On convex functions on the duals of Δ_2-Orlicz spaces

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ABSTRACT

In the dual L^{Φ^*} of a Δ_2-Orlicz space L^{Φ}, we show that a proper (resp. finite) convex function is lower semicontinuous (resp. continuous) for the Mackey topology $\tau(L^{\Phi^*}, L^{\Phi})$ if and only if on each order interval $[-\zeta, \zeta] = \{\xi : -\zeta \leq \xi \leq \zeta\}$ ($\zeta \in L^{\Phi^*}$), it is lower semicontinuous (resp. continuous) for the topology of convergence in probability. For this purpose, we provide the following Komlós type result: every norm bounded sequence $(\xi_n)_n$ in L^{Φ^*} admits a sequence of forward convex combinations $\bar{\xi}_n \in \mathrm{conv}(\xi_n, \xi_{n+1}, ...)$ such that $\sup_n |\bar{\xi}_n| \in L^{\Phi^*}$ and $\bar{\xi}_n$ converges a.s.

Key Words: Orlicz spaces, Mackey topology, Komlós's theorem, convex functions, order closed sets, risk measures

1 Introduction

Notation. We use the usual probabilistic notation. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ stands for the space of (classes modulo equality $\mathbb{P}$-a.s. of) finite measurable functions equipped with the complete metrisable vector topology τ_{L^0} of convergence in $\mathbb{P}$ (in probability). As usual, we identify a measurable function with the class it generates. We write $\mathbb{E}[\xi] := \int_\Omega \xi d\mathbb{P}$ whenever it makes sense, and $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty]$, denote the standard Lebesgue spaces.

Problems in financial mathematics often involve convex functions on the dual E' of a Banach space E. Dealing with such f, the lower semicontinuity (lsc) and continuity for the Mackey topology $\tau(E', E)$ are basic; the former ($\Leftrightarrow \sigma(E', E)$-lsc) is necessary and sufficient (by the Hahn-Banach theorem) for the dual representation

$$f(x') = \sup_{x \in E}(\langle x, x' \rangle - f^*(x)),\ x' \in E'; \quad \text{where } f^*(x) = \sup_{x' \in E'}(\langle x, x' \rangle - f(x'))$$

Generally speaking, $\tau(E', E)$ is not easy to deal with, but its restrictions to bounded sets often have a nice description. The best known case is $L^\infty = (L^1)'$: on bounded sets, $\tau(L^\infty, L^1)$ coincides with the topology of L^0, a fortiori metrisable (this result is due to Grothendieck; see [11], pp.222-223). Hence by the Krein-Šmulian theorem, we have

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Proposition 1.1. *For proper convex functions f on L^∞, the following are equivalent:*

(1) f is $\sigma(L^\infty, L^1)$-lsc, equivalently $\tau(L^\infty, L^1)$-lsc;
(2) f is sequentially $\tau(L^\infty, L^1)$-lsc;
(3) f is lsc on bounded sets for the topology of convergence in probability.

The following result for the $\tau(L^\infty, L^1)$-continuity is also known for convex risk measures (e.g. [12, 6]), and it remains true for finite convex functions; but we could not find a relevant reference, so we include a short proof in the Appendix.

Proposition 1.2. *For any convex function $f : L^\infty \to \mathbb{R}$, the following are equivalent:*

(1) f is $\tau(L^\infty, L^1)$-continuous;
*(2) f is **sequentially** $\tau(L^\infty, L^1)$-continuous;*
(3) f is continuous for the topology of convergence in probability on bounded sets.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a (finite coercive) ***Young function***, i.e. an even convex function with $\Phi(0) = 0$ and $\lim_{x\to+\infty} \frac{\Phi(x)}{x} = +\infty$. Then the associated Orlicz space

$$L^\Phi := \{\xi \in L^0 : \exists\lambda > 0 \text{ with } \mathbb{E}[\Phi(\lambda|\xi|)] < \infty\} = \textstyle\bigcup_n n\mathbb{B}_\Phi,$$

where $\mathbb{B}_\Phi := \{\xi \in L^0 : \mathbb{E}[\Phi(|\xi|)] \leq 1\}$, endowed with $\|\xi\|_\Phi := \inf\{\lambda > 0 : \xi \in \lambda\mathbb{B}_\Phi\}$ is a Banach lattice with the closed unit ball $\mathbb{B}_\Phi$ and the ***a.s. pointwise order***. The conjugate $\Phi^*(y) := \sup_x(xy - \Phi(x))$ is again a (finite coercive) Young function, so the Orlicz space L^{Φ^*} is similarly defined. A Young function Φ is said to satisfy the ***Δ_2-condition***, written $\Phi \in \Delta_2$, if $\limsup_{x\to\infty} \Phi(2x)/\Phi(x) < \infty$, or equivalently

$$p_\Phi := \inf_{x\geq 0} p_\Phi(x) := \inf_{x\geq 0}\left(\sup_{y>x} \frac{y\Phi'(y)}{\Phi(y)}\right) < \infty, \tag{1.1}$$

where Φ' is the left-derivative of Φ (see [19], Th. II.2.3). In this case, the norm dual $(L^\Phi)'$ of L^Φ is isometrically isomorphic to L^{Φ^*} with bilinear form $\langle \xi, \eta\rangle = \mathbb{E}[\xi\eta]$ and the norm $\|\xi\|_{(\Phi^*)} := \sup_{\eta\in\mathbb{B}_\Phi} \mathbb{E}[\eta\xi]$ which is equivalent to $\|\cdot\|_{\Phi^*}$; more precisely $\|\xi\|_{\Phi^*} \leq \|\xi\|_{(\Phi^*)} \leq 2\|\xi\|_{\Phi^*}$, and $\mathbb{E}[\eta\xi] \leq \|\eta\|_\Phi\|\xi\|_{(\Phi^*)}$. In particular, if $\Phi^* \in \Delta_2$ as well, L^Φ is reflexive; the condition is also necessary if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. In the sequel, unless otherwise mentioned, we suppose $\Phi \in \Delta_2$.

Our basic question is whether the $\tau(L^{\Phi^*}, L^\Phi)$-lower semicontinuity and continuity of convex functions are still characterised by sequential convergence in probability on bounded sets. At this point, we note that there are two possible interpretations of "bounded sets"; norm bounded sets, and ***order bounded*** sets, that is, those $A \subset L^{\Phi^*}$ contained in an ***order interval*** $[-\zeta, \zeta] := \{\xi : -\zeta \leq \xi \leq \zeta\}$, $\zeta \in L^{\Phi^*}_+$, i.e. ***dominated in*** L^{Φ^*}. Every order bounded set is norm bounded, and in L^∞, the two notions of boundedness coincide. This paper is concerned with the latter; we ask if the Mackey lsc and continuity of convex functions f on L^{Φ^*} are characterised by sequential ***dominated*** convergence in $\mathbb{P}$, or explicitly if those are equivalent respectively to

$$\xi_n \to \xi \text{ in } \mathbb{P} \text{ and } \exists\zeta \in L^{\Phi^*}_+ \text{ with } \forall n, |\xi_n| \leq \zeta \;\Rightarrow\; f(\xi) \leq \liminf_n f(\xi_n); \tag{1.2}$$

$$\xi_n \to \xi \text{ in } \mathbb{P} \text{ and } \exists\zeta \in L^{\Phi^*}_+ \text{ with } \forall n, |\xi_n| \leq \zeta \;\Rightarrow\; f(\xi) = \lim_n f(\xi_n). \tag{1.3}$$

