On Continuity Correction for First-Passage Times in A Flexible Jump Diffusion Model with Application to Option Pricing

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Abstract. This paper studies first passage times of crossing a flat boundary under the mixed-exponential jump diffusion models. We establish a continuity correction for the joint distribution of a first passage time and its corresponding stopped process when the event of crossing is monitored only discretely. The work is done by significantly extending the method in Keener (2013) from diffusion models to jump diffusion models. Unlike the exiting literature, the newly proposed correction forms (with respect to the boundary level) are now getting involved not only the diffusion parameter but also the jump parameters. Numerical results indicate that the new way to correct does improve the approximation performance especially when the monitoring frequency is low and the boundary level is close. Similar results are obtained when applying this new method to the pricing of discrete path-dependent options.

keywords: boundary crossing, overshoot, continuity correction, Laplace transform, discrete options, jump-diffusion models.

1. Introduction

In this paper, we establish one continuity correction for first-passage times under the mixed-exponential jump-diffusion models (MEM) of Cai and Kou (2011), and apply the obtained result to the pricing and hedging of discrete barrier and lookback options. The main subject is the joint moment generating function (MGF) of a first-passage time and its related stopped process, which is also the key to option pricing of path-dependent type. We choose the MEM as our working model for two reasons. First, it can be used to approximate any Levy process of finite activity; see, e.g., Botta and Harris, 1986) and, second, it also admits many analytical formula for problems with path-dependent feature. In other words, the MEM fits in with a wide range of applications. The famous double-exponential jump-diffusion model (DEM) of Kou (2002) is just one special case of the MEM.

Continuity correction is a technique in the approximation of a discrete random variable by its continuous counterpart which can be studied analytically.[1] To the best of our knowledge, this idea was firstly introduced to the field of "Continuous-time Finance" by Broadie *et al.* (1997), who studied the pricing of barrier options when the event of barrier hitting is monitored only at some equally-spaced points of time. They realized that significant price errors will arise if one dose not make suitable adjustments before directly applying the closed-form pricing formula derived under the assumption of continuous monitoring. The continuity correction just presents in the adjustment of the barrier level.

Their work was then completed by Kou (2003) and Hörfelt (2003) to cover all types of options with single-barrier and single asset. The approach of continuity correction also can be applied to other path-dependent options; see Broadie *et al.* (1999) for lookback options and Lai et al. (2007) for American options. Note that all the studies mentioned above were conducted under the Black-Scholes model (BSM). Dia and Lamberton (2011) and Fuh *et al.* (2013) then separately extended the correction results to jump diffusion models. Regardless of the models, the common correction way is to shift away the barrier level to account for less boundary-hitting likelihood under a discrete monitoring scheme, in comparison with a continuous correspondent.

Nevertheless, by extending the method in Keener (2013) from PDE (partial differential equation) to PIDE (partial integro-differential equation), we achieve a different approximation in which the shift amount is no longer consonant. Instead, it should be varying across different, say, scenarios. Moreover, unlike the existing results, the correction form

[1] A classic example is the approximation $P(X \leq k) \approx P(Y \leq k + 0.5)$ for a binomial variable X and an associated normal variable Y. The addition of 0.5 to any level k is just what we mean by "continuity correction."

now involves the jump parameters in addition to the diffusion volatility. Simulation results further show that our new approach of correction does have a better numerical performance, especially when the monitoring frequency is low or the boundary level is not far way initially.

The phenomenon of discordant correction can be understood more intuitively when one narrows the focus from a broader set of MEM to a narrower one of DEM. To start, notice that boundary crossing will be observed first under continuously monitoring. In the meantime, there are two possible scenarios: (i) the observed object coincides with the boundary (no overshooting); and (ii) the observed object exceeds the boundary (with jumping). So, it is easy to comprehend that the probability that we still see the event fo boundary crossing at the succeeding discrete monitoring point of time will be lower in the former scenario than the latter. Accordingly, the correction amount shall be larger in the former scenario than the latter. This is one aspect through which one can understand our result, and thus we hereafter call our new approach a *"scenario-dependent" correction* (SDC).

Note that our approximation is still of first order, for the convergence rate is still $o(1/\sqrt{\Delta t})$. Here Δt stands for the interval length between two monitoring points of time. Howison and Steinberg (2007) had obtained a second order approximation for option pricing with rate $o(1/\Delta t)$ under the Black-Scholes model, by appealing to the perturbation method in PDE analysis. However, the main drawback of their approach is the lack of rigorous proofs. Also, the extension to the calculation of "Greeks" is not as straightforward as the correction method like ours.

The rest of the paper is organized as follows. Section 2 provides the problem formulation where we introduce our working model and overview the methodology. Main results can then be found in Section 3, in which we also apply the new correction to the pricing and the delta-hedging of a discrete up-and-in put and a discrete floating-strike lookback put. Numerical examples are given in Section 4, and the final section concludes. All proofs are deferred to the Appendix.

