)^n\right] = n(n-1)\left(\int_0^{S_{i,j}} (x - S_{i,j})^{n-2} P_j(x)dx + \int_{S_{i,j}}^{\infty} (x - S_{i,j})^{n-2} C_j(x)dx\right), \quad n \geq 2 \quad (25)$$

where $P_j(x)$ is a forward price of a European put option at time *j* with an exercise price *x* and a maturity *T*. Similarly, $C_j(x)$ is a forward price of a European call option defined as $P_j(x)$.

Now let us discuss about estimation of the realized joint cumulants. For the estimation, we require implied comoments. However, unlike the case of implied moments of a security's price, implied comoments between securities' prices are not obtainable with only individual European options. Instead, utilizing some exotic options makes it possible to get the implied comoments. For example, when we have continuum of basket options or spread options, Equation (25) with *n*=2 makes us to get $\mathrm{var}_j(S_{1,T} + S_{2,T})$ or $\mathrm{var}_j(S_{1,T} - S_{2,T})$, respectively. One of these two in addition to variance of each individual security yields covariance as follows:

$$\mathrm{cov}_j(S_{1,T}, S_{2,T}) = \pm \frac{\mathrm{var}_j(S_{1,T} \pm S_{2,T}) - \mathrm{var}_j(S_{1,T}) - \mathrm{var}_j(S_{2,T})}{2} \quad (26)$$

Similarly, if we have continuum of both basket options and spread options in addition to the individual options, we can get implied third moments of both $S_{1,T} + S_{2,T}$ and $S_{1,T} - S_{2,T}$. Therefore, we can obtain implied third comoment due to Equation (27) with $A = S_{1,T} - S_{1,j}$ and $B = S_{2,T} - S_{2,j}$.

$$E_j[A^2B] = \frac{E_j[(A+B)^3] - E_j[(A-B)^3] - 2E_j[B^3]}{6} \quad (27)$$

Practically, basket options (and spread options), composed with an individual security and a market index, are not traded enough. However, as Kempf, Korn and Saßning (2015) point, index options are already basket options because an index is a portfolio of individual securities. Thus, the following two assumptions make it possible to get covariance between returns of an individual stock and an index. One of the assumptions is that asset returns follow an index model with time varying α and β. Then, at each time t_j, conditional distribution between a stock index ($S_{M,T}$) and an each stock price ($S_{i,T}$) are represented as follows:

$$S_{i,T} = \alpha_{i,j} + \beta_{i,j} S_{M,T} + \varepsilon_{i,j}, i \in \{1, \cdots, I\}, 0 \leq t_j < t_N = T. \quad (28)$$

The second assumption is that ratio of systematic risk over total risk is same for all securities at each time t_j with a value ρ_j. The combination of these two assumptions yields that $\beta_{i,j}^2 V_{M,j} = \rho_j V_{i,j}$ or

$$\beta_{i,j} = \sqrt{\rho_j \frac{V_{i,j}}{V_{M,j}}}. \tag{29}$$

where $V_{i,j} = \mathrm{var}_j(S_{i,T})$ and $V_{M,j} = \mathrm{var}_j(S_{M,T})$. Because the beta of the index portfolio is one, $\sum_{i=1}^{I} w_{i,j}\beta_{i,j} = 1$ holds where $w_{i,j}$ is the weight of security i within the index at time t_j. This condition with Equation (29) yields

$$\rho_j = \frac{V_{M,j}}{\left(\sum_{i=1}^{I} w_{i,j}\sqrt{V_{i,j}}\right)^2}. \tag{30}$$

Hence, the covariance between the price and the index are represented as follows:

$$\begin{aligned} C_{i,j} &\equiv \mathrm{cov}_j(S_{i,T}, S_{M,T}) \\ &= \beta_{i,j} V_{M,j} \\ &= \sqrt{\rho_j V_{i,j} V_{i,M}} \\ &= \frac{\sqrt{V_{i,j}}}{\sum_{i=1}^{I} w_{i,j}\sqrt{V_{i,j}}} V_{M,j} \end{aligned} \tag{31}$$

Then, from Equation (14), the realized third comoment between the individual return and the index return is represented as follows:

$$cTM_{M,M,i}^{real} \equiv \sum_{j=1}^{N} \left\{ \left(\Delta S_{M,j}\right)^2 \Delta S_{i,j} + \Delta S_{i,j}\Delta V_{M,j} + 2\Delta S_{M,j}\Delta C_{i,j} \right\}. \tag{32}$$

To standardize this, we adopt coskewness of Kraus and Litzenberger (1976) which is defined as

$$\gamma_i \equiv \frac{cTM_{M,M,i}}{TM_M} \tag{33}$$

where TM_M is the third moment of $S_{M,T}$.[14] Accordingly, we define the realized coskewness as $\gamma_i^{real} \equiv \frac{cTM_{M,M,i}^{real}}{TM_M^{real}}$ with

$$TM_M^{real} \equiv \sum_{j=1}^{N} \left\{ \left(\Delta S_{M,j}\right)^3 + 3\Delta S_{M,j} \Delta V_{M,j} \right\} \tag{34}$$

Additionally, one can show that implied gamma at time zero γ_i^{imp} is same with $\beta_{i,0}$ under our assumption.

IV. Empirical Results

The empirical analysis of this study is two folds. The first part investigates behaviors of the cumulants of S&P 500 returns such as predictability of historical or implied cumulants to the realized cumulants, and the second is about relations between lagged cumulants of individual stock returns and subsequent returns for the components of Dow Jones Industrial Average (DJIA). For the analysis, we use implied volatilities, prices of underlying securities, dividends, and risk-free rate from January 1996 to August 2014. We get those of S&P 500, and the components of DJIA from Option Metrics in Wharton Research Data Services (WRDS). To get continuum of option prices for each strike price, we use the methodology of Carr and Wu (2009) and Neuberger (2012), after options with zero bid price are deleted.

IV.1. Cumulants of the S&P 500 returns

Table 2 shows descriptive statistics of cumulants of the S&P 500 returns. It shows that each realized value is closer to sample values than implied values for monthly and quarterly returns,

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[14] Under the definition, they show the relation $E[R_i]=\beta_i\lambda_1+\gamma_i\lambda_2$, where R_i is a return of asset i, and λ_1 and λ_2 are constants.

although standard deviations of realized values are greater than those of implied values. However, in the case of annual returns, implied values are closer to sample values with small standard deviations. In addition, returns become less negatively skewed and less leptokurtic as time to maturity increases. This pattern may be related to i.i.d. returns because moment of n-period return is n times moment of 1-period return when return of multi-period is additive and each 1-period return is i.i.d.. However, as shown in Table 3, adjusted skewness and adjusted kurtosis are not appears to be constant.[15] It can be from ignored compounding of arithmetic return but it holds even when the compound effect is small; they are different even in the 1-month and 3-month comparison. It implies that sum of nth order returns of sub-periods cannot generate nth order moment of a full period.

[Table 2 about here]

[Table 3 about here]

Since the real probability measure is different from the risk neutral measure, the process of the price is not genuine martingale. Therefore, using the implied second and third moments to get the realized fourth cumulants arises a question of whether the fourth cumulants are reliable. To clarify the validity of lower order implied moments, Table 4 represents time series regression of each realized value on implied and lagged realized values. According to Table 4.A, both implied and lagged realized moments are significant in the univariate regression of the second and the third moments. In addition, implied moments are significant even in the two-variable regression of the second and the third moments while the significances of lagged realized moments vanish. Therefore using the implied second and third moments to yield the realized fourth cumulants is justified in some sense. Now let us deal with standardized moments, which are skewness and kurtosis. Both implied and lagged realized terms are significant in the regression of both skewness and kurtosis. Therefore, lagged realized kurtosis provides some information to future realized kurtosis.

[15] When monthly returns are i.i.d., each moment of n-month return is proportional to n. Accordingly, if the returns are i.i.d., skewness is proportional to $1/\sqrt{n}$ and non-excess kurtosis is proportional to $1/n$. Therefore, we define adjusted skewness and adjusted kurtosis by $\sqrt{n}$ times sample skewness of n-month return and n times non-excess kurtosis of n-month return minus 3, respectively. Then, adjust skewness and adjusted kurtosis should be irrelevant to the n if the returns are i.i.d..

The characteristics of the second and the third moments are also valid in the quarterly and semiannual returns; both the implied and lagged realized moments are significant in the univariate regression, and implied moments are significant in the two variable regression. However, they are insignificant in the annual analysis. In addition, implied skewness and kurtosis are significant as like the monthly case. But lagged realized terms are mostly insignificant.

[Table 4 about here]

IV.2. (Joint) cumulants of returns and subsequent returns

This section investigates relations between cumuanlts of returns and subsequent returns on a month-end by month-end basis. Because expirations of the options are not the end of the month, as a proxy, we use interpolated 30-day volatilities of options from volatility surface of Option Metrics for each day. This analysis, based on the end of the month, provides similar results to the analysis based on the expiration of the options. However, this makes it easy to get risk adjusted returns based on Fama and French (1993).

[Table 5 about here]

Table 5 shows average of regression results about comoments. Panel A is about average of time series regression for each security, and Panel B is about average of cross sectional regression for each time. These show that implied comoments have the greatest determinant coefficient among the univariate regressions in the both of time series and cross sectional analysis. In addition, it shows that using the implied covariance to yield the realized third comoment is reasonable.

[Table 6 about here]

Table 6 compares portfolio's return and moments after it is constructed based on implied or realized moments. Panel A represents return, moments, and comoments after it is constructed based on the rank of implied variance. It shows that portfolios keep their order of variance. In other words, a portfolio with the greatest (smallest) implied variance precedes the greatest (smallest) realized variance and the difference between the realized variances is significant. However, the difference of the returns is insignificant. Panel B – Panel G shows the similar results; portfolios keep their order of moments with significant differences but the differences of returns are insignificant. Despite the insignificance of the differences of returns, sizes of the differences are not economically ignorable. For the robustness, Table 7 provides the performances with controlling risk of the other sources while the portfolios in the Table 6 are constructed without controlling risk of the other sources.

[Table 7 about here]

Panel A of Table 7 shows that a portfolio constructed from small implied volatility takes high return. However, it is vague whether the result is based on the idiosyncratic volatility risk solely because implied beta or implied gamma is equivalent to the implied variance under the assumption of market model. To decompose the effects among idiosyncratic volatility, beta, and gamma, we construct the portfolios based on lagged realized moments. Panel D, E, and F show that all of idiosyncratic risk, beta, and gamma are linked to the returns of portfolios; portfolios with low variance, beta, and gamma are along with high return which is consistent with the Panel A. Likewise, other panels show the link between higher order moments and returns of portfolios. Although Panel B and C show insignificant difference of the zero cost portfolios, the size of the return is not economically ignorable. In addition, Panel G and H show that portfolios with low skewness and high kurtosis are along with high return. Hence the results, which use the realized moments, are generally in line with the literature; more specifically, Frazzini and Pedersen (2014) show that high beta is linked with low return, and Harvey and Siddique (2000) show that low gamma is linked to high return, and Conrad, Dittmar and Ghysels (2013) and Amaya et al. (2015) show that low skewness or high kurtosis is linked to high return.

V. Concluding Remark

Although many theoretical and empirical studies show that the third and the fourth (co)moments are related to returns of securities, estimation of realized higher order moments is not as simple as estimation of the second moments. In this paper, we propose the realized third comoment and the realized fourth (joint) cumulants. Different from previous studies, our realized estimates reflect characteristics of volatility of volatility as well as jump contributions. In addition, we could show that volatility clustering increases the fourth cumulant, and a negative correlation between skewness and returns reduces the fourth cumulant, and contagion effect reduces the third comoment.

Lack of exotic options may limits the accessibility of the implied and realized comoments. However, we present an alternative solution and it is supported by predictability of the implied lower-order comoments about the realized comoments. This solution may be incomplete but we can get the complete measure of higher order moments when derivative market expands in the future.

In addition, we conduct several empirical tests about the realized moments. All the realized moments are explained with implied moments with greater determinant coefficients. It implies both the realized and implied moments are well functioning. Finally, the relations between realized moments of returns and subsequent returns coincide with the literature. Since the realized moments make it possible to understand the distribution of a return of asset for a specific period, we hope these estimators to be applied in the future research.

Appendix A: Proofs of Proposition 1 and 2.

Proof of Proposition 1.

Let us consider a vector valued process $\{(S_{1,t}, S_{2,t}, M_t) : t = 0,1,2\}$ and assume[16]

$$(S_1, S_2, M) : (0,0,m) \to \begin{pmatrix} (s_{1,1}, s_{2,1}, \alpha) & \to & (s_{1,1}+\eta_1, s_{2,1}+\eta_2, \vec{0}) & \Pr = \pi_1 \\ (s_{1,2}, s_{2,2}, \vec{0}) & \to & (s_{1,2}, s_{2,2}, \vec{0}) & \Pr = \pi_2 \\ \vdots & & \vdots & \vdots \\ (s_{1,n}, s_{2,n}, \vec{0}) & \to & (s_{1,n}, s_{2,n}, \vec{0}) & \Pr = \pi_n \end{pmatrix} \tag{A1}$$
$$t: \quad 0 \quad \to \quad 1 \quad \to \quad 2$$

with $\sum_{j=1}^{n} \pi_j = 1$, $\sum_{j=1}^{n} \pi_j s_{i,j} = 0$, $E[\eta_i] = 0$, $E\left[\eta_1^k \eta_2^l\right] = \alpha_{k,l}$, and

$$m_{k,l} = \begin{cases} \pi_1 \alpha_{k,l} + \sum_{j=1}^{n} \pi_j s_{1j}^k s_{2,j}^l, & \text{if } k+l=2 \\ \pi_1(\alpha_{k,l} + l_+ \alpha_{k-1,l} s_{1,1} + k_+ \alpha_{k,l-1} s_{2,1}) + \sum_{j=1}^{n} \pi_j s_{1,j}^k s_{2,j}^l, & \text{if } k+l=3 \end{cases} \tag{A2}$$

where $x_+ = \max(x,0)$, $\alpha = (\alpha_{2,0}, \alpha_{1,1}, \alpha_{0,2}, \alpha_{3,0}, \alpha_{2,1}, \alpha_{1,2}, \alpha_{0,3})$, and $m = (m_{2,0}, \cdots, m_{0,3})$. Then, S_1 and S_2 are martingale, and the process $M = (M_{2,0}, M_{1,1}, M_{0,2}, M_{3,0}, M_{2,1}, M_{1,2}, M_{0,3})$ satisfies Equation (A3).

$$M_{k,l,t} = E_t\left[(S_{1,2} - S_{1,t})^k (S_{2,2} - S_{2,t})^l\right]. \tag{A3}$$

The Aggregation Property implies $g(0,...,0) = 0$ and

$$\begin{aligned} &E\left[\pi_1 g\left(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -m\right)\right] + \sum_{j=2}^{n} \pi_j g\left(s_{1,j}, s_{2,j}, -m\right) \\ &= \pi_1 g\left(s_{1,1}, s_{2,1}, \alpha - m\right) + E\left[\pi_1 g\left(\eta_1, \eta_2, -\alpha\right)\right] + \sum_{j=2}^{n} \pi_j g\left(s_{1,j}, s_{2,j}, -m\right) \end{aligned} \tag{A4}$$

[16] In Appendix A, *s* and *m* represent realizations of *S* and *M*, respectively. They are irrelevant to the ln(*S*) or generalized moments from the log prices.

or

$$E\left[g\left(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -m\right)\right]=g\left(s_{1,1}, s_{2,1}, \alpha-m\right)+E\left[g\left(\eta_1, \eta_2, -\alpha\right)\right]. \tag{A5}$$

Equation (A5) holds for any pair $(s_{1,1}, s_{2,1})$ including (0,0). Therefore, we have:

$$E\left[g\left(\eta_1, \eta_2, -m\right)\right]=g\left(0,0, \alpha-m\right)+E\left[g\left(\eta_1, \eta_2, -\alpha\right)\right]. \tag{A6}$$

Differentiating Equation (A6) with respect to m_{k-2} yields Equation (A7)

$$E\left[g_k\left(\eta_1, \eta_2, -m\right)\right]=g_k\left(0,0, \alpha-m\right) \tag{A7}$$

where g_k is a partial differentiation with respect to the (k-2)th term of the M. i.e. $g_3=\dfrac{\partial g}{\partial M_{2,0}}$, $g_4=\dfrac{\partial g}{\partial M_{1,1}}$, … , $g_9=\dfrac{\partial g}{\partial M_{0,3}}$.