The question of lower semicontinuity is equivalent to ask if the following is sufficient for the $\sigma(L^{\Phi^*}, L^{\Phi})$-closedness of convex sets $C \subset L^{\Phi^*}$ (the necessity is clear):

$$\forall \zeta \in L^{\Phi^*},\ C \cap [-\zeta, \zeta] \text{ is closed in } L^0. \tag{1.4}$$

Note that (1.2) (resp. (1.3)) is equivalent to the ***order lower semicontinuity*** (resp. ***continuity***) in the ***Riesz space*** L^{Φ^*}, and (1.4) is the ***order closedness*** in L^{Φ^*}; we can replace the convergence in $\mathbb{P}$ by a.s. convergence, and since L^{Φ^*} is super Dedekind complete, the order lsc, continuity and closedness are the same as the sequential ones, while the sequential order convergence is the dominated a.s. convergence. Also, L^{Φ} is isomorphic to the Riesz space of order continuous linear functionals on L^{Φ^*} (order continuous dual), so the question of lower semicontinuity is asking if every order lsc proper convex function on L^{Φ^*} is represented by ***order continuous*** linear functionals.

This question was first asked by [5] in the context of convex risk measures. They argued that the weak* topology $\sigma(L^{\Phi^*}, L^{\Phi})$ has the following property:

$$\xi_\alpha \overset{\sigma(L^{\Phi^*}, L^{\Phi})}{\longrightarrow} \xi \Rightarrow \begin{cases} \exists(\alpha_n)_n,\ \exists \zeta_n \in \mathrm{conv}(\xi_{\alpha_n}, \xi_{\alpha_{n+1}}, \dots) \text{ s.t.} \\ \sup_n |\zeta_n| \in L^{\Phi^*} \text{ and } \zeta_n \to \xi \text{ a.s.,} \end{cases} \tag{C}$$

and concluded that (1.4) implies the $\sigma(L^{\Phi^*}, L^{\Phi})$-closedness. Unfortunately, this is not correct; (C) holds (if and) ***only if*** L^{Φ} is reflexive ([10]). For, $(\zeta_n)_n$ in (C) converges in $\sigma(L^{\Phi^*}, L^{\Phi})$, thus ($C$) would imply that every point in the weak* closure $\bar{C}^*$ is a ***sequential*** weak* limit of points of C, i.e. $\bar{C}^*$ coincides with the ***sequential weak* closure*** $C^{(1)} := \{\xi : \xi = w^*\text{-}\lim_n \xi_n \text{ with } (\xi_n)_n \subset C\}$. On the other hand, every non-reflexive Banach space admits a convex set C in the dual with $\bar{C}^* \supsetneq C^{(1)}$ ([18, Th. 2], see also [17] for the history of problem of sequential weak* closures which goes back to Banach [4]).

Nevertheless, we shall show that convex sets $C \subset L^{\Phi^*}$ satisfying (1.4) are indeed $\sigma(L^{\Phi^*}, L^{\Phi})$-closed. For this purpose, we give a Komlós type result with an extra property that a resulting sequence of convex combinations is ***order bounded*** (Theorem 3.6 and its practical version Corollary 3.10), which serves as a substitute for (C), proving the claim on closedness. Consequently, $\tau(L^{\Phi^*}, L^{\Phi})$-lsc of a proper convex function on L^{Φ^*} is indeed equivalent to (1.2). We also prove the equivalence of $\tau(L^{\Phi^*}, L^{\Phi})$-continuity and (1.3).

2 Mackey Topology on Orlicz Spaces

The following criterion for $\sigma(L^{\Phi}, L^{\Phi^*})$-compact sets is known (e.g. [19], Th. IV.5.1), but we include a short proof in the Appendix. Here the Δ_2-condition is not necessary.

Lemma 2.1. *(Regardless of $\Phi \in \Delta_2$,) a set $A \subset L^{\Phi}$ is relatively $\sigma(L^{\Phi}, L^{\Phi^*})$-compact if and only if for each $\xi \in L^{\Phi^*}$, $A\xi := \{\eta\xi : \eta \in A\}$ is uniformly integrable.*

Lemma 2.2. *$\tau(L^{\Phi^*}, L^{\Phi})$ is finer than the restriction of τ_{L^0} to L^{Φ^*}, and*

$$\forall \zeta \in L^{\Phi^*},\ \tau(L^{\Phi^*}, L^{\Phi})|_{[-\zeta,\zeta]} = \tau_{L^0}|_{[-\zeta,\zeta]}. \tag{2.1}$$

In particular, $\tau(L^{\Phi^*}, L^{\Phi})$ is metrisable on order bounded sets. If $\Phi \in \Delta_2$, we have

$$\sigma(L^{\Phi^*}, L^{\Phi})|_{\mathbb{B}_{\Phi^*}} \subset \tau_{L^0}|_{\mathbb{B}_{\Phi^*}} \subset \tau(L^{\Phi^*}, L^{\Phi})|_{\mathbb{B}_{\Phi^*}}. \tag{2.2}$$

Proof. The (image in L^Φ) of $\mathbb{B}_{L^\infty}$ is $\sigma(L^\Phi, L^{\Phi^*})$-compact, which defines a Mackey continuous seminorm $\xi \mapsto \sup_{\eta\in\mathbb{B}_{L^\infty}} |\mathbb{E}[\xi\eta]| = \mathbb{E}[|\xi|] \geq \mathbb{E}[|\xi| \wedge 1]$, thus $\tau(L^{\Phi^*}, L^\Phi)$ is finer than the restriction of τ_{L^0}. On the other hand, if a sequence $(\xi_n)_n$ is dominated by $\zeta \in L^{\Phi^*}$ and null in $\mathbb{P}$, then for any $\sigma(L^\Phi, L^{\Phi^*})$-compact set $A \subset L^\Phi$, one has $\sup_{\eta\in A} \mathbb{P}(|\eta| \vee |\zeta| > N) \leq \frac{1}{N}(\sup_{\eta\in A} \|\eta\|_\Phi + \|\zeta\|_{\Phi^*}) \to 0$, and

$$p_A(\xi_n) := \sup_{\eta\in A} |\mathbb{E}[\eta\xi_n]| \leq \sup_{\eta\in A} \mathbb{E}[|\eta\zeta| \mathbb{1}_{\{|\eta|\vee|\zeta|>N\}}] + N^2\mathbb{E}[|\xi_n| \wedge 1],\ \forall N \in \mathbb{N}.$$

Using a standard diagonalisation procedure, we have $p_A(\xi_n) \to 0$, and we see that $\tau(L^{\Phi^*}, L^\Phi)|_{[-\zeta,\zeta]} \subset \tau_{L^0}|_{[-\zeta,\zeta]}$ since τ_{L^0} is metrisable. Finally, if $\Phi \in \Delta_2$, so $L^{\Phi^*} = (L^\Phi)'$, $\mathbb{B}_{\Phi^*}$ is $\sigma(L^{\Phi^*}, L^\Phi)$-compact, thus $\eta\mathbb{B}_{\Phi^*}$, $\eta \in L^\Phi$, are uniformly integrable. Thus $\xi_n \in \mathbb{B}_{\Phi^*}$ and $\xi_n \to \xi$ in $\mathbb{P}$ imply $\mathbb{E}[\eta\xi_n] \to \mathbb{E}[\eta\xi]$ ($\forall\eta \in L^\Phi$), i.e. $\xi_n \to \xi$ in $\sigma(L^{\Phi^*}, L^\Phi)$; thus we have (2.2). □

Remark 2.3. On $\mathbb{B}_{\Phi^*}$, $\tau(L^{\Phi^*}, L^\Phi)$ is not generally the same as the topology of L^0. For example, if $A_n \in \mathcal{F}$ are disjoint with $\mathbb{P}(A_n) > 0$, $\xi_n = \mathbb{P}(A_n)^{-1/2}\mathbb{1}_{A_n}$ form a sequence in $\mathbb{B}_{L^2}$, null in $\mathbb{P}$, but $\|\xi_n\|_2 \equiv 1$, while $\tau(L^2, L^2)$ is the norm topology. ♦

Proposition 2.4. *If $\Phi \in \Delta_2$, the following are equivalent for all convex $C \subset L^{\Phi^*}$:*

(1) C is $\sigma(L^{\Phi^*}, L^\Phi)$-closed;
*(2) C is **sequentially** $\sigma(L^{\Phi^*}, L^\Phi)$-closed;*
(3) for each $\lambda > 0$, $C \cap \lambda\mathbb{B}_{\Phi^*}$ is closed in L^0, or equivalently, $\xi_n \in C$ ($\forall n$), $\xi_n \to \xi$ in $\mathbb{P}$ and $\sup_n \|\xi_n\|_{\Phi^*} < \infty$ imply $\xi \in C$.

Proof. By the Krein-Šmulian theorem, (1) ⇔ $C \cap \lambda\mathbb{B}_{\Phi^*}$, $\lambda > 0$, are $\sigma(L^{\Phi^*}, L^\Phi)$-closed, and the three kinds of closedness are the same for $C \cap \lambda\mathbb{B}_{\Phi^*}$ by (2.2). □

3 Main Results

In the sequel, we suppose without mention that $\Phi \in \Delta_2$ so that $L^{\Phi^*} = (L^\Phi)'$.

3.1 Komlós-Type Results

The classical Komlós theorem [13] asserts that any norm bounded sequence $(\xi_n)_n$ in L^1 admits a subsequence $(\xi_{n_k})_k$ as well as a $\xi \in L^1$ such that for any further subsequence $(\xi_{n_{k(i)}})_i$, the sequence of ***Cesàro means*** $\frac{1}{N}\sum_{i\leq N} \xi_{n_{k(i)}}$ converges a.s. to ξ. Note that any norm bounded sequence in L^{Φ^*} is bounded in L^1. The core of our analysis consists of a few ramifications of Komlós's theorem using the stronger boundedness in L^{Φ^*}.

We start with some preliminary observations. Recall that $\Phi \in \Delta_2$ if and only if $p_\Phi = \inf_{x\geq 0} p_\Phi(x) < \infty$ where $p_\Phi(x) = \sup_{y>x} \frac{y\Phi'(y)}{\Phi(y)}$ (see (1.1)). Let $q_\Phi := \lim_{x\to\infty} \frac{p_\Phi(x)}{p_\Phi(x)-1} = \frac{p_\Phi}{p_\Phi - 1} > 1$ with the convention $1/0 = \infty$.

Lemma 3.1. *For any* $1 \le q < q_\Phi$, L^{Φ^*} *has* ***an upper q-estimate****, that is, there exists a constant* $C_{q,\Phi^*} > 0$ *such that for any* $n \in \mathbb{N}$ *and disjointly supported* $\xi_1, ..., \xi_n \in L^{\Phi^*}$ *(i.e.* $\xi_k = \xi_k \mathbb{1}_{A_k}$ *with* $A_k \in \mathcal{F}$ *pairwise disjoint),*

$$\left\| \sum_{k\le n} \xi_k \right\|_{(\Phi^*)} \le C_{q,\Phi^*} \left(\sum_{k\le n} \|\xi_k\|^q_{(\Phi^*)} \right)^{1/q}. \tag{3.1}$$

Proof. The case $q = 1$ is trivial (we can take $C_{q,\Phi^*} = 1$), and note that $1 < q < \frac{p_\Phi}{p_\Phi - 1}$ if and only if $q = \frac{p}{p-1}$ for some $p \in (p_\Phi(x_0), \infty)$ and $x_0 > 0$. Fix such q, p and x_0. Then $\Psi(x) := \frac{\Phi(x_0)}{x_0} x \mathbb{1}_{[0,x_0]}(x) + \Phi(x)\mathbb{1}_{(x_0,\infty)}(x)$ is a Δ_2-Young function with $\Psi(x) = \Phi(x)$ for $x \ge x_0$, thus $L^\Psi = L^\Phi$ with equivalent norms (since $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, see [19], Th. V.1.3); hence there exists a $C > 0$ such that

$$C^{-1}\|\cdot\|_{(\Psi^*)} \le \|\cdot\|_{(\Phi^*)} \le C\|\cdot\|_{(\Psi^*)}. \tag{3.2}$$

Moreover, $\Psi(x) > 0$ for $x > 0$ and $p_\Psi(0) = 1 \vee p_\Phi(x_0) = p_\Phi(x_0) < p < \infty$; in particular, for any $\lambda \ge 1$ and $x > 0$, $\log \frac{\Psi(\lambda x)}{\Phi(x)} = \int_1^\lambda \frac{tx\Psi'(tx)}{\Psi(tx)} \frac{dt}{t} \le p \log \lambda$, hence

$$\Psi(\lambda x) \le \lambda^p \Psi(x) \text{ for } x > 0,\ \lambda \ge 1. \tag{3.3}$$

Therefore $1 = \mathbb{E}[\Psi(\eta/\|\eta\|_\Psi)] \le \|\eta\|_\Psi^{-p}\mathbb{E}[\Psi(\eta)]$ for $0 < \|\eta\|_\Psi \le 1$, where the first equality is another consequence of $\Phi \in \Delta_2$. Hence we have

$$\|\eta\|_\Psi \le \mathbb{E}[\Psi(\eta)]^{1/p} \text{ for all } \eta \in \mathbb{B}_\Psi. \tag{3.4}$$

Now if $\xi_k = \xi_k \mathbb{1}_{A_k} \in L^{\Phi^*} = L^{\Psi^*}$ with $A_k \in \mathcal{F}$ disjoint, then for any $\eta \in \mathbb{B}_\Psi$,