When we construct m to be α,[17] we have

$$E\left[g_k\left(\eta_1, \eta_2, -\alpha\right)\right]=g_k\left(0,0,0,0,0\right). \tag{A8}$$

Therefore, g_k is represented as follows:

$$\begin{aligned} g_k(s_1, s_2, M) = a_{k,0} &+ A_{k,1}(M)s_1 + A_{k,2}(M)s_2 + A_{k,3}(M)(s_1^2 + M_{2,0}) \\ &+ A_{k,4}(M)(s_1 s_2 + M_{1,1}) + A_{k,5}(M)(s_2^2 + M_{0,2}) + A_{k,6}(M)(s_1^3 + M_{3,0}) \\ &+ A_{k,7}(M)(s_1^2 s_2 + M_{2,1}) + A_{k,8}(M)(s_1 s_2^2 + M_{1,2}) + A_{k,9}(M)(s_2^3 + M_{0,3}) \end{aligned} \tag{A9}$$

[17] Regardless the value of α, by making π_1 close to zero and using large n in Equation (A1), we can construct arbitrary m.

where $a_{k,0}$ is a constant and $A_{k,1},\cdots,A_{k,9}$ are functions of *M*. Then, substituting $m_{-(l-2)}=\alpha_{-(l-2)}$ and (A9) into the (A7) yields following:[18]

$$A_{k,l}(-m)(\alpha_{l-2}-m_{l-2})=A_{k,l}(0,...0,\alpha_{l-2}-m_{l-2},0,..0)(\alpha_{l-2}-m_{l-2})\text{ , } l=3,..,9. \tag{A10}$$

Since π, $s_{i,j}$, α are arbitrary, $A_{k,3},\cdots,A_{k,9}$ are constants. Therefore, we use notations $a_{k,3},\cdots,a_{k,9}$ instead of $A_{k,3},\cdots,A_{k,9}$.

$$\begin{aligned} g_k(s_1,s_2,M)=a_{k,0}+A_{k,1}(M)s_1+A_{k,2}(M)s_2+a_{k,3}(s_1^2+M_{2,0}) \\ +a_{k,4}(s_1s_2+M_{1,1})+a_{k,5}(s_2^2+M_{0,2})+a_{k,6}(s_1^3+M_{3,0}) \\ +a_{k,7}(s_1^2s_2+M_{2,1})+a_{k,8}(s_1s_2^2+M_{1,2})+a_{k,9}(s_2^3+M_{0,3}) \end{aligned} \tag{A11}$$

Now let us simplify functions $A_{k,1}$ and $A_{k,2}$. Differentiating (A5) with respect to m_{k-2} yields:

$$E\left[g_k\left(s_{1,1}+\eta_1,s_{2,1}+\eta_2,-m\right)\right]=g_k\left(s_{1,1},s_{2,1},\alpha-m\right). \tag{A12}$$

Substituting (A11) into (A12) yields

$$\begin{aligned} &\left(A_{k,1}(\alpha-m)-A_{k,1}(-m)-3a_{k,6}\alpha_{2,0}-2a_{k,7}\alpha_{1,1}-a_{k,8}\alpha_{0,2}\right)s_{1,1} \\ &=-\left(A_{k,2}(\alpha-m)-A_{k,2}(-m)-a_{k,7}\alpha_{2,0}-2a_{k,8}\alpha_{1,1}-3a_{k,9}\alpha_{0,2}\right)s_{2,1} \end{aligned} \tag{A13}$$

Because Equation (A13) is valid for arbitrary α and s,

$$A_{k,1}(M)=3a_{k,6}M_{2,0}+2a_{k,7}M_{1,1}+a_{k,8}M_{0,2}+a_{k,1} \tag{A14}$$

$$A_{k,2}(M)=a_{k,7}M_{2,0}+2a_{k,8}M_{1,1}+3a_{k,9}M_{0,2}+a_{k,2} \tag{A15}$$

for some constants $a_{k,1}$ and $a_{k,2}$. Therefore we obtain the following

[18] α_{-i} represents α without the i^{th} element. For example, $\alpha_{-1}=(\alpha_{1,1},\alpha_{0,2},\ldots,\alpha_{0,3})$. m_{-i} is defined similarly.

$$
\begin{aligned}
g_k(s_1,s_2,M) = {} & a_{k,0} + a_{k,1}s_1 + a_{k,2}s_2 + a_{k,3}(s_1^2 + M_{2,0}) + a_{k,4}(s_1 s_2 + M_{1,1}) \\
& + a_{k,5}(s_2^2 + M_{0,2}) + a_{k,6}(s_1^3 + M_{3,0} + 3M_{2,0}s_1) \\
& + a_{k,7}(s_1^2 s_2 + M_{2,1} + 2M_{1,1}s_1 + M_{2,0}s_2) \\
& + a_{k,8}(s_1 s_2^2 + M_{1,2} + M_{0,2}s_1 + 2M_{1,1}s_2) \\
& + a_{k,9}(s_2^3 + M_{0,3} + 3M_{0,2}s_2)
\end{aligned}
\tag{A16}
$$

When $k=3$, integrating (A16) with respect to M_{k-2} yields Equation (A17).

$$
\begin{aligned}
g(s_1,s_2,M) = {} & a_{3,0}M_{2,0} + a_{3,1}M_{2,0}s_1 + a_{3,2}M_{2,0}s_2 + a_{3,3}(s_1^2 M_{2,0} + \frac{1}{2}M_{2,0}^2) \\
& + a_{3,4}(s_1 s_2 M_{2,0} + M_{1,1}M_{2,0}) + a_{3,5}(s_2^2 M_{2,0} + M_{0,2}M_{2,0}) \\
& + a_{3,6}(s_1^3 M_{2,0} + M_{3,0}M_{2,0} + \frac{3}{2}M_{2,0}^2 s_1) \\
& + a_{3,7}(s_1^2 s_2 M_{2,0} + M_{2,1}M_{2,0} + 2M_{1,1}M_{2,0}s_1 + \frac{1}{2}M_{2,0}^2 s_2) \\
& + a_{3,8}(s_1 s_2^2 M_{2,0} + M_{1,2}M_{2,0} + M_{2,0}M_{0,2}s_1 + 2M_{2,0}M_{1,1}s_2) \\
& + a_{3,9}(M_{2,0}s_2^3 + M_{2,0}M_{0,3} + 3M_{2,0}M_{0,2}s_2) \\
& + A_{3,10}(M_{-1})
\end{aligned}
\tag{A17}
$$

Similarly, we can get alternative forms of $g(s_1,s_2,M)$ adopting $k=4...9$. When we combine these forms, $g(s_1,s_2,M)$ can be represented as

$$
g(s_1,s_2,M) = g_M(M;s_1,s_2) + g_s(s_1,s_2)
\tag{A18}
$$

where g_M is a multivariate polynomial whose coefficients are (multivariate polynomial) functions of s_1 and s_2 with a condition of $g_M(\vec{0};s_1,s_2)=0$ and g_s is a function of s_1 and s_2 with $g_s(0,0)=0$ because $g(0,0,\vec{0})=0$.

Substituting (A18) and $\vec{m}=\vec{\alpha}$ into Equation (A5) and multiplying $2/k^2$ yields:

$$
\begin{aligned}
& \frac{2}{k^2}\left(E\left[g_s\left(s_{1,1}+\eta_1, s_{2,1}+\eta_2\right)\right] - g_s(s_{1,1},s_{2,1}) - E\left[g_s\left(\eta_1,\eta_2\right)\right]\right) \\
& = \frac{2}{k^2}\left(E\left[g_M\left(-\vec{\alpha}; s_{1,1}+\eta_1, s_{2,1}+\eta_2\right)\right] - g_M(\vec{0};s_{1,1},s_{2,1}) - E\left[g_M\left(-\vec{\alpha};\eta_1,\eta_2\right)\right]\right)
\end{aligned}
\tag{A19}
$$

When we substitute $(\eta_1,\eta_2)=\begin{pmatrix} (k,0) & \Pr=1/2 \\ (-k,0) & \Pr=1/2 \end{pmatrix}$ into the (A19), the left hand side of (A19)

converges to $\frac{\partial^2 g_s(s_1,s_2)}{\partial s_1^2}-\frac{\partial^2 g_s(0,0)}{\partial s_1^2}$ as $k\to 0$ because $g_s(0,0)=0$. Therefore

$$\frac{\partial^2 g_s(s_1,s_2)}{\partial s_1^2}=\frac{\partial^2 g_s(0,0)}{\partial s_1^2}+g_{p1}(s_1,s_2) \tag{A20}$$

with some polynomial g_{p1} because g_M is a multivariate polynomial. Similarly adopting $(\eta_1,\eta_2)=\begin{pmatrix} (0,k) & \Pr=1/2 \\ (0,-k) & \Pr=1/2 \end{pmatrix}$ yields

$$\frac{\partial^2 g_s(s_1,s_2)}{\partial s_2^2}=\frac{\partial^2 g_s(0,0)}{\partial s_2^2}+g_{p2}(s_1,s_2) \tag{A21}$$

with some polynomial g_{p2}.

Now consider an alternative form of (A19)

$$\begin{aligned}&\frac{1}{2k_1k_2}\left(E\left[g_s\left(s_{1,1}+\eta_1,s_{2,1}+\eta_2\right)\right]-g_s(s_{1,1},s_{2,1})-E\left[g_s\left(\eta_1,\eta_2\right)\right]\right)\\&=\frac{1}{2k_1k_2}\left(E\left[g_M\left(-\vec{\alpha};s_{1,1}+\eta_1,s_{2,1}+\eta_2\right)\right]-g_M(\vec{0};s_{1,1},s_{2,1})-E\left[g_M\left(-\vec{\alpha};\eta_1,\eta_2\right)\right]\right)\end{aligned} \tag{A22}$$

Next, when we substitute $(\eta_1,\eta_2)=\begin{pmatrix} (k_1,k_2) & \Pr=1/2 \\ (-k_1,-k_2) & \Pr=1/2 \end{pmatrix}$ and $(\eta_1,\eta_2)=\begin{pmatrix} (k_1,-k_2) & \Pr=1/2 \\ (-k_1,k_2) & \Pr=1/2 \end{pmatrix}$

into Equation (A22) and subtract each other we get

$$\frac{\partial^2 g_s(s_1,s_2)}{\partial s_1\partial s_2}=\frac{\partial^2 g_s(0,0)}{\partial s_1\partial s_2}+g_{p3}(s_1,s_2) \tag{A23}$$

as $(k_1,k_2)\to(0,0)$ with some polynomial g_{p3}. (A20), (A21), and (A23) implies that $g(s_1,s_2,M)$ is a polynomial of s_1, s_2, $M_{2,0}$, …, and $M_{0,3}$.

Now let us substitute (l_1S_1, l_2S_2) into (S_1, S_2) for the function *g*. Since *g* satisfies the Aggregation Property for any l_1 and l_2, each coefficient of $l_1^{k_1} l_2^{k_2}$ also satisfies the Aggregation Property. Hence, for the coefficients of $l_1^{k_1} l_2^{k_2}$, we can construct a spanning set of functions that has the Aggregation Property and it is represented in Table A1 for $k_1 \ge k_2$.

[Table A1 about here]

Note that, in the case of $(k_1, k_2) = (4,0)$, $M_{2,0}^2$ and $M_{2,0}s_1^2$ come together as the form $\left(\frac{1}{2}M_{2,0}^2 + M_{2,0}s_1^2\right)$ rather than represented separately. It is due to the form of Equation (A17). Some of the other combined terms are from alternatives of (A17) that are omitted in this paper.

According to Neuberger, s_1, s_1^2, $M_{2,0}$ and $s_1^3 + 3s_1M_{2,0}$ satisfies the Aggregation Property. In addition, every $M_{i,j}$ also satisfies the Aggregation Property by definition of $M_{i,j}$. Now let us consider a case of $(k_1, k_2) = (2,1)$. Substituting

$$g(s_1, s_2, M) = b_1 s_1^2 s_2 + b_2 s_1 M_{1,1} + b_3 s_2 M_{2,0} + b_4 M_{2,1} \tag{A24}$$

into Equation (A5) yields

$$b_1(2s_{1,1}\alpha_{1,1} + s_{2,1}\alpha_{2,0}) = b_2 s_{1,1}\alpha_{1,1} + b_3 s_{2,1}\alpha_{2,0} \tag{A25}$$

Therefore $b_2 = 2b_3 = 2b_1$ and $b_3 = b_1$ because α and *s* are arbitrary numbers. It implies that expression (A26) is a candidate for a function with the Aggregation Property.

$$b_1(s_1^2 s_2 + 2s_1 M_{1,1} + s_2 M_{2,0}) + b_4 M_{2,1} \tag{A26}$$

Similarly we can try for the other pairs of (k_1, k_2) and the result is arranged in the Table A2. Without loss of generality, we can let $(s,t,u,T) = (0,1,2,3)$ in Equation (2) and

$$(S_1(\tau), S_2(\tau)) = \left(\sum_{j=1}^{\tau} R_{1,j}, \sum_{j=1}^{\tau} R_{2,j} \right) \tag{A27}$$

for $\tau = 0,1,2,3$ and $R_{i,j}$ such that $E_{j-1}[R_{i,j}] = 0$. Then each element of the Table A2 satisfies Equation (3).

■

Proof of Proposition 2.

Let us consider the securities that pay $S_{1,T}$, $S_{2,T}$, $S_{1,T}^2$, $S_{1,T}S_{2,T}$, $S_{1,T}^3$, $S_{1,T}^2 S_{2,T}$, or $S_{1,T}S_{2,T}^2$ at T. Then the price of each security at time j is $S_{1,j}$, $S_{2,j}$, $S_{1,j}^2 + M_{2,0,j}$, $S_{1,j}S_{2,j} + M_{1,1,j}$, $S_{1,j}^3 + 3S_{1,j}M_{2,0,j} + M_{3,0,j}$, $S_{1,j}^2 S_{2,j} + 2S_{1,j}M_{1,1,j} + S_{2,j}M_{2,0,j} + M_{2,1,j}$, or $S_{1,j}S_{2,j}^2 + 2S_{2,j}M_{1,1,j} + S_{1,j}M_{0,2,j} + M_{1,2,j}$, respectively.