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k\le n} \xi_k\right)\eta\right] &\le \sum_{k\le n} \|\xi_k\|_{(\Psi^*)} \|\eta \mathbb{1}_{A_k}\|_\Psi \overset{(3.4)}{\le} \sum_{k\le n} \|\xi_k\|_{(\Psi^*)} \mathbb{E}[\Psi(\eta)\mathbb{1}_{A_k}]^{1/p} \\ &\le \left(\sum_{k\le n} \|\xi_k\|^q_{(\Psi^*)}\right)^{1/q} \left(\sum_{k\le n} \mathbb{E}[\Psi(\eta)\mathbb{1}_{A_k}]\right)^{1/p} \le \left(\sum_{k\le n} \|\xi_k\|^q_{(\Psi^*)}\right)^{1/q},\end{aligned}$$

since $\sum_{k\le n} \mathbb{E}[\Psi(\eta)\mathbb{1}_{A_k}] \le \mathbb{E}[\Psi(\eta)] \le 1$. Taking the supremum over $\eta \in \mathbb{B}_\Psi$,

$$\frac{1}{C}\left\|\sum_{k\le n} \xi_k\right\|_{(\Phi^*)} \overset{(3.2)}{\le} \left\|\sum_{k\le n} \xi_k\right\|_{(\Psi^*)} \le \left(\sum_{k\le n} \|\xi_k\|^q_{(\Psi^*)}\right)^{\frac{1}{q}} \overset{(3.2)}{\le} C\left(\sum_{k\le n} \|\xi_k\|^q_{(\Phi^*)}\right)^{\frac{1}{q}}.$$

□

Corollary 3.2. *If* $(\xi_n)_n$ *is a norm bounded disjointly supported sequence in* L^{Φ^*}*, then*

$$\sup_n \left|\frac{\xi_1 + \cdots + \xi_n}{n}\right| \in L^{\Phi^*} \quad \textit{and} \quad \left\|\frac{\xi_1 + \cdots + \xi_n}{n}\right\|_{\Phi^*} \to 0.$$

Proof. Let $\xi_n = \xi_n \mathbb{1}_{A_n}$ with $A_n \in \mathcal{F}$ disjoint, $a := \sup_n \|\xi_n\|_{(\Phi^*)} < \infty$, $1 < q < q_\Phi$, and $C = C_{q,\Phi^*}$ as in Lemma 3.1. Put $\bar{\xi}_n := \frac{\xi_1 + \cdots + \xi_n}{n}$. Then $\|\bar{\xi}_n\|_{(\Phi^*)} \le aC\,(nn^{-q})^{1/q} = aCn^{\frac{1}{q}-1} \to 0$. Next, observe that

$$\sup_{n\le N} |\bar{\xi}_n| = \sum_{k\le N}\left(\sup_{n\le N} |\bar{\xi}_n|\right)\mathbb{1}_{A_k} = \sum_{k\le N}\left(\sup_{k\le n\le N} \frac{1}{n}|\xi_k|\right)\mathbb{1}_{A_k} = \sum_{k\le N} \frac{1}{k}|\xi_k|,$$

while $\left\|\sum_{k\le N} \frac{1}{k}\xi_n\right\|_{(\Phi^*)} \le aC\left(\sum_{k\le N} \frac{1}{k^q}\right)^{1/q} \le aC\left(\sum_{k=1}^\infty \frac{1}{k^q}\right)^{1/q} < \infty$, hence $\|\sup_n |\bar{\xi}_n|\|_{(\Phi^*)} = \sup_N \|\sup_{n\le N} |\bar{\xi}_n|\|_{(\Phi^*)} < \infty$. □

A sequence $(\bar{\xi}_n)_n$ is called a sequence of ***forward convex combinations of*** $(\xi_n)_n$ if $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \geq n)$ for each n. Noting that $\bar{\xi}_n = \frac{\xi_{n+1}+\cdots+\xi_{2n}}{n} = 2\frac{\xi_1+\cdots+\xi_{2n}}{2n} - \frac{\xi_1+\cdots+\xi_n}{n} \in \mathrm{conv}(\xi_n, \xi_{n+1}, ...)$, we get:

Corollary 3.3. *Any norm bounded disjoint sequence $(\xi_n)_n$ in L^{Φ^*} has an order bounded and norm null sequence of forward convex combinations $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \geq n)$.*

Since any subsequence of norm bounded disjoint sequence is again bounded and disjoint, the same conclusion holds for any subsequence; thus

Corollary 3.4. *Any norm bounded disjoint sequence in L^{Φ^*} is $\sigma(L^{\Phi^*}, (L^{\Phi^*})')$-null.*

Remark 3.5. The last two corollaries could be derived also from the fact that the norm dual E' of a Banach lattice E has order continuous norm iff every norm bounded disjoint sequence in E is $\sigma(E, E')$-null ([20, Th. 116.1] or [14, Th. 2.4.14]). In L^{Φ^*}, $(\xi_n)_n$ is disjoint in the lattice sense iff it is disjointly supported, while $(L^{\Phi^*})' = L^\Phi \oplus (L^{\Phi^*})^\sim_s$, where $(L^{\Phi^*})^\sim_s$ is the band of singular linear functionals. The projections of $(L^{\Phi^*})'$ onto L^Φ and onto $(L^{\Phi^*})^\sim_s$ are order continuous (e.g. [3]). But $(L^{\Phi^*})^\sim_s$ is an AL space, hence has order continuous norm (regardless of Δ_2; e.g. [20, Th. 133.6]), thus $\Phi \in \Delta_2$ implies that $\|\cdot\|_{(L^{\Phi^*})'}$ is order continuous ($\Leftrightarrow \|\cdot\|_{(L^{\Phi^*})'}|_{L^\Phi} = \|\cdot\|_{(\Phi)}$ is order continuous), so any bounded disjoint sequence is weakly null. ♦

We now come to the core of the paper. We first state the basic version of our Komlós type result, then a few (practically more useful) consequences will follow.

Theorem 3.6. *If $(\xi_n)_n$ is a norm bounded sequence in L^{Φ^*}, converging in $\mathbb{P}$ to some $\xi \in L^{\Phi^*}$, then there exists a subsequence $(\xi_{n_k})_k$ such that for any further subsequence $(\xi_{n_{k(i)}})_i$, the Cesàro means $\frac{1}{N}\sum_{k\leq N}\xi_{n_{k(i)}}$ converge in order to ξ, i.e.*

$$\sup_N \left|\frac{1}{N}\sum_{i\leq N}\xi_{n_{k(i)}}\right| \in L^{\Phi^*} \text{ and } \frac{1}{N}\sum_{i\leq N}\xi_{n_{k(i)}} \xrightarrow{N} \xi \text{ a.s.} \tag{3.5}$$

Here the original bounded sequence $(\xi_n)_n$ is supposed to converge in $\mathbb{P}$, which is needed to ensure that the Cesàro means themselves of any subsequence converge in order. Without this a priori assumption, we still have a slightly weaker conclusion.