Equipped with $M_{k,l,N} = 0$ for each k and l, we have the following equality:

$$\begin{aligned}
(S_{1,T} - S_{1,0})^2 (S_{2,T} - S_{2,0}) - \sum_{j=1}^{N} \left((\Delta S_{1,j})^2 \Delta S_{2,j} + \Delta S_{2,j} \Delta M_{2,0,j} + 2\Delta S_{1,j} \Delta M_{1,1,j} \right) \\
= -2\sum_{j=0}^{N-1} (S_{1,j}S_{2,j} - S_{1,0}S_{2,0} - M_{1,1,j}) \Delta S_{1,j+1} \\
- \sum_{j=0}^{N-1} (S_{1,j}^2 - S_{1,0}^2 - M_{2,0,j}) \Delta S_{2,j+1} \\
+ \sum_{j=0}^{N-1} (S_{2,j} - S_{2,0}) \Delta (S_{1,j+1}^2 + M_{2,0,j+1}) \\
+ 2\sum_{j=0}^{N-1} (S_{1,j} - S_{1,0}) \Delta (S_{1,j+1}S_{2,j+1} + M_{1,1,j+1})
\end{aligned} \tag{A28}$$

Left hand side of (A28) describes the difference between receiving leg and paying leg of the third comoment swap described in Table 1. In addition, right hand side of Equation (A28) describes strategy of a self-financing portfolio which is managed with securities that pay $S_{1,T}$, $S_{2,T}$, $S_{1,T}^2$, or $S_{1,T}S_{2,T}$ at T. Therefore (A28) shows the fairness and replicability of the third comoment swap.

To deal with the properties about the fourth moment swap and non-zero fourth moment swap, we present the following equality:

$$
\begin{aligned}
&(S_{1,T}-S_{1,0})^4-3a(S_{1,T}-S_{1,0})^2M_{2,0,0}\\
&\qquad -\sum_{j=1}^{N}\left\{(\Delta S_{1,j})^4+6(\Delta S_{1,j})^2\Delta M_{2,0,j}+4\Delta S_{1,j}\Delta M_{3,0,j}+3(\Delta M_{2,0,j})^2\right\}\\
&\qquad =4\sum_{j=0}^{N-1}(M_{3,0,j}-3S_{1,j}M_{2,0,j}+\frac{3}{2}aS_{1,0}M_{2,0,0}+S_{1,j}^3-S_{1,0}^3)\Delta S_{1,j+1}\\
&\qquad +6\sum_{j=0}^{N-1}(M_{2,0,j}-\frac{1}{2}aM_{2,0,0}-S_{1,j}^2+S_{1,0}^2)\Delta(S_{1,j+1}^2+M_{2,0,j+1})\\
&\qquad +4\sum_{j=0}^{N-1}(S_{1,j}-S_{1,0})\Delta(S_{1,j+1}^3+3S_{1,j+1}M_{2,0,j+1}+M_{3,0,j+1})\\
&\qquad +3(1-a)M_{2,0,0}^2
\end{aligned}
\tag{A29}
$$

Left hand side of Equation (A29) with a=0 describes the difference between receiving leg and paying leg of the non-zero fourth moment swap. Since the right hand side of Equation (A29) describes strategy of a self-financing portfolio with initial cost $3M_{2,0,0}^2$ when a=0, we see the fairness and replicability of the non-zero fourth moment swap. Similarly, we can see the properties about the fourth moment swap, non-zero asymmetric fourth comoment swap, asymmetric fourth comoment swap, non-zero symmetric fourth comoment swap, and symmetric fourth comoment swap from Equation (A29) with a=1, Equation (A30) with (a_1,a_2)=(0,0), Equation (A30) with (a_1,a_2)=(a,1-a), Equation (A31) with (a_3,a_4,a_5)=(0,0,0), and Equation (A31) with (a_3,a_4,a_5)=(a,1-a,1), respectively.

$$
\begin{aligned}
&(S_{1,T}-S_{1,0})^3(S_{2,T}-S_{2,0})-3a_1(S_{1,T}-S_{1,0})(S_{2,T}-S_{2,0})M_{2,0,0}-3a_2(S_{1,T}-S_{1,0})^2M_{1,1,0}\\
&\quad-\sum_{j=1}^{N}\left\{(\Delta S_{1,j})^3\Delta S_{2,j}+\Delta S_{2,j}\Delta M_{3,0,j}+3(\Delta S_{1,j})^2\Delta M_{1,1,j}\right\}\\
&\quad-\sum_{j=1}^{N}\left\{3\Delta S_{1,j}\Delta S_{2,j}\Delta M_{2,0,j}+3\Delta M_{2,1,j}\Delta S_{1,j}+3\Delta M_{2,0,j}\Delta M_{1,1,j}\right\}\\
&\quad=3\sum_{j=0}^{N-1}\begin{pmatrix}M_{2,1,j}-2S_{1,j}M_{1,1,j}-S_{2,j}M_{2,0,j}+2a_2S_{1,0}M_{1,1,0}\\+a_1S_{2,0}M_{2,0,0}+S_{1,j}^2S_{2,j}-S_{1,0}^2S_{2,0}\end{pmatrix}\Delta S_{1,j+1}\\
&\quad+\sum_{j=0}^{N-1}(M_{3,0,j}-3S_{1,j}M_{2,0,j}+3a_1S_{1,0}M_{2,0,0}+S_{1,j}^3-S_{1,0}^3)\Delta S_{2,j+1}\\
&\quad+3\sum_{j=0}^{N-1}(M_{1,1,j}-a_2M_{1,1,0}-S_{1,j}S_{2,j}+S_{1,0}S_{2,0})\Delta(S_{1,j+1}^2+M_{2,0,j+1})\\
&\quad+3\sum_{j=0}^{N-1}(M_{2,0,j}-a_1M_{2,0,0}-S_{1,j}^2+S_{1,0}^2)\Delta(S_{1,j+1}S_{2,j+1}+M_{1,1,j+1})\\
&\quad+\sum_{j=0}^{N-1}(S_{2,j}-S_{2,0})\Delta(S_{1,j+1}^3+3S_{1,j+1}M_{2,0,j+1}+M_{3,0,j+1})\\
&\quad+3\sum_{j=0}^{N-1}(S_{1,j}-S_{1,0})\Delta\begin{pmatrix}S_{1,j+1}^2S_{2,j+1}+2S_{1,j+1}M_{1,1,j+1}\\+S_{2,j+1}M_{2,0,j+1}+M_{2,1,j+1}\end{pmatrix}\\
&\quad+3(1-a_1-a_2)M_{2,0,0}M_{1,1,0}
\end{aligned}
\tag{A30}
$$

$$
\begin{aligned}
&(S_{1,T}-S_{1,0})^2(S_{2,T}-S_{2,0})^2 - a_3(S_{1,T}-S_{1,0})^2 M_{0,2,0} \\
&\qquad - a_4(S_{2,T}-S_{2,0})^2 M_{2,0,0} - 2a_5(S_{1,T}-S_{1,0})(S_{2,T}-S_{2,0})M_{1,1,0} \\
&\qquad - \sum_{j=1}^{N}\left\{((\Delta S_{1,j})^2+\Delta M_{2,0,j})((\Delta S_{2,j})^2+\Delta M_{0,2,j})+2(\Delta M_{1,1,j})^2\right\} \\
&\qquad - \sum_{j=1}^{N}\left\{4\Delta M_{1,1,j}\Delta S_{1,j}\Delta S_{2,j}+2\Delta M_{1,2,j}\Delta S_{1,j}+2\Delta M_{2,1,j}\Delta S_{2,j}\right\} \\
&\qquad\ = 2\sum_{j=0}^{N-1}\begin{pmatrix} M_{1,2,j}-2S_{2,j}M_{1,1,j}-S_{1,j}M_{0,2,j}+a_5S_{2,0}M_{1,1,0} \\ +a_3S_{1,0}M_{0,2,0}+S_{1,j}S_{2,j}^2-S_{1,0}S_{2,0}^2 \end{pmatrix}\Delta S_{1,j+1} \\
&\qquad\ + 2\sum_{j=0}^{N-1}\begin{pmatrix} M_{2,1,j}-2S_{1,j}M_{1,1,j}-S_{2,j}M_{2,0,j}+a_5S_{1,0}M_{1,1,0} \\ +a_4S_{2,0}M_{2,0,0}+S_{1,j}^2S_{2,j}-S_{1,0}^2S_{2,0} \end{pmatrix}\Delta S_{2,j+1} \\
&\qquad\ + \sum_{j=0}^{N-1}(M_{0,2,j}-a_3M_{0,2,0}-S_{2,j}^2+S_{2,0}^2)\Delta(S_{1,j+1}^2+M_{2,0,j+1}) \\
&\qquad\ + \sum_{j=0}^{N-1}(M_{2,0,j}-a_4M_{2,0,0}-S_{1,j}^2+S_{1,0}^2)\Delta(S_{2,j+1}^2+M_{0,2,j+1}) \\
&\qquad\ + 4\sum_{j=0}^{N-1}\begin{pmatrix} M_{1,1,j}-\dfrac{1}{2}a_5M_{1,1,0} \\ -S_{1,j}S_{2,j}+S_{1,0}S_{2,0} \end{pmatrix}\Delta(S_{1,j+1}S_{2,j+1}+M_{1,1,j+1}) \\
&\qquad\ + 2\sum_{j=0}^{N-1}(S_{2,j}-S_{2,0})\Delta\begin{pmatrix} S_{1,j+1}^2S_{2,j+1}+2S_{1,j+1}M_{1,1,j+1} \\ +S_{2,j+1}M_{2,0,j+1}+M_{2,1,j+1} \end{pmatrix} \\
&\qquad\ + 2\sum_{j=0}^{N-1}(S_{1,j}-S_{1,0})\Delta\begin{pmatrix} S_{2,j+1}^2S_{1,j+1}+2S_{2,j+1}M_{1,1,j+1} \\ +S_{1,j+1}M_{0,2,j+1}+M_{1,2,j+1} \end{pmatrix} \\
&\qquad\ + (1-a_3-a_4)M_{2,0,0}M_{0,2,0}+2(1-a_5)M_{1,1,0}^2
\end{aligned}
\tag{A31}
$$

■

Appendix B: Proofs of Propositions 3 and 5 and Corollary 4.

In Appendix B, we deal with log price *s* and generalized moment *m* instead of price *S* and moment *M*. However, we keep the notations in Appendix A although required conditions are changed. The changed conditions are represented between Equations (B1) and (B2)

Common property B.

Consider a vector valued process $\{(S_{1,t}, S_{2,t}, \vec{M}_t) : t = 0,1,2\}$. The moment vector process $\vec{M}$ is defined as $(M_{2,0}, M_{1,1}, M_{0,2}, M_{3,0}, M_{2,1}, M_{1,2}, M_{0,3})$. In addition assume

$$(S_1, S_2, \overrightarrow{M}) : (0,0,\vec{m}) \rightarrow \begin{cases} (s_{1,1}, s_{2,1}, \vec{\alpha}) & \rightarrow & (s_{1,1}+\eta_1, s_{2,1}+\eta_2, \vec{0}) & \Pr = \pi_1 \\ (s_{1,2}, s_{2,2}, \vec{0}) & \rightarrow & (s_{1,2}, s_{2,2}, \vec{0}) & \Pr = \pi_2 \\ \vdots & & \vdots & \\ (s_{1,n}, s_{2,n}, \vec{0}) & \rightarrow & (s_{1,n}, s_{2,n}, \vec{0}) & \Pr = \pi_n \end{cases} \tag{B1}$$

with $\sum_{j=1}^{n} \pi_j = 1$, $\sum_{j=1}^{n} \pi_j \exp(s_{i,j}) = 0$, $E[\exp(\eta_i)] = 1$, $E\left[f^{k,l}(\eta_1, \eta_2)\right] = \alpha_{k,l}$,

$\vec{\alpha} = (\alpha_{2,0}, \cdots, \alpha_{0,3})$, and $\vec{m} = \vec{M}(0)$

where

$$m_{k,l} = \pi_1 E[f^{k,l}(s_{1,1}+\eta_1, s_{2,1}+\eta_2)] + \sum_{j=2}^{n} \pi_j f^{k,l}(s_{1,j}, s_{2,j}) \tag{B2}$$

and $f^{k,l}$ is a generalized moment function such that $f^{k,l}(0,0) = 0$ and $\lim_{(a,b)\to(0,0)} \frac{f^{k,l}(a,b)}{a^k b^l} = 1$, $f^{l,k}(a,b) = f^{k,l}(b,a)$, and $f^{k}(a) = f^{k,0}(a,b)$.

Again, $g(0,\ldots,0)=0$ and Equations (A5) - (A12) hold here. Let us represent some of them:

$$E\lfloor g(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -\vec{m})\rfloor = g(s_{1,1}, s_{2,1}, \vec{\alpha}-\vec{m}) + E[g(\eta_1,\eta_2,-\vec{\alpha})] \quad \text{(B3)}$$

$$E[g_k(\eta_1,\eta_2,-\vec{m})] = g_k(0,0,\vec{\alpha}-\vec{m}) \quad \text{(B4)}$$

$$\begin{aligned} g_k(s_1,s_2,M) &= a_{k,0} + A_{k,1}(M)(e^{s_1}-1) + A_{k,2}(M)(e^{s_2}-1) + a_{k,3}(f^2(s_1)+M_{2,0}) \\ &+ a_{k,4}(f^{1,1}(s_1,s_2)+M_{1,1}) + a_{k,5}(f^2(s_2)+M_{0,2}) + a_{k,6}(f^3(s_1)+M_{3,0}) \\ &+ a_{k,7}(f^{2,1}(s_1,s_2)+M_{2,1}) + a_{k,8}(f^{2,1}(s_2,s_1)+M_{1,2}) + a_{k,9}(f^3(s_2)+M_{0,3}) \end{aligned} \quad \text{(B5)}$$

$$E\lfloor g_k(s_{1,1}+\eta_1, s_{2,1}+\eta_2, -\vec{m})\rfloor = g_k(s_{1,1}, s_{2,1}, \vec{\alpha}-\vec{m}) \quad \text{(B6)}$$

Substituting (B5) into (B6) and differentiating with respect to m_l yields

$$\frac{\partial A_{k,1}(-\vec{m})}{\partial m_l} = \frac{\partial A_{k,1}(\vec{a}-\vec{m})}{\partial m_l}, \quad \frac{\partial A_{k,2}(-\vec{m})}{\partial m_l} = \frac{\partial A_{k,2}(\vec{a}-\vec{m})}{\partial m_l}, \quad l=3,..9 \quad \text{(B7)}$$

Therefore each $A_{k,*}(M)$ is an affine function. Accordingly (B5) is represented as follows:

$$\begin{aligned} g_k(s_1,s_2,M) &= a_{k,0} + (b_{k,0} + b_{k,1}M_{2,0} + b_{k,2}M_{1,1} + \ldots + b_{k,7}M_{0,3})(e^{s_1}-1) \\ &+ (c_{k,0} + c_{k,1}M_{2,0} + c_{k,2}M_{1,1} + \ldots + c_{k,7}M_{0,3})(e^{s_2}-1) \\ &+ a_{k,3}(f^2(s_1)+M_{2,0}) + a_{k,4}(f^{1,1}(s_1,s_2)+M_{1,1}) + a_{k,5}(f^2(s_2)+M_{0,2}) \\ &+ a_{k,6}(f^3(s_1)+M_{3,0}) + a_{k,7}(f^{2,1}(s_1,s_2)+M_{2,1}) \\ &+ a_{k,8}(f^{2,1}(s_2,s_1)+M_{1,2}) + a_{k,9}(f^3(s_2)+M_{0,3}) \end{aligned} \quad \text{(B8)}$$

Proof of Proposition 3.