Theorem 3.7. *Any norm bounded sequence $(\xi_n)_n$ in L^{Φ^*} admits a subsequence $(\xi_{n_k})_k$ as well as $\xi \in L^{\Phi^*}$ such that for any subsequence $(\xi_{n_{k(i)}})_i$, the sequence of Cesàro means $\frac{1}{N}\sum_{k\leq N}\xi_{n_{k(i)}}$ has a subsequence order convergent to ξ, i.e. there is a sequence $(N_l)_l$ with $\sup_l \left|\frac{1}{N_l}\sum_{i\leq N_l}\xi_{n_{k(i)}}\right| \in L^{\Phi^*}$ and $\frac{1}{N_l}\sum_{i\leq N_l}\xi_{n_{k(i)}} \to \xi$ a.s.*

Lemma 3.8 (cf. [16]). *If $\xi_n \to 0$ in $\mathbb{P}$ and if $(\Phi^*(\xi_n))_n$ is uniformly integrable, there exists a subsequence $(\xi_{n_k})_k$ such that $\sup_k |\xi_{n_k}| \in L^{\Phi^*}$.*

Proof. The assumption implies $\mathbb{E}[\Phi^*(\xi_n)] \to 0$, so there is a subsequence $(\xi_{n_k})_k$ such that $\sum_k \mathbb{E}[\Phi^*(\xi_{n_k})] < \infty$. Noting that $\Phi^*(|\eta| \vee |\eta'|) = \Phi^*(\eta)\mathbb{1}_{\{|\eta|>|\eta'|\}} + \Phi^*(\eta')\mathbb{1}_{\{|\eta|\leq|\eta'|\}} \leq \Phi^*(\eta) + \Phi^*(\eta')$, a simple induction and the monotone convergence theorem show that

$$\mathbb{E}\left[\Phi^*\left(\sup_k |\xi_{n_k}|\right)\right] \leq \lim_m \mathbb{E}\left[\Phi^*\left(\sup_{k\leq m}|\xi_{n_k}|\right)\right] \leq \lim_m \sum_{k\leq m}\mathbb{E}[\Phi^*(\xi_{n_k})] \leq \sum_{k=1}^\infty \mathbb{E}[\Phi^*(\xi_{n_k})] < \infty.$$

Hence $\sup_k |\xi_{n_k}| \in L^{\Phi^*}$. □

Proof of Theorems 3.6 and 3.7. Let $(\xi_n)_n$ be a norm bounded sequence in L^{Φ^*} which is also bounded in L^1. In view of Komlós's theorem, we can find a subsequence, still denoted by $(\xi_n)_n$, and a $\xi \in L^1$ such that the Cesàro means of any further subsequence converges a.s. to ξ; then $\xi \in L^{\Phi^*}$ by Fatou's lemma. We can normalise $(\xi_n)_n$ so that $\xi = 0$ and $\|\xi_n\|_{\Phi^*} \leq 1$ ($\Leftrightarrow \mathbb{E}[\Phi^*(\xi_n)] \leq 1$). Then the Kadec–Pełczyński theorem (e.g. [1, Lemma 5.2.8]) applied to the bounded sequence $(\Phi^*(\xi_n))_n$ yields a subsequence $(\zeta_n)_n$ of (ξ_n) as well as a disjoint sequence $(A_n)_n$ in $\mathcal{F}$ such that $(\Phi^*(\zeta_n \mathbb{1}_{A_n^c}))_n$ is uniformly integrable. Let $\zeta_n^r := \zeta_n \mathbb{1}_{A_n^c}$ and $\zeta_n^s := \zeta_n \mathbb{1}_{A_n}$ so that $\zeta_n = \zeta_n^r + \zeta_n^s$.

Now if the original sequence $(\xi_n)_n$ converges in $\mathbb{P}$ (to 0 by the reduction above), then $(\zeta_n)_n \subset (\xi_n)_n$ as well as $(\zeta^r)_n$ are null in $\mathbb{P}$. Since $(\Phi^*(\zeta_n^r))_n$ is uniformly integrable, Lemma 3.8 yields a subsequence $(n_k)_k$ of positive integers such that $\eta' := \sup_k |\zeta_{n_k}^r| \in L^{\Phi^*}$. On the other hand, $(\zeta_n^s)_n$ (and any of its subsequence) is a norm bounded disjoint sequence, hence Corollary 3.2 shows that for any subsequence $(k(i))_i$,

$$\sup_N \left| \frac{1}{N} \sum_{i \leq N} \zeta_{n_{k(i)}} \right| \leq \sup_N \left| \frac{1}{N} \sum_{i \leq N} \zeta^r_{n_{k(i)}} \right| + \sup_N \left| \frac{1}{N} \sum_{i \leq N} \zeta^s_{n_{k(i)}} \right| \leq \eta' + \sup_N \left| \frac{1}{N} \sum_{i \leq N} \zeta_{n_{k(i)}} \right| \in L^{\Phi^*}.$$

Since $\frac{1}{N} \sum_{i \leq N} \zeta_{n_{k(i)}} \to 0$ a.s. by construction, we have Theorem 3.6.

Next, if $(\zeta_n)_n$ is not null in $\mathbb{P}$, we can no longer hope for a "universal bound" for the regular part $(\zeta_n^r)_n$. However, once a subsequence $(n_k)_k$ is chosen we get

$$\bar{\zeta}_N := \frac{1}{N} \sum_{k \leq N} \zeta_{n_k} = \frac{1}{N} \sum_{k \leq N} \zeta^r_{n_k} + \frac{1}{N} \sum_{k \leq N} \zeta^s_{n_k} =: \bar{\zeta}^r_N + \bar{\zeta}^s_N \to 0 \text{ in } \mathbb{P},$$

by the construction of $(\zeta_n)_n$. Again by Corollary 3.2, $(\bar{\zeta}_N^s)_N$ is order bounded and norm null. In particular, $\bar{\zeta}^r_N = \bar{\zeta}_N - \bar{\zeta}^s_N \to 0$ in $\mathbb{P}$, and $(\Phi^*(\bar{\zeta}^r_N))_N$ is uniformly integrable since Φ^* is convex. Thus by Lemma 3.8, we find a subsequence $(N(i))_i$ such that $(\bar{\zeta}^r_{N(i)})_i$, hence $(\bar{\zeta}_{N(i)})_i = (\bar{\zeta}^r_{N(i)} + \bar{\zeta}^s_{N(i)})_i$ too, are order bounded. □

Note that with the notation and reduction in the proof, any subsequence $(\bar{\zeta}_{N_k})_k$ of $(\bar{\zeta}_N)_N$, not necessarily the one just constructed, inherits the property that $\bar{\zeta}^r_{N_k} \to 0$ in $\mathbb{P}$ and $(\Phi^*(\bar{\zeta}^r_{N_k}))_k$ is uniformly integrable; thus $(\bar{\zeta}_{N_k})_k$ contains a further subsequence that converges in order, hence in $\tau(L^{\Phi^*}, L^{\Phi})$ by (2.1). This yields:

Corollary 3.9. *Any norm bounded sequence $(\xi_n)_n$ in L^{Φ^*} admits a subsequence $(\xi_{n_k})_k$ as well as $\xi \in L^{\Phi^*}$ such that for any further subsequence $(n_{k(i)})_i$, $\frac{1}{N} \sum_{i \leq N} \xi_{n_{k(i)}} \to \xi$ in $\tau(L^{\Phi^*}, L^{\Phi})$.*

Note that convex combinations of convex combinations are convex combinations, and any norm bounded sequence in L^{Φ^*} admits a sequence of forward convex combinations that converges a.s. to some $\xi \in L^{\Phi^*}$ by Komlós's theorem (cf. the first part of the proof of Theorems 3.6 and 3.7). We thus deduce the following practically most useful corollary of Theorem 3.6.