We use the Common property B with omitting all terms related the second security. Accordingly, we ignore the S_2 and restrict M to be (M_2, M_3) with $M_2 = M_{2,0}$ and $M_3 = M_{3,0}$. Then integrating (B8) with respect to M_2 yields

$$
\begin{aligned}
g(s_1, M_2, M_3) = a_{1,0}M_2 + (b_{1,0}M_2 + b_{1,1}M_2^2/2 + b_{1,3}M_3M_2)(e^{s_1} - 1) \\
+ a_{1,3}(M_2 f^2(s_1) + M_2^2/2) + a_{1,6}M_2(f^3(s_1) + M_3) \\
+ g^1(s_1, M_3)
\end{aligned}
\tag{B9}
$$

Similarly, we can get alternative form of (B9) by integrating (B8) with respect to M_3. By combining (B9) and the alternative form, we obtain the following form

$$
\begin{aligned}
g(s, M_2, M_3) = a_1M_2 + a_2M_3 + (a_3M_2 + a_4M_3 + a_5M_2^2 + a_6M_2M_3 + a_7M_3^2)(e^s - 1) \\
+ a_8(M_2^2 + 2M_2 f^2(s)) + a_9(M_3^2 + 2M_3 f^3(s)) \\
+ 2a_{10}(M_2M_3 + f^2(s)M_3 + f^3(s)M_2) + g^s(s)
\end{aligned}
\tag{B10}
$$

for some constants $a_1, \cdots, a_{10}$ and a function g^s such that $g^s(0) = 0$. Substituting (B10) and $\eta_p = \begin{cases} \eta, & \Pr = p \\ 0, & \Pr = 1 - p \end{cases}$ into (B3) yields

$$
\begin{aligned}
0 = p\left(
\begin{array}{l}
(e^{s_1} - 1)\left(a_3\alpha_2 + a_4\alpha_3 - 2a_5m_2\alpha_2 + a_6(-m_2\alpha_3 - m_3\alpha_2) - 2a_7m_3\alpha_3\right) \\
+ 2a_8\left((f^2(s_1 + \eta) - f^2(s_1) - \alpha_2)m_2 + \alpha_2 f^2(s_1)\right) \\
+ 2a_9\left((f^3(s_1 + \eta) - f^3(s_1) - \alpha_3)m_3 + \alpha_3 f^3(s_1)\right) \\
+ 2a_{10}\left(\begin{array}{l} (f^2(s_1 + \eta) - f^2(s_1) - \alpha_2)m_3 + \alpha_3 f^2(s_1) \\ + (f^3(s_1 + \eta) - f^3(s_1) - \alpha_3)m_2 + \alpha_2 f^3(s_1) \end{array}\right) \\
- E[g^s(s_1 + \eta)] + g^s(s_1) + E[g^s(\eta)]
\end{array}
\right) \\
+ p^2(e^{s_1} - 1)\left(a_5\alpha_2^2 + a_6\alpha_2\alpha_3 + a_7\alpha_3^2\right)
\end{aligned}
\tag{B11}
$$

Because we can set p arbitrary, coefficients of p and p^2 are zero. Therefore,

$$
a_5\alpha_2^2 + a_6\alpha_2\alpha_3 + a_7\alpha_3^2 = 0\,. \tag{B12}
$$

It implies that $a_5 = a_6 = a_7 = 0$ because η is arbitrary function with $E\left[e^{\eta}\right] = 1$.

Accordingly we have the following relation:

$$
\begin{aligned}
0 = {} & (e^{s_1}-1)\left(a_3\alpha_2 + a_4\alpha_3\right) \\
& + 2a_8\left((f^2(s_1+\eta) - f^2(s_1) - \alpha_2)m_2 + \alpha_2 f^2(s_1)\right) \\
& + 2a_9\left((f^3(s_1+\eta) - f^3(s_1) - \alpha_3)m_3 + \alpha_3 f^3(s_1)\right) \\
& + 2a_{10}\begin{pmatrix} (f^2(s_1+\eta) - f^2(s_1) - \alpha_2)m_3 + \alpha_3 f^2(s_1) \\ +(f^3(s_1+\eta) - f^3(s_1) - \alpha_3)m_2 + \alpha_2 f^3(s_1) \end{pmatrix} \\
& - E[g^s(s_1+\eta)] + g^s(s_1) + E[g^s(\eta)]
\end{aligned}
\tag{B13}
$$

In addition, coefficients of m_2 and m_3 are zero because we can set them arbitrary. Thus we have:

$$
0 = a_8(f^2(s_1+\eta) - f^2(s_1) - \alpha_2) + a_{10}(f^3(s_1+\eta) - f^3(s_1) - \alpha_3) \tag{B14}
$$

$$
0 = a_{10}(f^2(s_1+\eta) - f^2(s_1) - \alpha_2) + a_9(f^3(s_1+\eta) - f^3(s_1) - \alpha_3) \tag{B15}
$$

We have three cases that satisfy both (B14) and (B15).

Condition B.1

i) $a_8 = a_9 = a_{10} = 0$,

ii) $\exists f^3$ such that $\forall (s_1, \eta)$, $f^3(s_1+\eta) - f^3(s_1) - \alpha_3 = 0$ and $a_8 = a_{10} = 0$

iii) $\exists (a, f^2, f^3)$ s.t. $\forall (s_1, \eta)$, $(f^2(s_1+\eta) - f^2(s_1) - \alpha_2) + a(f^3(s_1+\eta) - f^3(s_1) - \alpha_3) = 0$

with $a_{10} = a_8 a$ and $a_9 = a^2 a_8$.

First, to check about the condition B1.ii), substituting $\eta = \begin{cases} \log(1+k), & \Pr = 0.5 \\ \log(1-k), & \Pr = 0.5 \end{cases}$ into $\frac{2}{k^2}\left(f^3(s_1+\eta) - f^3(s_1) - \alpha_3\right) = 0$ and taking the limit for $k \to 0$ yields:

$$
f^{3}{}''(s_1) - f^{3}{}'(s_1) = 0 \tag{B16}
$$

Then $f^3(s_1) = b_1 s_1 + b_2 e^{s_1}$ is a solution of (B16) for some constants b_1 and b_2. However, then f^3 cannot be a generalized (3,0)-comoment function. It implies that the condition B1.ii) is impossible. Now let us check the condition B1.iii).

Let $f^a(x) = f^2(x) + af^3(x)$ and $\eta = \begin{cases} \log(1+k), & \Pr = 0.5 \\ \log(1-k), & \Pr = 0.5 \end{cases}$. Then like the method above, we can show that

$$f^{a}{''}(s_1) - f^{a}{'}(s_1) - 2 = 0 \tag{B17}$$

Since f^2 and f^3 are a generalized (2,0)-comoment function and a generalized (3,0)-comoment function, respectively, we have the following solution

$$f^2(s) = 2(e^s - s - 1) - af^3(s) \tag{B18}$$

for some generalized (3,0)-comoment function f^3. Therefore (B13) is arranged as follows:

$$\begin{aligned} E[g^s(s_1+\eta)] - g^s(s_1) - E[g^s(\eta)] = {} & (e^{s_1} - 1)(a_3\alpha_2 + a_4\alpha_3) \\ & + 2a_8(\alpha_2 + a\alpha_3)(f^2(s_1) + af^3(s_1)) \end{aligned} \tag{B19}$$

Again, by letting $\eta = \begin{cases} \log(1+k), & \Pr = 0.5 \\ \log(1-k), & \Pr = 0.5 \end{cases}$ and taking limit, g^s is represented as follows:

$$g^s(s) = a_9 s + a_{10}(e^s - 1) + 4a_8 s^2 + (8a_8 + 2a_3)se^s \tag{B20}$$

$$\begin{aligned} g(s, M_2, M_3) = {} & a_1 M_2 + a_2 M_3 + (a_3 M_2 + a_4 M_3)(e^s - 1) \\ & + a_8((M_2 + aM_3)^2 + 2(M_2 + aM_3)(f^2(s) + af^3(s))) \\ & + a_9 s + a_{10}(e^s - 1) + 4a_8 s^2 + (8a_8 + 2a_3)se^s \end{aligned} \tag{B21}$$

Then, (B18) implies that

$$\begin{aligned} g(s, M_2, M_3) = {} & a_1 M_2 + a_2 M_3 + (a_3 M_2 + a_4 M_3)(e^s - 1) \\ & + a_8(M_2 + aM_3 - 2s)^2 + 4a_8(M_2 + aM_3)(e^s - 1) \\ & + a_9 s + a_{10}(e^s - 1) + (8a_8 + 2a_3)se^s \end{aligned} \tag{B22}$$

or

$$g(s,M_2,M_3)=d_1M_2+d_2M_3+d_3M_3e^s+d_4(M_2+aM_3-2s)^2 \\ +d_5(M_2+aM_3+2s)e^s+d_6s+d_7(e^s-1) \tag{B23}$$

where $d_1=a_1-a_3$, $d_2=a_2-a_4$, $d_3=a_4-aa_3$, $d_4=a_8$, $d_5=a_3+4a_8$, $d_6=d_9$, and $d_7=d_{10}$.[19] Then substituting it into (B3) yields

$$d_4(4s_1+2(m_2+am_3-\alpha_2-a\alpha_3))(E[2\eta]+\alpha_2+a\alpha_3) \\ +(e^{s_1}-1)\big(d_5(E[2\eta e^\eta]-\alpha_2-a\alpha_3)-d_3\alpha_3\big)=0 \tag{B24}$$

Since s_1 is arbitrary, we have the following cases

Condition B.2

i) $d_3=d_4=d_5=0$

ii) $d_3=d_5=0$ & $E[2\eta]+\alpha_2+a\alpha_3=0$

iii) $d_4=0$ & $E[2\eta e^\eta]-\alpha_2-h\alpha_3=0$ with $h=a+d_3/d_5$

When the Condition B.2.ii) holds, $f^2(s)+af^3(s)=2(e^s-s-1)$. And because iii) and ii) are exclusive, Condition B.2.iii) is equivalent to $d_3=d_4=0$ with

$$E[2\eta e^\eta]-\alpha_2-a\alpha_3=0\,. \tag{B25}$$

(B25) is equivalent to

[19] Note that form (B23) include the condition B.1.i).

$$f^2(s) + af^3(s) = 2(se^s - e^s + 1) \tag{B26}$$

Arranging all above yields the equation and the condition of Proposition 3. Without loss of generality, we can let $(s,t,u,T) = (0,1,2,3)$ in Equation (2) and

$$S(\tau) = \exp\left(\sum_{j=1}^{\tau} r_j\right) \tag{B27}$$

for $\tau = 0,1,2,3$ and r_j such that $E_{j-1}\lfloor \exp(r_j) \rfloor = 1$. Then g above satisfies Equation (3).

■

Proof of Corollary 4.

If a function is a realized (4,0)-comoment element, it should be decomposed as

$$g(s_1, m_{2,0}, m_{3,0}) = (e^{s_1} - 1)\eta(s_1, m_{2,0}, m_{3,0}) + g^r(s_1)$$

such that $g^r(s_1) = O(s_1^4)$ because of the restriction, $E\lfloor e^{\Delta s_1} \rfloor = 1$. Now let us investigate the each condition in Proposition 3. At the first condition, if h_3 or h_4 are not zero, they cannot be eliminated. Therefore, $h_3 = h_4 = 0$. However, $h_1(e^{s_1} - 1) + h_2 s_1$ is at most $O(s_1^2)$ as $s_1 \to 0$.

At the second condition, if h_5 is zero, it is a case of the first condition. Therefore, it suffices to show the case of nonzero h_5. However, if h_5 is not zero, $m_{2,0}^2$ is not eliminated with zero expectation.

Similarly, at the third condition, investigating nonzero h_6 is enough. However, if h_6 is not zero, $h_3 = -h_6$ and $h_4 = -ah_6$ should hold to eliminate $m_{2,0}$ and $m_{3,0}$. And then, the remaining term is at most $O(s_1^3)$ as $s_1 \to 0$.

■

Proof of Proposition 5.

Proof of Proposition 5 is similar to Proof of Proposition 3. For a convenience, let us replace some notations. At the Common property B, let us restrict the $M = (V_1, V_2, V_c)$ with $V_1 = M_{2,0}$, $V_2 = M_{0,2}$, and $V_c = M_{1,1}$. In addition, f and f_c replace f^2 and $f^{1,1}$, respectively. Then integrating (B8) with respect to V_1 yields

$$\begin{aligned} g(s_1,s_2,V_1,V_2,V_c) = a_{1,0}V_1 &+ (b_{1,0}V_1 + b_{1,1}V_1^2/2 + b_{1,2}V_1V_c + b_{1,3}V_1V_2)(e^{s_1}-1) \\ &+ (c_{1,0}V_1 + c_{1,1}V_1^2/2 + c_{1,2}V_1V_c + c_{1,3}V_1V_2)(e^{s_2}-1) \\ &+ a_{1,3}(V_1 f^2(s_1) + V_1^2/2) + a_{1,4}V_1(f^{1,1}(s_1,s_2)+V_c) \\ &+ a_{1,5}V_1(f^2(s_2)+V_2) + g^1(s_1,s_2,V_2,V_c) \end{aligned} \tag{B28}$$

Similarly, we can get alternative form of (B28) by integrating (B8) with respect to V_2 or V_c. By combining (B28) and the alternatives, we obtain the following form

$$
\begin{aligned}
g(s_1,s_2,V_1,V_2,V_c) = {} & b_0V_c + b_1V_1 + b_2V_2 + (e^{s_1}-1)(b_3V_c + b_4V_1 + b_5V_2 + b_6V_cV_1 \\
& + b_7V_cV_2 + b_8V_1V_2 + b_9V_c^2 + b_{10}V_1^2 + b_{11}V_2^2) \\
& + (e^{s_2}-1)(b_{12}V_c + b_{13}V_1 + b_{14}V_2 + b_{15}V_cV_1 \\
& + b_{16}V_cV_2 + b_{17}V_1V_2 + b_{18}V_c^2 + b_{19}V_1^2 + b_{20}V_2^2) \\
& + b_{21}(f(s_1)V_c + V_1V_c + V_1 f_c(s_1,s_2)) \\
& + b_{22}(f(s_2)V_c + V_2V_c + f_c(s_1,s_2)V_2) \\
& + b_{23}(f(s_2)V_1 + V_1V_2 + f(s_1)V_2) \\
& + b_{24}(2f(s_1,s_2) + V_c)V_c + b_{25}(2f(s_1) + V_1)V_1 \\
& + b_{26}(2f(s_2) + V_2)V_2 + g^s(s_1,s_2)
\end{aligned}
\tag{B29}
$$

with $g^s(0,0) = 0$. Let us substitute (B29) into equation (B3). Then

$$
\begin{aligned}
0 = {} & (e^{s_{11}}-1)\begin{pmatrix} b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2 + b_6(\alpha_c\alpha_1 - \alpha_1 v_c - \alpha_c v_1) \\ + b_7(\alpha_2\alpha_c - \alpha_2 v_c - \alpha_c v_2) + b_8(\alpha_1\alpha_2 - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_9(\alpha_c^2 - 2\alpha_c v_c) + b_{10}(\alpha_1^2 - 2\alpha_1 v_1) + b_{11}(\alpha_2^2 - 2\alpha_2 v_2) \end{pmatrix} \\
& + (e^{s_2}-1)\begin{pmatrix} b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2 + b_{15}(\alpha_c\alpha_1 - \alpha_1 v_c - \alpha_c v_1) \\ + b_{16}(\alpha_2\alpha_c - \alpha_2 v_c - \alpha_c v_2) + b_{17}(\alpha_1\alpha_2 - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_{18}(\alpha_c^2 - 2\alpha_c v_c) + b_{19}(\alpha_1^2 - 2\alpha_1 v_1) + b_{20}(\alpha_2^2 - 2\alpha_2 v_2) \end{pmatrix} \\
& + b_{21}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_c + f(s_{11})\alpha_c + \alpha_1 f_c(s_{11},s_{21}) \\ + (E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_1 \end{pmatrix} \\
& + b_{22}\begin{pmatrix} (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_c + f(s_{21})\alpha_c + \alpha_2 f_c(s_{11},s_{21}) \\ + (E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_2 \end{pmatrix} \\
& + b_{23}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_2 + f(s_{21})\alpha_1 + f(s_{11})\alpha_2 \\ + (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_1 \end{pmatrix} \\
& + 2b_{24}((E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11},s_{21}) - \alpha_c)v_c + f_c(s_{11},s_{21})\alpha_c) \\
& + 2b_{25}((E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_1 + f(s_{11})\alpha_1) \\
& + 2b_{26}((E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_2 + f(s_{21})\alpha_2) \\
& - E[g^s(s_{11}+\eta_1, s_{21}+\eta_2)] + g^s(s_1,s_2) + E[g^s(\eta_1,\eta_2)]
\end{aligned}
\tag{B30}
$$

Let

$$
(\eta_1^p, \eta_2^p) = \begin{cases} (\eta_1,\eta_2) & \Pr = p \\ (0,0) & \Pr = 1-p \end{cases}
\tag{B31}
$$

Then, for $i \in \{1,2\}$, $E\left[e^{\eta_i^p}\right] = 1$, $E[f(\eta_i^p)] = \alpha_i p$ and $E[f_c(\eta_1^p, \eta_2^p)] = \alpha_c p$. Therefore, substituting (η_1^p, η_2^p) into (η_1, η_2) of the previous equation yields

$$
\begin{aligned}
0 = {} & p\left(e^{s_{11}} - 1\right)\begin{pmatrix} b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2 + b_6(\alpha_c\alpha_1 p - \alpha_1 v_c - \alpha_c v_1) \\ + b_7(\alpha_2\alpha_c p - \alpha_2 v_c - \alpha_c v_2) + b_8(\alpha_1\alpha_2 p - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_9(\alpha_c^2 p - 2\alpha_c v_c) + b_{10}(\alpha_1^2 p - 2\alpha_1 v_1) + b_{11}(\alpha_2^2 p - 2\alpha_2 v_2) \end{pmatrix} \\
& + p\left(e^{s_2} - 1\right)\begin{pmatrix} b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2 + b_{15}(\alpha_c\alpha_1 p - \alpha_1 v_c - \alpha_c v_1) \\ + b_{16}(\alpha_2\alpha_c p - \alpha_2 v_c - \alpha_c v_2) + b_{17}(\alpha_1\alpha_2 p - \alpha_2 v_1 - \alpha_1 v_2) \\ + b_{18}(\alpha_c^2 p - 2\alpha_c v_c) + b_{19}(\alpha_1^2 p - 2\alpha_1 v_1) + b_{20}(\alpha_2^2 p - 2\alpha_2 v_2) \end{pmatrix} \\
& + pb_{21}\begin{pmatrix} (E[f(s_{11} + \eta_1)] - f(s_{11}) - \alpha_1)v_c + f(s_{11})\alpha_c + \alpha_1 f_c(s_{11}, s_{21}) \\ + (E[f_c(s_{11} + \eta_1, s_{21} + \eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_1 \end{pmatrix} \\
& + pb_{22}\begin{pmatrix} (E[f(s_{21} + \eta_2)] - f(s_{21}) - \alpha_2)v_c + f(s_{21})\alpha_c + \alpha_2 f_c(s_{11}, s_{21}) \\ + (E[f_c(s_{11} + \eta_1, s_{21} + \eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_2 \end{pmatrix} \\
& + pb_{23}\begin{pmatrix} (E[f(s_{11} + \eta_1)] - f(s_{11}) - \alpha_1)v_2 + f(s_{21})\alpha_1 + f(s_{11})\alpha_2 \\ + (E[f(s_{21} + \eta_2)] - f(s_{21}) - \alpha_2)v_1 \end{pmatrix} \\
& + 2pb_{24}((E[f_c(s_{11} + \eta_1, s_{21} + \eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_c + f_c(s_{11}, s_{21})\alpha_c) \\
& + 2pb_{25}((E[f(s_{11} + \eta_1)] - f(s_{11}) - \alpha_1)v_1 + f(s_{11})\alpha_1) \\
& + 2pb_{26}((E[f(s_{21} + \eta_2)] - f(s_{21}) - \alpha_2)v_2 + f(s_{21})\alpha_2) \\
& - pE[g^s(s_{11} + \eta_1, s_{21} + \eta_2)] + pg^s(s_1, s_2) + pE[g^s(\eta_1, \eta_2)]
\end{aligned}
\tag{B32}
$$

Because (B32) holds for arbitrary p , the coefficient of p^2 should be zero.

$$
\begin{aligned}
0 = {} & \left(e^{s_{11}} - 1\right)\left(b_6\alpha_c\alpha_1 + b_7\alpha_2\alpha_c + b_8\alpha_1\alpha_2 + b_9\alpha_c^2 + b_{10}\alpha_1^2 + b_{11}\alpha_2^2\right) \\
& + \left(e^{s_2} - 1\right)\left(b_{15}\alpha_c\alpha_1 + b_{16}\alpha_2\alpha_c + b_{17}\alpha_1\alpha_2 + b_{18}\alpha_c^2 + b_{19}\alpha_1^2 + b_{20}\alpha_2^2\right)
\end{aligned}
\tag{B33}
$$

Since s_{11} and s_{21} are arbitrary, the following holds:

$$
\begin{aligned}
0 &= b_6\alpha_c\alpha_1 + b_7\alpha_2\alpha_c + b_8\alpha_1\alpha_2 + b_9\alpha_c^2 + b_{10}\alpha_1^2 + b_{11}\alpha_2^2 \\
0 &= b_{15}\alpha_c\alpha_1 + b_{16}\alpha_2\alpha_c + b_{17}\alpha_1\alpha_2 + b_{18}\alpha_c^2 + b_{19}\alpha_1^2 + b_{20}\alpha_2^2
\end{aligned}
\tag{B34}
$$

Because α_c is arbitrary, given α_1 and α_2 ,

$$b_9 = b_6\alpha_1 + b_7\alpha_2 = b_8\alpha_1\alpha_2 + b_{10}\alpha_1^2 + b_{11}\alpha_2^2 = 0\,. \tag{B35}$$

According to the similar logic with the α_1 and α_2, the followings hold.

$$b_6 = b_7 = \cdots = b_{11} = 0 \text{ and } b_{15} = b_{16} = \cdots = b_{20} = 0\,. \tag{B36}$$

Because, at Equation (B32), coefficient of p is zero, we have:

$$\begin{aligned}
0 &= \left(e^{s_{11}} - 1\right)\left(b_3\alpha_c + b_4\alpha_1 + b_5\alpha_2\right) + \left(e^{s_2} - 1\right)\left(b_{12}\alpha_c + b_{13}\alpha_1 + b_{14}\alpha_2\right) \\
&+ b_{21}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_c + f(s_{11})\alpha_c + \alpha_1 f_c(s_{11}, s_{21}) \\ + (E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_1 \end{pmatrix} \\
&+ b_{22}\begin{pmatrix} (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_c + f(s_{21})\alpha_c + \alpha_2 f_c(s_{11}, s_{21}) \\ + (E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_2 \end{pmatrix} \\
&+ b_{23}\begin{pmatrix} (E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_2 + f(s_{21})\alpha_1 + f(s_{11})\alpha_2 \\ + (E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_1 \end{pmatrix} \\
&+ 2b_{24}((E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c)v_c + f_c(s_{11}, s_{21})\alpha_c) \\
&+ 2b_{25}((E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1)v_1 + f(s_{11})\alpha_1) \\
&+ 2b_{26}((E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2)v_2 + f(s_{21})\alpha_2) \\
&- E[g^s(s_{11}+\eta_1, s_{21}+\eta_2)] + g^s(s_1, s_2) + E[g^s(\eta_1, \eta_2)]
\end{aligned} \tag{B37}$$

Because v_c is arbitrary, coefficient of v_c is zero.

$$\begin{aligned}
&b_{21}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) \\
&+ 2b_{24}(E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] - f_c(s_{11}, s_{21}) - \alpha_c) = 0
\end{aligned} \tag{B38}$$

Now consider a random variable η_3 with $\eta_3 \overset{d}{\sim} \eta_2$ and $E[f_c(\eta_1, \eta_3)] \neq E[f_c(\eta_1, \eta_2)]$. Then,

$$\begin{aligned}
&b_{21}(E[f(s_{11}+\eta_1)] - f(s_{11}) - \alpha_1) + b_{22}(E[f(s_{21}+\eta_2)] - f(s_{21}) - \alpha_2) \\
&+ 2b_{24}(E[f_c(s_{11}+\eta_1, s_{21}+\eta_3)] - f_c(s_{11}, s_{21}) - E[f_c(\eta_1, \eta_3)]) = 0
\end{aligned} \tag{B39}$$

By subtracting these two equations, one can see that $b_{24} = 0$ or

$$E[f_c(s_{11}+\eta_1, s_{21}+\eta_2)] = E[f_c(s_{11}+\eta_1, s_{21}+\eta_3)] + E[f_c(\eta_1, \eta_2)] - E[f_c(\eta_1, \eta_3)] \tag{B40}$$

When we substitute

$$(\eta_1,\eta_2)=\begin{cases}(\log(1+\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\\ (\log(1-\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\end{cases},$$

$$(\eta_1,\eta_3)=\begin{cases}(\log(1+\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\\ (\log(1-\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\end{cases}$$

into Equation (B40) and multiply $2/(\ln(1+\sqrt{k})-\ln(1-\sqrt{k}))$ to the both hand side of the equation, and take the limit with $k\to 0$, we get

$$f_{c12}(s_{11},s_{21})=1 \tag{B41}$$

Hence

$$f_c(s_{11},s_{21})=s_{11}s_{21}+F_1(s_{11})+F_2(s_{21}) \tag{B42}$$

for some functions F_1 and F_2. In addition the condition of $\lim_{s_{11},s_{21}\to 0,0}\frac{f_c(s_{11},s_{21})}{s_{11}s_{21}}=1$ provides $f_c(s_{11},s_{21})=s_{11}s_{21}$. Therefore, (B40) implies $f_c(s_{11},s_{21})=s_{11}s_{21}$. When one substitute function f_c into previous of previous equation (B42), we get the following equation:

$$\begin{aligned}&b_{21}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)+b_{22}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)\\&+2b_{24}(s_{11}E[\eta_2]+s_{21}E[\eta_1])=0\end{aligned} \tag{B43}$$

When $s_{11}=0$, (B43) is changed to $b_{22}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)+2b_{24}s_{21}E[\eta_1]=0$. Because η_1 can be chosen independently on s_{21} and η_2,

$$b_{24}=0. \tag{B44}$$

Instead of Equation (B38), let us consider the coefficients of v_1 and v_2. Then adopting same logic from (B37) to (B44) yields

$$b_{21}=b_{22}=0. \tag{B45}$$

Because the coefficient of v_1 is at Equation (B37), the equations (B44) and (B45) implies:

$$b_{23}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)+2b_{25}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)=0 \tag{B46}$$

Substituting $s_{11}=0$ or $s_{21}=0$ into the (B46) yields:

$$b_{23}(E[f(s_{21}+\eta_2)]-f(s_{21})-\alpha_2)=b_{25}(E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1)=0 \tag{B47}$$

Similarly, we can get an alternative form of (B47) by using the coefficient of v_2. The combination between these two yields the following

$$(E[f(s+\eta)]-f(s)-E[f(\eta)])=0 \quad \text{or} \quad b_{23}=b_{25}=b_{26}=0. \tag{B48}$$

Here, $E[f(s_{11}+\eta_1)]-f(s_{11})-\alpha_1=0$ is equivalent to

$$f(x)=2(e^x-1-x) \tag{B49}$$

by Neuberger (2012). In sum, (B37) with (B44), (B45), and (B48) yields

$$\begin{aligned}0=&\left(b_3\alpha_c+b_4\alpha_1+b_5\alpha_2\right)\left(e^{s_{11}}-1\right)+\left(b_{12}\alpha_c+b_{13}\alpha_1+b_{14}\alpha_2\right)\left(e^{s_2}-1\right)\\&+b_{23}\left(f(s_{21})\alpha_1+f(s_{11})\alpha_2\right)+2b_{25}f(s_{11})\alpha_1+2b_{26}f(s_{21})\alpha_2\\&-E[g^s(s_{11}+\eta_1,s_{21}+\eta_2)]+g^s(s_{11},s_{21})+E[g^s(\eta_1,\eta_2)]\end{aligned} \tag{B50}$$

Substituting $\eta_1=\begin{cases}\log(1+\sqrt{k}) & \Pr=1/2\\ \log(1-\sqrt{k}) & \Pr=1/2\end{cases}$ and $\eta_2=0$ into the (B50) and taking limit yields:

$$\begin{aligned}0=&2b_4\left(e^{s_{11}}-1\right)+2b_{13}\left(e^{s_{21}}-1\right)+4b_{23}(e^{s_{21}}-1-s_{21})+8b_{25}(e^{s_{11}}-1-s_{11})\\&-g^s_{11}(s_{11},s_{21})+g^s_1(s_{11},s_{21})+g^s_{11}(0,0)-g^s_1(0,0)\end{aligned} \tag{B51}$$