Corollary 3.10. *Any norm bounded sequence $(\xi_n)_n$ in L^{Φ^*} admits a sequence of forward convex combinations $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \geq n)$ as well as a $\xi \in L^{\Phi^*}$ such that $\bar{\xi}_n \to \xi$ in order, i.e. $\sup_n |\bar{\xi}_n| \in L^{\Phi^*}$ and $\bar{\xi}_n \to \xi$ a.s.*

On the other hand, Cesàro means of Cesàro means are not Cesàro means, of course. At the moment, it is not clear if one can drop the a priori assumption of convergence in $\mathbb{P}$ in Theorem 3.6, or equivalently whether the Cesàro means in Theorem 3.7 are order bounded without passing to further subsequence. This question will be studied in future work.

3.2 Closedness of Convex Sets

Now we deduce from Corollary 3.10 that

Theorem 3.11. *A **convex** subset $C \subset L^{\Phi^*}$ is $\sigma(L^{\Phi^*}, L^{\Phi})$-closed if and only if for every $\zeta \in L^{\Phi^*}$, the intersection $C \cap [-\zeta, \zeta]$ is closed in L^0 (i.e. order closed).*

Proof. The necessity is clear since $[-\zeta, \zeta]$ is closed in L^0 and $\tau(L^{\Phi^*}, L^{\Phi})|_{[-\zeta,\zeta]} = \tau_{L^0}|_{[-\zeta,\zeta]}$. For the sufficiency, it suffices that for each $\lambda > 0$, $C \cap \lambda\mathbb{B}_{\Phi^*}$ is closed in L^0 (Proposition 2.4). Take a sequence $(\xi_n)_n$ in $C \cap \lambda\mathbb{B}_{\Phi^*}$ and suppose $\xi_n \to \xi$ in $\mathbb{P}$. Fatou's lemma already implies that $\xi \in \lambda\mathbb{B}_{\Phi^*}$. Corollary 3.10 provides us with a sequence $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \geq n)$ with $\zeta := \sup_n |\bar{\xi}_n| \in L^{\Phi^*}$, and $\bar{\xi}_n \to \xi$ a.s. The new sequence $(\bar{\xi}_n)_n$ lies in $C \cap [-\zeta, \zeta]$ by convexity. But $C \cap [-\zeta, \zeta]$ is τ_{L^0}-closed, hence $\xi \in C \cap [-\zeta, \zeta] \cap \lambda\mathbb{B}_{\Phi^*}$. □

To the best of our knowledge, this criterion for the weak*-closedness is only known for ***solid sets*** (i.e. $A \subset L^{\Phi^*}$ with $\zeta \in A$ and $|\xi| \leq |\zeta| \Rightarrow \xi \in A$); see [2, Th. 4.20]. But convex functions with solid lower level sets are symmetric, so exclude all non-trivial monotone convex functions, especially convex risk measures. Also, since $\sigma(L^{\Phi^*}, L^{\Phi})|_{[-\zeta,\zeta]} \subset \tau_{L^0}|_{[-\zeta,\zeta]} = \tau(L^{\Phi^*}, L^{\Phi})|_{[-\zeta,\zeta]}$, $\zeta \in L^{\Phi^*}$ (by (2.1) and (2.2)), the condition is also equivalent to: $C \cap [-\zeta, \zeta]$, $\zeta \in L^{\Phi^*}$, are $\sigma(L^{\Phi^*}, L^{\Phi})$-closed.

Remark 3.12. After our results were presented in Vienna Congress on Mathematical Finance, 12–14 September 2016 (https://fam.tuwien.ac.at/events/vcmf2016/), and after a discussion with Niushan Gao, he and his collaborators [9] came up with their own proof of Theorem 3.11. They used a different technique which in our opinion will not yield a Komlós type theorem. The problem to get a Komlós type theorem was suggested by Hans Föllmer during the aforementioned Vienna conference. ♦

While the Mackey and weak* closed convex sets in the dual of a Banach space are the same, ***sequentially*** Mackey closed convex sets need not be (sequentially) weak* closed. For instance, $A = \{(\alpha_n)_n \in \ell^1 : \alpha_1 = \sum_{n\geq 2} \alpha_n\}$ is norm closed but not sequentially weak* closed in $\ell^1 = (c_0)'$ (see [4]), while $\tau(\ell^1, c_0)$ and norm convergences are equivalent for sequences; thus A is sequentially Mackey closed. In our situation, however, since $\tau(L^{\Phi^*}, L^{\Phi})|_{[-\zeta,\zeta]} = \tau_{L^0}|_{[-\zeta,\zeta]}$, $\zeta \in L^{\Phi^*}$, are metrisable, Theorem 3.11 implies that

Corollary 3.13. ***Sequentially*** *$\tau(L^{\Phi^*}, L^{\Phi})$-closed convex sets in L^{Φ^*} are $\tau(L^{\Phi^*}, L^{\Phi})$-closed.*

Now the dual representation of proper convex functions on L^{Φ^*}, or equivalently the $\sigma(L^{\Phi^*}, L^{\Phi})$-lsc ($\Leftrightarrow \tau(L^{\Phi^*}, L^{\Phi})$-lsc), is characterised as follows.

Theorem 3.14. *For a proper convex function f on L^{Φ^*}, the following are equivalent:*

(1) f is $\sigma(L^{\Phi^*}, L^{\Phi})$-lsc, or equivalently $f(\xi) = \sup_{\eta \in L^{\Phi}}(\mathbb{E}[\eta\xi] - f^*(\eta))$, $\xi \in L^{\Phi^*}$;
*(2) f is **sequentially** $\tau(L^{\Phi^*}, L^{\Phi})$-lsc;*
(3) f is τ_{L^0}-lsc on every order interval $[-\zeta, \zeta]$ ($\zeta \in L^{\Phi^*}$), or equivalently order lsc: $f(\xi) \le \liminf_n f(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L^{Φ^*}, i.e. $\exists \zeta \in L^{\Phi^*}_+$ with $|\xi_n| \le \zeta$ for all n.