Similarly, when use $\eta_2=\begin{cases}\log(1+\sqrt{k}) & \Pr=1/2\\ \log(1-\sqrt{k}) & \Pr=1/2\end{cases}$ and $\eta_1=0$, we get the following relation

$$\begin{aligned}0=&2b_5\left(e^{s_{11}}-1\right)+2b_{14}\left(e^{s_{21}}-1\right)+4b_{23}(e^{s_{11}}-1-s_{11})+8b_{26}(e^{s_{21}}-1-s_{21})\\&-g^s_{22}(s_{11},s_{21})+g^s_2(s_{11},s_{21})+g^s_{22}(0,0)-g^s_2(0,0)\end{aligned} \tag{B52}$$

Now let us consider η_1 and η_2 that are dependent each other. If we substitute (B53) or (B54) into Equation (B50) and subtract each other, then then taking limits yields the (B55).

$$(\eta_1,\eta_2)=\begin{cases}(\log(1+\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\\(\log(1-\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\end{cases}\tag{B53}$$

$$(\eta_1,\eta_2)=\begin{cases}(\log(1+\sqrt{k}),\log(1-\sqrt{k})) & \Pr=1/2\\(\log(1-\sqrt{k}),\log(1+\sqrt{k})) & \Pr=1/2\end{cases}\tag{B54}$$

$$0=b_3\left(e^{s_{11}}-1\right)+b_{12}\left(e^{s_{21}}-1\right)-g_{12}^s(s_{11},s_{21})+g_{12}^s(0,0)\tag{B55}$$

Then the solutions of the (B51), (B52), and (B55) are given as

$$g^s(x,y)=b_3(e^x-x)y+b_{12}(e^y-y)x+h_1(x)+h_2(y)+b_{27}xy\tag{B56}$$

$$\begin{aligned}&g^s(x,y)=2b_4(e^x x-e^x+x)-2b_{13}(e^y-1)x-4b_{23}x(e^y-y-1)\\&+4b_{25}(2e^x x-2e^x+x^2+4x)+e^x h_3(y)+h_4(y)+b_{28}x\end{aligned}\tag{B57}$$

$$\begin{aligned}&g^s(x,y)=2b_{14}(e^y y-e^y+y)-2b_5(e^x-1)y-4b_{23}y(e^x-x-1)\\&+4b_{26}(2e^y y-2e^y+y^2+4y)+e^y h_5(x)+h_6(x)+b_{29}y\end{aligned}\tag{B58}$$

for some functions $h_1,\cdots,h_6$ and constants b_i. Therefore $g^s(x,y)$ is a linear combination of $e^x y, e^y x, xy, e^x x, e^x, x^2, x, e^y y, e^y, y^2, y$ and 1. Consistency about coefficients of $e^x y$ and $e^y x$ requires $b_5=-\frac{1}{2}b_3-2b_{23}$ and $b_{13}=-\frac{1}{2}b_{12}-2b_{23}$. Because $g^s(0,0)$ is zero, g and g^s are given by

$$\begin{aligned}g(s_1,s_2,V_1,V_2,V_c)&=b_0V_c+b_1V_1+b_2V_2+\left(b_3V_c+b_4V_1-\left(\frac{1}{2}b_3+2b_{23}\right)V_2\right)\left(e^{s_1}-1\right)\\&+\left(b_{12}V_c-\left(\frac{1}{2}b_{12}+2b_{23}\right)V_1+b_{14}V_2\right)\left(e^{s_2}-1\right)\\&+b_{23}\left(2\left(e^{s_2}-s_2-1\right)V_1+V_1V_2+2\left(e^{s_1}-s_1-1\right)V_2\right)\\&+b_{25}\left(4\left(e^{s_1}-s_1-1\right)+V_1\right)V_1+b_{26}\left(4\left(e^{s_2}-s_2-1\right)+V_2\right)V_2+g^s(s_1,s_2)\end{aligned}\tag{B59}$$

$$\begin{aligned}g^s(x,y)&=d_1(e^x-1)+d_2x+d_3(e^y-1)+d_4y+4b_{23}xy+4b_{25}x^2+4b_{26}y^2\\&+b_3e^x y+b_{12}e^y x+(2b_4+8b_{25})e^x x+(2b_{14}+8b_{26})e^y y\end{aligned}\tag{B60}$$

(B59) and (B60) are arranged as

$$
\begin{aligned}
g\left(s_1,s_2,V_1,V_2,V_c\right) = {} & d_1(e^{s_{11}}-1)+d_2 s_{11}+d_3(e^{s_{21}}-1)+d_4 s_{21}+d_5V_1+d_6V_2+d_7V_c \\
& +d_8(V_1-2s_1)^2+d_9(V_2-2s_2)^2+d_{10}(V_1-2s_1)(V_2-2s_2) \\
& +d_{11}e^{s_1}\left(2V_c-V_2+2s_2\right)+d_{12}e^{s_2}\left(2V_c-V_1+2s_1\right) \\
& +d_{13}e^{s_1}(V_1+2s_1)+d_{14}e^{s_2}(V_2+2s_2)
\end{aligned}
\tag{B61}
$$

where $d_5=b_1-b_4+\frac{1}{2}b_{12}-4b_{25}$, $d_6=b_2+\frac{1}{2}b_3-b_{14}-4b_{26}$, $d_7=b_0-b_3-b_{12}$, $d_8=b_{25}$, $d_9=b_{26}$, $d_{10}=b_{23}$, $d_{11}=\frac{1}{2}b_3$, $d_{12}=\frac{1}{2}b_{12}$, $d_{13}=b_4+4b_{25}$, $d_{14}=b_{14}+4b_{26}$.[20]

Substituting it into equation (B3) yields the following:

$$
\begin{aligned}
0 = {} & 2d_8\left(-v_1-2s_{11}+\alpha_1\right)\left(\alpha_1+2E[\eta_1]\right)+2d_9\left(-v_2-2s_{21}+\alpha_2\right)\left(\alpha_2+2E[\eta_2]\right) \\
& +d_{10}\left((-v_1-2s_{11}+\alpha_1)(\alpha_2+2E[\eta_2])+(-v_2-2s_{21}+\alpha_2)(\alpha_1+2E[\eta_1])\right) \\
& +(e^{s_{11}}-1)\left(d_{13}(\alpha_1-E[2\eta_1e^{\eta_1}])+d_{11}(2\alpha_c-2E[\eta_2e^{\eta_1}]-\alpha_2)\right) \\
& +(e^{s_{21}}-1)\left(d_{14}(\alpha_2-E[2\eta_2e^{\eta_2}])+d_{12}\left(2\alpha_c-2E[\eta_1e^{\eta_2}]-\alpha_1\right)\right)
\end{aligned}
\tag{B62}
$$

Since coefficients of v_1 and v_2 are zero, $E[f(\eta)]=E[-2\eta]$ or $d_8=d_9=d_{10}=0$. In addition, because s_{11},s_{21} are arbitrary,

$$
\begin{aligned}
0 &= d_{13}(\alpha_1-E[2\eta_1e^{\eta_1}])+d_{11}(2\alpha_c-2E[\eta_2e^{\eta_1}]-\alpha_2) \\
0 &= d_{14}(\alpha_2-E[2\eta_2e^{\eta_2}])+d_{12}\left(2\alpha_c-2E[\eta_1e^{\eta_2}]-\alpha_1\right)
\end{aligned}
\tag{B63}
$$

(1) If d_{11} is not zero, for some constants k_1 and k_2, the following holds

[20] The ten coefficients, $b_0,\cdots,b_5,b_{12},b_{14},b_{23},b_{25}$, and b_{26} are replaced with $d_5,\cdots,d_{14}$. More precisely, (d_5,d_8,d_{12},d_{13}) replace (b_1,b_4,b_{12},b_{25}). (d_6,d_9,d_{11},d_{14}) replace (b_2,b_3,b_{14},b_{26}). d_{10} replaces b_{23}. And d_7 replaces b_0 given b_3 and b_{12}.

$$f_c(\eta_1,\eta_2)=\eta_2 e^{\eta_1}+\frac{1}{2}\alpha_2+\frac{d_{13}}{2d_{11}}(2\eta_1 e^{\eta_1}-\alpha_1)+k_1\left(e^{\eta_1}-1\right)+k_2\left(e^{\eta_2}-1\right)$$

Then $\frac{d_{13}}{2d_{11}}(2\eta_1 e^{\eta_1}-\alpha_1)+k_1\left(e^{\eta_1}-1\right)=0$ and $\frac{1}{2}\alpha_2+k_2\left(e^{\eta_2}-1\right)=-\eta_2$ because $\frac{f_c(x,y)}{xy}\to 1$ as $x,y\to 0$. Accordingly, $k_2=-1$ because $\frac{f(x)}{x^2}\to 1$ as $x,y\to 0$. And it implies $k_1=d_{13}=0$. Therefore, $f_c(\eta_1,\eta_2)=\eta_2(e^{\eta_1}-1)$, $f(\eta)=2\left(e^{\eta}-\eta-1\right)$. Then $d_{12}=d_{14}=0$.

(2) Similarly, if d_{12} is not zero, $f_c(\eta_1,\eta_2)=\eta_1(e^{\eta_2}-1)$, $f(\eta)=2\left(e^{\eta}-\eta-1\right)$ and $d_{11}=d_{13}=d_{14}=0$

(3) or $d_{11}=d_{12}=d_{13}=d_{14}=0$, $f(\eta)=2\left(e^{\eta}-\eta-1\right)$ with arbitrary function f_c.

(4) $d_8=d_9=d_{10}=d_{11}=d_{12}=0$, $f(\eta)=2(\eta e^{\eta}-e^{\eta}+1)$ with arbitrary function f_c.

(5) $d_8=d_9=d_{10}=d_{11}=d_{12}=d_{13}=d_{14}=0$ with arbitrary functions f and f_c.

■

Table 1. Higher order moment swaps

Type of swap	Receiving at T	Paying at $j \in \{1,\cdots,N\}$	Cost at 0
Third comoment	$(S_{1,T}-S_{1,0})^2(S_{2,T}-S_{2,0})$	$(\Delta S_{1,j})^2\Delta S_{2,j}+\Delta S_{2,j}\Delta M_{2,0,j}+2\Delta S_{1,j}\Delta M_{1,1j}$	0
Non-zero fourth moment	$(S_{1,T}-S_{1,0})^4$	$(\Delta S_{1,j})^4+6(\Delta S_{1,j})^2\Delta M_{2,0,j}+4\Delta S_{1,j}\Delta M_{3,0,j}+3(\Delta M_{2,0,j})^2$	$3M_{2,0,0}^2$
Fourth moment	$(S_{1,T}-S_{1,0})^4-3(S_{1,T}-S_{1,0})^2 M_{2,0,0}$	$(\Delta S_{1,j})^4+6(\Delta S_{1,j})^2\Delta M_{2,0,j}+4\Delta S_{1,j}\Delta M_{3,0,j}+3(\Delta M_{2,0,j})^2$	0
Non-zero asymmetric fourth comoment	$(S_{1,T}-S_{1,0})^3(S_{2,T}-S_{2,0})$	$(\Delta S_{1,j})^3\Delta S_{2,j}+\Delta S_{2,j}\Delta M_{3,0,j}+3\Delta M_{1,1,j}(\Delta S_{1,j})^2$ $+3\Delta M_{2,1,j}\Delta S_{1,j}+3\Delta M_{2,0,j}\Delta S_{1,j}\Delta S_{2,j}+3\Delta M_{2,0,j}\Delta M_{1,1,j}$	$3M_{2,0,0}M_{1,1,0}$
Asymmetric fourth comoment with *a*	$(S_{1,T}-S_{1,0})^3(S_{2,T}-S_{2,0})$ $-3a(S_{1,T}-S_{1,0})(S_{2,T}-S_{2,0})M_{2,0,0}$ $-3(1-a)(S_{1,T}-S_{1,0})^2 M_{1,1,0}$ with a constant *a*	$(\Delta S_{1,j})^3\Delta S_{2,j}+\Delta S_{2,j}\Delta M_{3,0,j}+3\Delta M_{1,1,j}(\Delta S_{1,j})^2$ $+3\Delta M_{2,1,j}\Delta S_{1,j}+3\Delta M_{2,0,j}\Delta S_{1,j}\Delta S_{2,j}+3\Delta M_{2,0,j}\Delta M_{1,1,j}$	0
Non-zero symmetric fourth comoment	$(S_{1,T}-S_{1,0})^2(S_{2,T}-S_{2,0})^2$	$(\Delta S_{1,j})^2+\Delta M_{2,0,j})((\Delta S_{2,j})^2+\Delta M_{0,2,j})+2(\Delta M_{1,1,j})^2$ $+4\Delta M_{1,1,j}\Delta S_{1,j}\Delta S_{2,j}+2\Delta M_{1,2,j}\Delta S_{1,j}+2\Delta M_{2,1,j}\Delta S_{2,j}$	$M_{2,0,0}M_{0,2,0}$ $+2M_{1,1,0}^2$
Symmetric fourth comoment with *a*	$(S_{1,T}-S_{1,0})^2(S_{2,T}-S_{2,0})^2$ $-2(S_{1,T}-S_{1,0})(S_{2,T}-S_{2,0})M_{1,1,0}$ $-a(S_{1,T}-S_{1,0})^2 M_{0,2,0}$ $-(1-a)(S_{2,T}-S_{2,0})^2 M_{2,0,0}$ with a constant *a*	$(\Delta S_{1,j})^2+\Delta M_{2,0,j})((\Delta S_{2,j})^2+\Delta M_{0,2,j})+2(\Delta M_{1,1,j})^2$ $+4\Delta M_{1,1,j}\Delta S_{1,j}\Delta S_{2,j}+2\Delta M_{1,2,j}\Delta S_{1,j}+2\Delta M_{2,1,j}\Delta S_{2,j}$	0

This table describes various (co)moment swaps. Each row represents structure of a swap. The second, the third, and the fourth column represent amount of receiving leg, paying leg, and initial cost, respectively. Paying leg consists of the terms of realized cumulant and the receiving leg is a product of total returns possibly with additional terms. Since each swap is constructed to be fair, some swaps require additional cost at time zero and they have a prefix, non-zero, at the name. The non-zero swaps are modified to be zero cost swaps by changing the receiving legs. As a result, expectation of receiving leg of a non-zero swap becomes (co)moment and expectation of receiving leg of a modified swap becomes (joint) cumulant. There are two kinds of comoment in the case of the fourth comoment. When the receiving leg is related to the product of square of returns, then it has an affix, symmetric, at the name; otherwise, it has an affix of asymmetric. In the case of the fourth comoments, there are various forms for zero cost swaps. Adopting a constant a allows these variations for each fourth comoments.

Table 2. Statistics of cumulants of the S&P 500 returns

This table represents descriptive statistics of cumulants, skewness, and kurtosis of S&P 500 returns from January 1996 to August 2014. Panel A shows the statistics of 30-day returns up to the last trading day of each option. The second column represents sample moments. The third and fourth column represent averages for the implied and realized values, respectively. Implied cumulants are calculated through Equation (25). Realized cumulants are calculated through the expressions (6), (8), and (16). Numbers in parentheses are standard deviations for each term. The other panels are similar to the Panel A except the time horizon and frequency of sample.

A. 30 days

	Sample	Implied	Realized
2nd cumulant	0.24	0.40	0.32
(×100)		(0.37)	(0.50)
3rd cumulant	-0.11	-0.34	-0.22
(×1000)		(0.46)	(0.57)
Skewness	-0.90	-1.37	-1.11
		(0.50)	(0.72)
4th cumulant	0.18	0.80	0.25
(×10000)		(1.27)	(0.97)
Kurtosis	3.04	5.94	4.33
		(4.02)	(5.87)

B. 90 days

	Sample	Implied	Realized
2nd cumulant	0.75	1.18	0.89
(×100)		(0.80)	(0.97)
3rd cumulant	-0.11	-1.42	-0.92
(×1000)		(1.13)	(1.49)
Skewness	-0.16	-1.17	-1.11
		(0.38)	(0.49)
4th cumulant	0.62	3.76	1.28
(×10000)		(3.30)	(2.33)
Kurtosis	1.11	3.35	2.93
		(1.95)	(2.51)

C. 180 days

	Sample	Implied	Realized
2nd cumulant	1.65	2.32	1.93
(×100)		(1.36)	(2.03)
3rd cumulant	0.05	-3.01	-2.12
(×1000)		(2.09)	(2.44)
Skewness	0.02	-0.91	-1.03
		(0.33)	(0.53)
4th cumulant	3.04	6.75	3.84
(×10000)		(5.44)	(13.05)
Kurtosis	1.11	1.67	2.10
		(1.08)	(2.03)

D. 360 days

	Sample	Implied	Realized
2nd cumulant	3.97	4.63	7.81
(×100)		(2.27)	(28.59)
3rd cumulant	1.97	-4.23	6.40
(×1000)		(4.89)	(81.53)
Skewness	0.25	-0.50	-0.81
		(0.40)	(0.58)
4th cumulant	-12.91	8.63	468.85
(×10000)		(16.02)	(3652.54)
Kurtosis	-0.82	0.62	1.27
		(0.61)	(1.63)

Table 3. Adjusted skewness and kurtosis

The adjusted skewness and kurtosis are defined to be irrelevant to the *n* if the returns are i.i.d.. More specifically, adjusted skewness represents monthly skewness which is calculated by skewness of *n*-month return; $\sqrt{n}$ times sample skewness of *n*-month returns of the Table 2. In addition, adjusted kurtosis represents *n* times non-excess kurtosis of *n*-month returns minus 3.

Months (*n*)	1	3	6	12
Adjusted skewness	-0.90	-0.28	0.06	0.86
Adjusted kurtosis	3.04	9.34	21.68	23.19

Table 4. Time series regression of cumulants of S&P 500 returns

Panel A represents the time series regression about cumulants of monthly returns of S&P 500 from January 1996 to August 2014. Each row represents the result of the regression with coefficients and t-values in parentheses. The first column represents the measure that we analyze. Within the measure, dependent variables are realized cumulants and the independent variables are implied or lagged realized cumulants. The other panels are similar to the Panel A except the time horizon.

A. 30 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.00	0.95		0.51
	(-1.23)	(6.31)		
	0.00		0.71	0.50
	(2.44)		(4.71)	
	0.00	0.54	0.37	0.55
	(-0.51)	(3.06)	(1.47)	
3rd cum.	0.00	0.82		0.43
	(0.98)	(4.00)		
	0.00		0.61	0.37
	(-2.02)		(2.46)	
	0.00	0.57	0.25	0.46
	(0.73)	(3.01)	(0.92)	
Skew	-0.20	0.67		0.22
	(-2.30)	(9.42)		
	-0.68		0.39	0.15
	(-7.59)		(4.68)	
	-0.20	0.52	0.19	0.24
	(-2.21)	(5.81)	(2.26)	
4th cum.	0.00	0.23		0.08
	(0.64)	(1.13)		
	0.00		-0.13	0.01
	(3.99)		(-0.46)	
	0.00	0.28	-0.24	0.13
	(0.83)	(1.62)	(-0.74)	
Kurt	1.48	0.48		0.10
	(3.25)	(6.27)		
	3.18		0.27	0.07
	(7.01)		(3.40)	
	1.38	0.38	0.16	0.12
	(3.01)	(4.94)	(2.31)	

B. 90 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.00	0.66		0.29
	(1.66)	(7.91)		
	0.00		0.48	0.22
	(4.66)		(3.85)	
	0.00	0.58	0.08	0.28
	(1.81)	(4.56)	(0.64)	
3rd cum.	0.00	0.56		0.17
	(-1.06)	(4.56)		
	0.00		0.35	0.11
	(-3.71)		(5.03)	
	0.00	0.48	0.08	0.16
	(-1.43)	(2.39)	(0.79)	
Skew	-0.17	0.80		0.39
	(-1.59)	(9.27)		
	-0.48		0.57	0.31
	(-4.69)		(6.49)	
	-0.17	0.59	0.22	0.40
	(-1.55)	(3.44)	(1.41)	
4th cum.	0.00	0.20		0.06
	(1.42)	(1.81)		
	0.00		0.32	0.09
	(2.93)		(2.15)	
	0.00	0.11	0.25	0.10
	(1.56)	(0.89)	(1.40)	
Kurt	0.39	0.75		0.34
	(1.01)	(6.81)		
	1.28		0.55	0.30
	(3.78)		(4.61)	
	0.26	0.51	0.32	0.40
	(0.76)	(3.53)	(2.74)	

C. 180 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.01	0.62		0.16
	(2.55)	(6.65)		
	0.01		0.27	0.06
	(5.18)		(2.52)	
	0.01	0.60	0.03	0.15
	(2.60)	(5.26)	(0.40)	
3rd cum.	0.00	0.53		0.19

	(-2.02)	(6.76)		
	0.00		0.29	0.07
	(-4.90)		(3.73)	
	0.00	0.63	-0.11	0.19
	(-1.77)	(4.49)	(-0.95)	
Skew	0.14	1.29		0.63
	(1.53)	(13.06)		
	-0.31		0.71	0.48
	(-3.45)		(9.44)	
	0.12	1.12	0.13	0.63
	(1.26)	(5.35)	(0.95)	
4th cum.	0.00	0.07		-0.01
	(0.89)	(0.18)		
	0.00		0.04	-0.01
	(2.14)		(0.47)	
	0.00	0.05	0.04	-0.03
	(0.87)	(0.14)	(0.48)	
Kurt	-0.30	1.43		0.57
	(-1.27)	(8.10)		
	0.87		0.60	0.32
	(3.18)		(4.13)	
	-0.30	1.25	0.15	0.57
	(-1.37)	(5.48)	(1.36)	

D. 360 days

	Intercept	Implied	Realized(-1)	Adj. R2
2nd cum.	0.02	1.18		-0.01
	(1.00)	(1.24)		
	0.08		-0.01	-0.02
	(2.19)		(-0.72)	
	0.02	1.27	-0.03	-0.02
	(0.90)	(1.21)	(-0.99)	
3rd cum.	0.01	0.47		-0.01
	(0.83)	(1.29)		
	0.01		0.00	-0.02
	(0.67)		(-0.31)	
	0.01	0.47	-0.01	-0.03
	(0.83)	(1.28)	(-0.45)	
Skew	-0.31	1.03		0.49
	(-3.66)	(9.11)		
	-0.39		0.58	0.32
	(-3.79)		(6.71)	
	-0.26	0.85	0.18	0.50

	(-2.87)	(5.16)	(1.47)	
4th cum.	0.08	-34.24		0.01
	(1.04)	(-0.91)		
	0.05		-0.02	-0.02
	(1.08)		(-1.12)	
	0.08	-36.47	-0.05	-0.01
	(1.03)	(-0.90)	(-1.00)	
Kurt	0.19	1.73		0.41
	(1.07)	(6.68)		
	0.87		0.37	0.10
	(3.81)		(3.08)	
	0.20	1.81	-0.05	0.40
	(1.11)	(5.90)	(-0.50)	

Table 5. Average of regression about comoments

Panel A represents average of time series regression results about comoments of monthly returns between S&P 500 and each stock contained in Dow Jones Industrial Average from January 1996 to August 2014. Each row represents the result of the regression with coefficients and t-values in parentheses. The first column represents the measure that we address. Within the measure, dependent variables are realized comoments and the independent variables are implied, lagged realized, and historical comoments; the implied and the realized moments are calculated as Section 3 describes and the historical comoments are calculated from the previous 24 monthly returns. Panel B is similar to the Panel A except that Panel B is about average of cross sectional regressions.

A. Average of time series regressions.

	Intercept	Implied	Realized(-1)	Historical	Adj. R2
covar	0.000	0.691			0.403
	(-1.304)	(11.510)			(13.477)
	0.001		0.495		0.322
	(15.013)		(11.042)		(11.394)
	0.002			0.568	0.037
	(5.899)			(4.992)	(3.641)
	0.000	0.521	0.134	0.010	0.429
	(2.246)	(7.532)	(3.247)	(0.057)	(13.846)
3rd comom	0.000	0.907			0.440
	(1.065)	(13.638)			(16.983)
	0.000		0.401		0.234
	(-18.145)		(9.255)		(10.566)
	0.000			0.676	0.053
	(-13.645)			(6.809)	(6.086)
	0.000	0.872	0.039	-0.184	0.466
	(1.212)	(13.435)	(1.155)	(-2.593)	(18.530)
beta	0.133	0.716			0.114
	(1.152)	(7.605)			(4.903)
	0.686		0.257		0.084
	(18.651)		(8.738)		(5.228)
	0.698			0.223	0.066
	(9.104)			(3.244)	(4.418)
	-0.022	0.586	0.111	0.150	0.167
	(-0.094)	(4.134)	(3.780)	(1.142)	(5.527)

gamma	0.310	0.648			0.045
	(2.092)	(5.382)			(2.966)
	0.952		0.042		0.005
	(26.977)		(1.835)		(0.520)
	0.992			-0.009	-0.004
	(26.816)			(-0.610)	(-0.688)
	0.339	0.644	-0.010	-0.027	0.035
	(2.225)	(4.913)	(-0.521)	(-1.167)	(2.141)

B. Average of cross sectional regressions.

	Intercept	Implied	Realized(-1)	Historical	Adj. R2
covar	0.000	0.668			0.282
	(-0.283)	(12.675)			(15.831)
	0.001		0.600		0.269
	(5.618)		(11.946)		(15.827)
	0.001			0.834	0.265
	(7.379)			(8.181)	(16.990)
	0.000	0.282	0.269	0.298	0.386
	(0.845)	(6.442)	(9.210)	(5.823)	(21.012)
3rd comom	0.000	0.617			0.358
	(-3.601)	(9.245)			(18.601)
	0.000		0.784		0.227
	(-6.649)		(3.331)		(12.863)
	0.000			0.107	0.075
	(-6.574)			(1.522)	(8.438)
	0.000	0.510	0.308	0.006	0.409
	(-3.713)	(10.603)	(2.942)	(0.203)	(21.448)
beta	0.036	0.861			0.282
	(0.748)	(18.166)			(15.831)
	0.455		0.508		0.269
	(20.007)		(21.210)		(15.827)
	0.485			0.459	0.265
	(23.011)			(22.536)	(16.990)
	0.109	0.374	0.240	0.212	0.386
	(2.539)	(8.198)	(12.797)	(12.072)	(21.012)
gamma	0.254	0.687			0.358
	(4.256)	(12.425)			(18.601)
	0.626		0.345		0.227
	(18.817)		(8.397)		(12.863)
	0.884			0.085	0.075
	(16.109)			(1.906)	(8.438)

	0.221	0.570	0.111	0.042	0.409
	(4.809)	(7.507)	(2.606)	(1.062)	(21.448)

Table 6. Return and realized moments of (co)moment portfolios

Panel A represents performance of portfolios that are constructed based on the rank of the implied volatility. We classify the firms in the DJIA index into the three groups, based on the model free implied variance at each month-end, with breakpoints 30% and 70%. Using the three groups, we make three equally weighted portfolios and zero cost portfolio which is denoted by 3-1. The numbers in the second column is the average of returns over the subsequent month. Similarly, the other columns present realized moments of return over the subsequent month. The last row represents t-value of the statistics for the 3-1 portfolio. Panel B and C are similarly constructed except that the portfolios are sorted based on the model free implied skewness or kurtosis at each month-end. The other Panels are similar except that the portfolios are sorted based on the realized moment of each month-end.

A. var_imp	return	volatility	beta	gamma	skewness	kurtosis
1 (lowest)	0.0060	0.0601	0.6696	0.7571	-0.2129	0.4505
2	0.0081	0.0764	0.9360	0.9829	-0.1533	0.3106
3 (highest)	0.0048	0.1045	1.2115	1.1614	-0.0187	0.3439
3-1	-0.0012	0.0444	0.5419	0.4042	0.1942	-0.1065
t(3-1)	-0.2795	15.7341	18.6191	9.9106	12.5295	-2.1609
B. skew_imp						
1	0.0031	0.0696	0.8378	0.9174	-0.2355	0.5461
2	0.0078	0.0772	0.9281	0.9559	-0.1415	0.3151
3	0.0082	0.0938	1.0535	1.0373	-0.0122	0.2428
3-1	0.0051	0.0242	0.2156	0.1199	0.2233	-0.3033
t(3-1)	1.4832	12.0778	9.1469	3.4865	15.5593	-5.6128
C. kurt_imp						
1	0.0056	0.0824	0.9939	1.0043	-0.1188	0.2503
2	0.0066	0.0776	0.9303	1.0021	-0.1376	0.3469
3	0.0071	0.0806	0.8946	0.8885	-0.1337	0.4957
3-1	0.0015	-0.0019	-0.0992	-0.1157	-0.0149	0.2454
t(3-1)	0.5322	-1.1838	-5.0005	-2.8297	-0.9675	4.7449
D. var_real(-1)						
1	0.0071	0.0631	0.7206	0.7985	-0.1972	0.4541
2	0.0075	0.0770	0.9240	0.9660	-0.1561	0.3687
3	0.0043	0.1005	1.1762	1.1415	-0.0311	0.2638
3-1	-0.0028	0.0374	0.4556	0.3430	0.1661	-0.1903
t(3-1)	-0.6844	13.3402	15.9114	5.6637	10.7673	-4.3224
E. β_real(-1)						
1	0.0072	0.0679	0.6813	0.7999	-0.1563	0.4433
2	0.0060	0.0756	0.9085	0.9738	-0.1476	0.3998
3	0.0063	0.0976	1.2352	1.1297	-0.0833	0.2319
3-1	-0.0008	0.0297	0.5539	0.3298	0.0730	-0.2114
t(3-1)	-0.2000	10.2679	19.7464	4.8118	4.6119	-5.5192
F. γ_real(-1)						

1	0.0068	0.0709	0.7527	0.8522	-0.1470	0.4466
2	0.0076	0.0758	0.9117	0.9583	-0.1524	0.3477
3	0.0046	0.0943	1.1598	1.0976	-0.0864	0.2992
3-1	-0.0022	0.0233	0.4072	0.2454	0.0605	-0.1475
t(3-1)	-0.6941	9.3553	15.1543	6.1452	3.9138	-3.4842
G. skew_real(-1)						
1	0.0081	0.0747	0.8937	0.9548	-0.2189	0.4011
2	0.0071	0.0783	0.9419	0.9596	-0.1430	0.3038
3	0.0038	0.0872	0.9796	0.9939	-0.0264	0.4030
3-1	-0.0043	0.0125	0.0859	0.0391	0.1924	0.0019
t(3-1)	-1.4363	5.0359	3.3964	1.3867	12.1968	0.0411
H. kurt_real(-1)						
1	0.0048	0.0811	0.9763	0.9747	-0.1265	0.2200
2	0.0061	0.0799	0.9569	0.9720	-0.1219	0.3555
3	0.0086	0.0787	0.8775	0.9580	-0.1467	0.5152
3-1	0.0038	-0.0024	-0.0988	-0.0167	-0.0202	0.2953
t(3-1)	1.5108	-1.9058	-5.2212	-0.4412	-1.3980	6.8189

Table 7. Fama and French 3 factor risk adjusted return

This table constructs the portfolios as the Table 6 describes. And then this table shows coefficients and t-values about time series regression of excess return of each portfolio on the Fama and French 3 factors; mkt, smb, and hml are market excess return, SMB factor, and HML factor, respectively.