For the $\tau(L^{\Phi^*}, L^{\Phi})$-continuity, we have

Theorem 3.15. *For any convex function $f : L^{\Phi^*} \to \mathbb{R}$, the following are equivalent:*

(1) f is $\tau(L^{\Phi^*}, L^{\Phi})$-continuous on L^{Φ^*};
*(2) f is **sequentially** $\tau(L^{\Phi^*}, L^{\Phi})$-continuous on L^{Φ^*};*
*(3) f is **sequentially** $\tau(L^{\Phi^*}, L^{\Phi})$-continuous on **closed balls** $\lambda\mathbb{B}_{\Phi^*}$ ($\lambda > 0$);*
*(4) f is **sequentially** $\tau(L^{\Phi^*}, L^{\Phi})$-continuous on **order intervals**;*
*(5) f is τ_{L^0}-continuous on **order intervals**, or equivalently order continuous, i.e. $f(\xi) = \lim_n f(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L^{Φ^*}.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial; (4) $\Leftrightarrow$ (5) since $\tau(L^{\Phi^*}, L^{\Phi})$ coincides on order bounded sets with τ_{L^0}. Suppose (5). Then, by Theorem 3.14, $f = f^{**}$, so by Moreau's theorem [15], it suffices that each $\Lambda_c := \{\eta \in L^{\Phi} : f^*(\eta) \le c\}$, $c \in \mathbb{R}$, is $\sigma(L^{\Phi}, L^{\Phi^*})$-compact. By Young's inequality, for any $\lambda > 0$, $\xi \in L^{\Phi^*}$ and $\eta \in \Lambda_c$,

$$|\mathbb{E}[\eta\xi\mathbb{1}_A]| = \mathbb{E}[\eta\xi\mathbb{1}_A] \vee \mathbb{E}[\eta(-\xi)\mathbb{1}_A] \le \frac{1}{\lambda}(f(\lambda\xi\mathbb{1}_A) \vee f(-\lambda\xi\mathbb{1}_A) + c) \tag{3.6}$$

which implies that $\Lambda_c\xi$, $\xi \in L^{\Phi^*}$, are uniformly integrable, thus Λ_c is $\sigma(L^{\Phi}, L^{\Phi^*})$-compact. For if $\Lambda_c\xi$ were not uniformly integrable, there would be $\varepsilon > 0$, $A_n \in \mathcal{F}$ and $\eta_n \in \Lambda_c$ such that $\mathbb{P}(A_n) \le 2^{-n}$ and $|\mathbb{E}[\eta_n\xi\mathbb{1}_{A_n}]| \ge \varepsilon$; here note that $\mathbb{E}[|\zeta|\mathbb{1}_A] \ge 2\varepsilon$ implies either $|\mathbb{E}[\zeta\mathbb{1}_{A\cap\{\zeta>0\}}]| \ge \varepsilon$ or $|\mathbb{E}[\zeta\mathbb{1}_{A\cap\{\zeta<0\}}]| \ge \varepsilon$ and $\mathbb{P}(A \cap \{\zeta \gtrless 0\}) \le \mathbb{P}(A)$. But since $|\lambda\xi\mathbb{1}_{A_n}| \le \lambda|\xi|$ and $\lambda\xi\mathbb{1}_{A_n} \to 0$ in $\mathbb{P}$ for each $\lambda > 0$, (5) and (3.6) together with a diagonal argument shows that $|\mathbb{E}[\eta_n\xi\mathbb{1}_{A_n}]| \to 0$, a contradiction. □

The property that f is (sequentially) τ_{L^0}-continuous on every closed ball implies (via (5)) the Mackey continuity of f. The converse implication holds for all finite convex functions if and only if $\tau(L^{\Phi^*}, L^{\Phi})|_{\mathbb{B}_{\Phi^*}} = \tau_{L^0}|_{\mathbb{B}_{\Phi^*}}$. Indeed, seminorms generating the Mackey topology are finite valued Mackey continuous convex functions. As we saw in Remark 2.3, this is not the case if $\Phi(x) = x^2$; more generally, it fails whenever $\Phi^* \in \Delta_2$ (then L^{Φ} is reflexive). Precisely when $\tau(L^{\Phi^*}, L^{\Phi})$ coincide with τ_{L^0} on $\mathbb{B}_{\Phi^*}$ is a subtle question which is left for further investigation.

Remark 3.16. In the proof of (5) $\Rightarrow$ (1), we only used the facts that $f = f^{**}$ and $f|_{[-\zeta,\zeta]}$ is τ_{L^0}-continuous at 0, from which we derived that f is $\tau(L^{\Phi^*}, L^{\Phi})$-continuous at 0. Thus if f is a priori supposed to be $\sigma(L^{\Phi^*}, L^{\Phi})$-lsc on L^{Φ^*} (or any of its equivalent properties in Theorem 3.14), and $f(\xi_0) < \infty$ (we can suppose $\xi_0 = 0$ by translation), the following remain equivalent: (1′) f is $\tau(L^{\Phi^*}, L^{\Phi})$-continuous at ξ_0, (2′) f sequentially $\tau(L^{\Phi^*}, L^{\Phi})$-continuous at ξ_0, (3′) $f(\xi) = \lim_n f(\xi_n)$ whenever $\xi_n \to \xi_0$ in $\tau(L^{\Phi^*}, L^{\Phi})$ and $\sup_n \|\xi_n\|_{\Phi^*} < \infty$, (4′) the same but with $|\xi_n| \le \zeta$ for some $\zeta \in L^{\Phi^*}$, (5′) the same but with $\xi_n \to \xi_0$ in $\mathbb{P}$ and $|\xi_n| \le \zeta$ for some $\zeta \in L^{\Phi^*}_+$. ♦

3.3 Application to Monetary Utility Functions

In utility theory, ***concave*** functions $u : L^{\Phi^*} \to \mathbb{R} \cup \{-\infty\}$ satisfying the following properties are called ***monetary utility functions*** (see e.g. [7, 8]):

$$u(0) = 0;\ \xi \in L^{\Phi^*},\ \xi \geq 0 \ \Rightarrow\ u(\xi) \geq 0; \tag{3.7}$$

$$a \in \mathbb{R},\ \xi \in L^{\Phi^*} \ \Rightarrow\ u(\xi + a) = u(\xi) + a. \tag{3.8}$$

Since $-u$ is a convex function, which is called a ***convex risk measure***, Theorems 3.14 and 3.15 with obvious change of sign characterise the basic regularities of u for the Mackey topology $\tau(L^{\Phi^*}, L^{\Phi})$. (3.7) and (3.8) then give an even better description.

Theorem 3.17. *A monetary utility function $u : L^{\Phi^*} \to \mathbb{R} \cup \{-\infty\}$ is $\sigma(L^{\Phi^*}, L^{\Phi})$-upper semicontinuous (or what is the same, $\tau(L^{\Phi^*}, L^{\Phi})$-upper semicontinuous) if and only if it is continuous from above:*

$$\xi_n \downarrow \xi \ \Rightarrow\ u(\xi) = \lim_n u(\xi_n). \tag{3.9}$$

In this case, the dual representation of u can be written as

$$u(\xi) = \inf\{\mathbb{E}_Q[\xi] + c(Q) : c(Q) < \infty\}, \tag{3.10}$$

where Q runs through probabilities absolutely continuous w.r.t. $\mathbb{P}$ with $dQ/d\mathbb{P} \in L^{\Phi}$, $c(Q) = (-u)^*(-dQ/d\mathbb{P})$ and $\mathbb{E}_Q[\xi] = \mathbb{E}[\xi dQ/d\mathbb{P}]$.