A. var_imp

	Intercept	MKT	SMB	HML	Adj. R2
1	0.002	0.632	-0.291	0.125	0.603
	(1.332)	(14.561)	(-3.266)	(2.047)	
2	0.002	0.974	-0.262	0.195	0.802
	(1.164)	(22.550)	(-4.078)	(3.317)	
3	-0.006	1.457	0.077	0.536	0.782
	(-2.417)	(22.227)	(0.886)	(5.559)	
3-1	-0.008	0.825	0.368	0.411	0.457
	(-2.626)	(9.949)	(2.710)	(3.367)	
B. skew_imp					
1	-0.002	0.832	-0.259	0.131	0.695
	(-1.283)	(19.125)	(-4.225)	(2.028)	
2	0.002	0.957	-0.222	0.190	0.785
	(1.028)	(23.678)	(-3.212)	(3.141)	
3	-0.001	1.279	-0.005	0.531	0.733
	(-0.561)	(18.955)	(-0.056)	(5.018)	
3-1	0.001	0.448	0.254	0.400	0.213
	(0.273)	(5.234)	(2.176)	(2.968)	
C. kurt_imp					
1	-0.002	1.086	-0.209	0.277	0.779
	(-1.025)	(23.439)	(-2.683)	(3.880)	
2	0.000	0.934	-0.163	0.207	0.802
	(0.300)	(24.613)	(-2.787)	(4.051)	
3	0.000	1.057	-0.137	0.366	0.669
	(-0.185)	(16.557)	(-1.657)	(4.089)	
3-1	0.001	-0.029	0.072	0.090	-0.007
	(0.471)	(-0.370)	(0.626)	(0.818)	
D. var_real(-1)					
1	0.003	0.708	-0.298	0.156	0.636
	(1.566)	(14.523)	(-3.319)	(2.266)	
2	0.001	0.970	-0.291	0.173	0.840
	(1.051)	(31.984)	(-7.558)	(3.326)	
3	-0.006	1.387	0.114	0.541	0.741
	(-2.294)	(19.808)	(1.169)	(4.794)	
3-1	-0.009	0.678	0.411	0.385	0.358

	(-2.663)	(7.396)	(3.083)	(2.693)	
E. β_real(-1)					
1	0.003	0.680	-0.224	0.126	0.572
	(1.637)	(13.065)	(-2.511)	(1.863)	
2	-0.001	0.965	-0.195	0.297	0.769
	(-0.418)	(23.806)	(-2.675)	(4.698)	
3	-0.004	1.423	-0.081	0.397	0.762
	(-1.406)	(18.277)	(-0.920)	(3.476)	
3-1	-0.006	0.744	0.143	0.271	0.334
	(-1.897)	(6.974)	(1.001)	(1.809)	
F. γ_real(-1)					
1	0.001	0.853	-0.224	0.144	0.679
	(0.778)	(15.463)	(-2.805)	(2.392)	
2	0.001	0.990	-0.216	0.275	0.799
	(0.593)	(25.051)	(-3.492)	(4.082)	
3	-0.004	1.214	-0.057	0.416	0.723
	(-1.735)	(19.612)	(-0.636)	(3.870)	
3-1	-0.006	0.360	0.167	0.271	0.140
	(-1.792)	(4.106)	(1.297)	(1.988)	
G. skew_real(-1)					
1	0.003	0.836	-0.255	0.117	0.673
	(1.664)	(17.148)	(-3.579)	(1.607)	
2	0.000	0.995	-0.188	0.309	0.804
	(0.136)	(27.431)	(-3.413)	(5.855)	
3	-0.005	1.227	-0.064	0.396	0.799
	(-2.608)	(19.122)	(-0.797)	(4.414)	
3-1	-0.008	0.390	0.190	0.279	0.199
	(-3.013)	(4.351)	(1.838)	(2.385)	
H. kurt_real(-1)					
1	-0.002	1.017	-0.054	0.346	0.785
	(-1.555)	(25.606)	(-0.732)	(6.021)	
2	-0.001	1.044	-0.182	0.257	0.798
	(-0.609)	(26.825)	(-3.018)	(4.785)	
3	0.002	0.980	-0.269	0.233	0.704
	(1.134)	(17.813)	(-3.916)	(2.719)	
3-1	0.005	-0.037	-0.215	-0.113	0.026
	(1.935)	(-0.580)	(-2.920)	(-1.184)	

Table A1. Elements of a spanning set of functions that satisfy Equation (A5) for each pair (k_1,k_2)

k_1+k_2	(k_1,k_2)	Elements of a spanning set of functions for each pair (k_1,k_2)
1	(1,0)	s_1
2	(2,0)	s_1^2, $M_{2,0}$
	(1,1)	$s_1 s_2$, $M_{1,1}$
3	(3,0)	s_1^3, $s_1 M_{2,0}$, $M_{3,0}$
	(2,1)	$s_1^2 s_2$, $s_1 M_{1,1}$, $s_2 M_{2,0}$, $M_{2,1}$
4	(4,0)	s_1^4, $s_1 M_{3,0}$, $\frac{1}{2}M_{2,0}^2 + M_{2,0}s_1^2$
	(3,1)	$s_1^3 s_2$, $M_{2,0}M_{1,1} + M_{1,1}s_1^2 + M_{2,0}s_1 s_2$, $M_{2,1}s_1$, $M_{3,0}s_2$
	(2,2)	$s_1^2 s_2^2$, $M_{2,0}M_{0,2} + M_{0,2}s_1^2 + M_{2,0}s_2^2$, $\frac{1}{2}M_{1,1}^2 + M_{1,1}s_1 s_2$, $M_{1,2}s_1$, $M_{2,1}s_2$
5	(5,0)	s_1^5, $M_{2,0}M_{3,0} + \frac{3}{2}M_{2,0}^2 s_1 + M_{3,0}s_1^2 + M_{2,0}s_1^3$
	(4,1)	$s_1^4 s_2$, $4M_{3,0}M_{1,1} + 4M_{1,1}s_1^3 + 4M_{3,0}s_1 s_2 + 12M_{2,0}M_{1,1}s_1 + 3M_{2,0}^2 s_2 + 6M_{2,0}M_{2,1} + 6M_{2,1}s_1^2 + 6M_{2,0}s_1^2 s_2$
	(3,2)	$s_1^3 s_2^2$, $M_{3,0}M_{0,2} + 6M_{2,1}M_{1,1} + 3M_{1,2}M_{2,0} + 3M_{2,0}M_{0,2}s_1 + 6M_{1,1}^2 s_1 + 6M_{2,0}M_{1,1}s_2 + M_{3,0}s_2^2 + 6M_{2,1}s_1 s_2 + 3M_{1,2}s_1^2 + M_{0,2}s_1^3$ $+ 6M_{1,1}s_1^2 s_2 + 3M_{2,0}s_1 s_2^2$
6	(6,0)	s_1^6, $\frac{1}{2}M_{3,0}^2 + 3M_{3,0}M_{2,0}s_1 + M_{3,0}s_1^3$
	(5,1)	$s_1^5 s_2$, $M_{3,0}M_{2,1} + 2M_{3,0}M_{1,1}s_1 + 3M_{2,0}M_{2,1}s_1 + M_{3,0}M_{2,0}s_2 + M_{2,1}s_1^3 + M_{3,0}s_1^2 s_2$
	(4,2)	$s_1^4 s_2^2$, $M_{3,0}M_{1,2} + M_{3,0}M_{0,2}s_1 + 3M_{1,2}M_{2,0}s_1 + 2M_{3,0}M_{1,1}s_2 + M_{1,2}s_1^3 + M_{3,0}s_1 s_2^2$, $\frac{1}{2}M_{2,1}^2 + 2M_{2,1}M_{1,1}s_1 + M_{2,1}M_{2,0}s_2 + M_{2,1}s_1^2 s_2$
	(3,3)	$s_1^3 s_2^3$, $M_{3,0}M_{0,3} + 3M_{2,0}M_{0,3}s_1 + 3M_{3,0}M_{0,2}s_2 + M_{0,3}s_1^3 + M_{3,0}s_2^3$, $M_{2,1}M_{1,2} + 2M_{1,1}M_{1,2}s_1 + M_{0,2}M_{2,1}s_1 + 2M_{2,1}M_{1,1}s_2 + M_{1,2}M_{2,0}s_2$ $+ M_{1,2}s_1^2 s_2 + M_{2,1}s_1 s_2^2$

Each row of this table represents elements of spanning set of functions that satisfies Equation (A5) and $g(l_1S_1, l_2S_2, M(l_1S_1, l_2S_2)) = l_1^{k_1} l_2^{k_2} g(S_1, S_2, M(S_1, S_2))$ for each pair (k_1,k_2) with $k_1 \geq k_2$. The cases of $k_2 > k_1$ are omitted because they are represented by symmetry.

Table A2. Elements of basis of functions that satisfies Equation (A5) for each pair (k_1,k_2)

k_1+k_2	(k_1,k_2)	Elements that satisfies Equation (A5)
1	(1,0)	s_1
2	(2,0)	s_1^2, $M_{2,0}$
	(1,1)	$s_1 s_2$, $M_{1,1}$
3	(3,0)	$s_1^3 + 3s_1 M_{2,0}$, $M_{3,0}$
	(2,1)	$s_1^2 s_2 + 2s_1 M_{1,1} + s_2 M_{2,0}$, $M_{2,1}$
4	(4,0)	$s_1^4 + 6s_1^2 M_{2,0} + 4s_1 M_{3,0} + 3M_{2,0}^2$
	(3,1)	$s_1^3 s_2 + s_2 M_{3,0} + 3(M_{2,0} M_{1,1} + M_{1,1} s_1^2 + M_{2,0} s_1 s_2 + M_{2,1} s_1)$
	(2,2)	$s_1^2 s_2^2 + M_{2,0} M_{0,2} + M_{0,2} s_1^2 + M_{2,0} s_2^2 + 2M_{1,1}^2 + 4M_{1,1} s_1 s_2 + 2M_{1,2} s_1 + 2M_{2,1} s_2$
5	(5,0)	N/A
	(4,1)	N/A
	(3,2)	N/A
6	(6,0)	N/A
	(5,1)	N/A
	(4,2)	N/A
	(3,3)	N/A

Each row of this table represents elements of basis of functions that satisfies the Aggregation Property and $g(l_1 S_1, l_2 S_2, M(l_1 S_1, l_2 S_2)) = l_1^{k_1} l_2^{k_2} g(S_1, S_2, M(S_1, S_2))$ for each pair (k_1,k_2).

References

Allen, F., and D. Gale. "Financial contagion." *Journal of Political Economy,* 108 (2000), 1-33.

Amaya, D., P. Christoffersen, K. Jacobs, and A. Vasquez. "Does realized skewness predict the cross-section of equity returns?" *Journal of Financial Economics,* 118 (2015), 135-167.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys. "Modeling and forecasting realized volatility." *Econometrica,* 71 (2003), 579-625.

Ang, A., R. J. Hodrick, Y. Xing, and X. Zhang. "The cross-section of volatility and expected returns." *Journal of Finance,* 61 (2006), 259-299.

Ang, A., and A. Timmermann. "Regime Changes and Financial Markets." *Annual Review of Financial Economics,* 4 (2012), 313-337.

Bakshi, G., N. Kapadia, and D. Madan. "Stock return characteristics, skew laws, and the differential pricing of individual equity options." *Review of Financial Studies,* 16 (2003), 101-143.

Baur, D. G. "Financial contagion and the real economy." *Journal of Banking & Finance,* 36 (2012), 2680-2692.

Bollerslev, T., M. Gibson, and H. Zhou. "Dynamic estimation of volatility risk premia and investor risk aversion from option-implied and realized volatilities." *Journal of Econometrics,* 160 (2011), 235-245.

Boyer, B., T. Mitton, and K. Vorkink. "Expected idiosyncratic skewness." *Review of Financial Studies,* 23 (2010), 169-202.

Carr, P., and L. Wu. "Variance risk premiums." *Review of Financial Studies,* 22 (2009), 1311-1341.

Cespa, G., and T. Foucault. "Illiquidity contagion and liquidity crashes." *Review of Financial Studies,* 27 (2014), 1615-1660.

Conrad, J., R. F. Dittmar, and E. Ghysels. "Ex ante skewness and expected stock returns." *Journal of Finance,* 68 (2013), 85-124.

Dittmar, R. F. "Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns." *Journal of Finance,* 57 (2002), 369-403.

Engle, R. F. "Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation." *Econometrica,* (1982), 987-1007.

Fama, E. F., and K. R. French. "Common risk factors in the returns on stocks and bonds." *Journal of Financial Economics,* 33 (1993), 3-56.

Forbes, K. J., and R. Rigobon. "No contagion, only interdependence: measuring stock market comovements." *Journal of Finance,* 57 (2002), 2223-2261.

Frazzini, A., and L. H. Pedersen. "Betting against beta." *Journal of Financial Economics,* 111 (2014), 1-25.

Ghose, D., and K. F. Kroner. "The relationship between GARCH and symmetric stable processes: Finding the source of fat tails in financial data." *Journal of Empirical Finance,* 2 (1995), 225-251.

Harvey, C. R., and A. Siddique. "Conditional skewness in asset pricing tests." *Journal of Finance,* 55 (2000), 1263-1295.

Jiang, G. J., and Y. S. Tian. "The model-free implied volatility and its information content." *Review of Financial Studies,* 18 (2005), 1305-1342.

Kempf, A., O. Korn, and S. Saßning. "Portfolio Optimization Using Forward-Looking Information." *Review of Finance,* 19 (2015), 467-490.

Kraus, A., and R. H. Litzenberger. "Skewness preference and the valuation of risk assets." *Journal of Finance,* 31 (1976), 1085-1100.

Neuberger, A. "Realized skewness." Working Paper, Warwick University (2011).

—. "Realized skewness." *Review of Financial Studies,* 25 (2012), 3423-3455.