Proof. The necessity is clear from Theorem 3.14 since $\xi_n \downarrow \xi$ implies $\xi_n \to \xi$ in order. For the sufficiency, we first show that (3.7)–(3.9) imply that u is monotone, i.e.

$$\xi, \eta \in L^{\Phi^*},\ \xi \leq \eta \ \Rightarrow\ u(\xi) \leq u(\eta) \tag{3.11}$$

We can suppose $u(\xi) = 0$ thanks to (3.8). For each $\varepsilon \in (0, 1)$, let $\alpha_\varepsilon = (1-\varepsilon)/\varepsilon$ so that $\zeta_\varepsilon := \eta + \varepsilon\xi^- + \alpha_\varepsilon(\eta + \varepsilon\xi^- - \xi) \geq 0$. Putting $\lambda_\varepsilon := \alpha_\varepsilon/(1+\alpha_\varepsilon) \in (0, 1)$, we have $\eta + \varepsilon\xi^- = \lambda_\varepsilon\xi + (1-\lambda_\varepsilon)\zeta_\varepsilon$, hence by the concavity, $u(\eta + \varepsilon\xi^-) \geq \lambda_\varepsilon u(\xi) + (1-\lambda_\varepsilon)u(\zeta_\varepsilon) \geq 0$. Then (3.9) shows that $u(\eta) = \lim_n u(\eta + n^{-1}\xi^-) \geq 0 = u(\xi)$. Now by Theorem 3.14 applied to the ***convex*** function $-u$, the $\sigma(L^{\Phi^*}, L^{\Phi})$-upper semicontinuity of u is equivalent to the property that $u(\xi) \geq \limsup_n u(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L^{Φ^*}; given the monotonicity (3.11) of u, this is equivalent to (3.9). That the dual representation of $f = -u$ together with (3.7) and (3.8) yields (3.10) is standard. □

Note that if u is finite valued ($\mathbb{R}$-valued), (3.7) and (3.8) still imply (3.11) without assuming (3.9). For $\varepsilon \mapsto u(\eta + \varepsilon\xi^-)$ is continuous as a finite valued convex function on $\mathbb{R}$. One can easily see also that any monetary utility function that is $\tau(L^{\Phi^*}, L^{\Phi})$-continuous at 0 is finite valued. For such u, Theorem 3.15 yields that

Theorem 3.18. *A monetary utility function $u : L^{\Phi^*} \to \mathbb{R}$ is $\tau(L^{\Phi^*}, L^{\Phi})$-continuous if (and only if) it is continuous from below, i.e. $\xi_n \uparrow \xi \Rightarrow u(\xi) = \lim_n u(\xi_n)$.*

Proof. Given that u is finite, monotone and convave, the continuity from below implies the continuity from above. For if $\xi_n \downarrow \xi$, then $u(\xi) \geq \frac{1}{2}u(\xi_n) + \frac{1}{2}u(2\xi - \xi_n)$ by the concavity, so the continuity from below and the monotonicity imply $0 \leq u(\xi_n) - u(\xi) \leq u(\xi) - u(2\xi - \xi_n) \downarrow 0$ since $2\xi - \xi_n \uparrow \xi$. In particular, u is $\sigma(L^{\Phi^*}, L^{\Phi})$-usc. On the other hand, again by the monotonicity, the continuity of u from below is equivalent to the property that $u(\xi) = \lim_n u(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L^{Φ^*}. The result now follows from Theorem 3.15. □

Appendix

Proof of Proposition 1.2. Only (3) $\Rightarrow$ (1) deserves a proof. (3) implies, by Proposition 1.1, $f = f^{**}$, and $|\mathbb{E}[\eta \mathbb{1}_A]| \le \frac{1}{n}(f(n\mathbb{1}_A) \vee f(-n\mathbb{1}_A) + c)$ for $A \in \mathcal{F}$ and $\eta \in L^1$ with $f^*(\eta) \le c$ by Young's inequality; thus (3) implies that $\{\eta \in L^1 : f^*(\eta) \le c\}$ is uniformly integrable, hence $\sigma(L^1, L^\infty)$-compact by the Dunford-Pettis theorem. Now Moreau's theorem [15] shows that f is $\tau(L^\infty, L^1)$-continuous. □

Proof of Lemma 2.1. For each $\xi \in L^{\Phi^*}$, the mapping $\eta \mapsto \eta\xi$, $(L^\Phi, \sigma(L^\Phi, L^{\Phi^*})) \to (L^1, \sigma(L^1, L^\infty))$ is continuous since $\xi\zeta \in L^1$, $\forall \zeta \in L^\infty$. Thus if A is relatively $\sigma(L^\Phi, L^{\Phi^*})$-compact, its image $A\xi$ is relatively weakly compact in L^1, i.e. uniformly integrable. For the converse, it suffices that the uniform integrability of $A\xi$ ($\xi \in L^{\Phi^*}$) implies that A is pointwise bounded in the algebraic dual $(L^{\Phi^*})^\#$ of L^{Φ^*} (this is clear), and its $\sigma((L^{\Phi^*})^\#, L^{\Phi^*})$-closure lies in L^Φ. If $(\eta_\alpha)_\alpha$ is a net in A with the pointwise limit $f(\xi) = \lim_\alpha \mathbb{E}[\eta_\alpha \xi]$, $f|_{L^\infty}$ is the pointwise limit of $\eta_\alpha|_{L^\infty} \in L^1$. Since A is uniformly integrable $\Leftrightarrow$ relatively weakly compact in L^1, there is an $\eta_0 \in L^1$ with $f(\xi) = \mathbb{E}[\eta_0\xi]$, $\forall \xi \in L^\infty$. For any $\xi \in L^{\Phi^*}$, $\zeta \in \mathbb{B}_{L^\infty}$ and $n \ge 1$,

$$\mathbb{E}[\eta_0\xi\zeta\mathbb{1}_{\{|\xi|\le n\}}] = f(\xi\zeta\mathbb{1}_{\{|\xi|\le n\}}) \le \sup_\alpha \mathbb{E}[\eta_\alpha\xi\zeta\mathbb{1}_{\{|\xi|\le n\}}] \le \sup_{\eta\in A} \mathbb{E}[|\eta\xi|] =: c_\xi < \infty,$$

so $\mathbb{E}[|\eta_0\xi|] = \sup_{n\ge 1, \zeta\in\mathbb{B}_{L^\infty}} \mathbb{E}[\eta_0\xi\zeta\mathbb{1}_{\{|\xi|\le n\}}] \le c_\xi < \infty$, hence $\eta_0 \in L^\Phi$. Finally, for each $\xi \in L^{\Phi^*}$, $|f(\xi) - f(\xi\mathbb{1}_{\{|\xi|\le n\}})| \le \sup_{\eta\in A} \mathbb{E}[|\eta\xi|\mathbb{1}_{\{|\xi|>n\}}] \to 0$ by the uniform integrability of $A\xi$, hence $f(\xi) = \lim_n \mathbb{E}[\eta_0\xi\mathbb{1}_{\{|\xi|\le n\}}] = \mathbb{E}[\eta_0\xi]$ since $\xi\mathbb{1}_{\{|\xi|\le n\}} \to \xi$ in $\mathbb{P}$ and $|\xi\mathbb{1}_{\{|\xi|\le n\}}| \le |\xi| \in L^{\Phi^*}$. □

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