2. Problem Formulation and Methodology

2.1. The model

Throughout this paper, the underlying process is assumed to have the form

$$X_t = \sigma W_t + \mu t + M_t, \tag{1}$$

where $\sigma > 0$ is the volatility, $\mu \in \mathbb{R}$ is the mean rate of log-return, W_t is a standard Brownian motion with $W_0 = 0$, and $M_t = \sum_{l=1}^{N_t} \Upsilon_l$ is a compound Poisson process with $M_0 = 0$. Here N_t is a homogeneous Poisson process with intensity λ, and Υ_l's are independent and identically distributed (*i.i.d.*) random variables with probability density function

$$f_\Upsilon(y) = p_u \sum_{i=1}^{n_u} p_i \eta_i e^{-\eta_i y} 1_{\{y \geq 0\}} + q_d \sum_{j=1}^{n_d} q_j \xi_j e^{\xi_j y} 1_{\{y<0\}}, \tag{2}$$

in which $p_u \geq 0$, $q_d = 1 - p_u \geq 0$,

$$\begin{aligned} p_i \in (-\infty, \infty) \;\; &\text{for all } i = 1, ..., n_u, \;\; \sum_{i=1}^{n_u} p_i = 1, \\ q_j \in (-\infty, \infty) \;\; &\text{for all } j = 1, ..., n_d, \;\; \sum_{j=1}^{n_d} q_j = 1, \\ \eta_i > 1 \;\; &\text{for all } i = 1, ..., n_u, \;\; \text{and} \\ \xi_j > 0 \;\; &\text{for all } j = 1, ..., n_d. \end{aligned}$$

From now on, $1_{\{\cdot\}}$ stands for an indicator function. Note that all sources of randomness in (1) are assumed to be independent, and $\{\mathcal{F}_t\}_{t\geq 0}$ will denote its natural filtration. Because p_i and q_j can be negative, the parameters should satisfy some conditions to qualify $f_\Upsilon(y)$ of (2) as a density function. Interested readers are referred to Cai and Kou (2011) for further suggestions.

By the celebrated *Lévy-Khintchine formula*, X_t admits the following characteristic function: for all $t \geq 0$,

$$\Phi_t(\theta) = E\left[e^{i\theta X_t}\right] = e^{\psi(\theta)\cdot t}, \quad \theta \in \mathbb{R}.$$

Here $i = \sqrt{-1}$ and $\psi(\theta)$ is called the *characteristic exponent* of a Lévy process. Now, define $G(\theta) = \psi(-i\theta)$ such that $G : \Theta \to \mathbb{R}$ is a real function with respect to some parameter space $\Theta \subseteq \mathbb{R}$ containing 0. It then has the expression

$$G(\theta) = \frac{1}{2}\sigma^2\theta^2 + \mu\theta + \lambda[\Phi_\Upsilon(-i\theta) - 1], \tag{3}$$

where

$$\Phi_\Upsilon(\theta) = E\left[e^{i\theta\Upsilon_1}\right] = p_u \sum_{k=1}^{n_u} \frac{p_k \eta_k}{\eta_k - i\theta} + q_d \sum_{j=1}^{n_d} \frac{q_j \xi_j}{\xi_j + i\theta}$$

is the characteristic function of the jump size variable Υ_1.

According to Theorem 3.1 of Cai and Kou (2011), for sufficiently large $\alpha > 0$, the equation $G(\theta) = \alpha$ has $(n_u + 1)$ positive roots, $\beta_{1,\alpha}, \cdots, \beta_{n_u+1,\alpha}$, and $(n_d + 1)$ negative roots, $\gamma_{1,\alpha}, \cdots, \gamma_{n_d+1,\alpha}$. Moreover, these roots meet the following relationship:

$$-\infty < \gamma_{n_d+1,\alpha} < \cdots < \gamma_{2,\alpha} < \gamma_{1,\alpha} < 0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \cdots < \beta_{n_u+1,\alpha} < +\infty. \tag{4}$$

In other words, there are $(n_u + n_d + 2)$ distinct roots. These quantities are highly close to the distribution of a first-passage time we are going to study below. Note finally that we also have $G(0) = 0$, and

$$\begin{aligned} \bar{\mu} = G'(0) &= \mu + \lambda \left(p_u \sum_{i=1}^{n_u} \frac{p_i}{\eta_i} - q_d \sum_{j=1}^{n_d} \frac{q_j}{\xi_j} \right); \\ \bar{\sigma}^2 = G''(0) &= \sigma^2 + 2\lambda \left(p_u \sum_{i=1}^{n_u} \frac{p_i}{\eta_i^2} + q_d \sum_{j=1}^{n_d} \frac{q_j}{\xi_j^2} \right) \end{aligned} \tag{5}$$

are, respectively, the total drift and variance rates of the model (1).

2.2. First-passage times

Motivated from the payoff of a typical barrier option, we are interested in the distribution of the following random time:

$$\tau = \tau(b, X) = \inf\{t \in \mathbb{R}_+ : X_t \geq b\} \tag{6}$$

for some constant boundary level b. Without loss of generality, we will assume $b > 0$ in the sequel. Indeed, τ is called a *first-passage time* mathematically and is our main focus in this study. Cai and Kou (2011, Theorem 3.3), together with (Cai and Sun, 2014, Eq. (3.17)), showed that the model (1) leads to an analytical joint MGF for τ and X_τ, which is

$$E\left[e^{-\alpha\tau + \theta X_\tau}\right] = \sum_{k=1}^{n_u+1} d_k(\alpha, \theta)\, e^{-b\,(\beta_{k,\alpha} - \theta)} \tag{7}$$

for sufficiently large $\alpha > 0$ and $\theta < \eta_1$. Here $\beta_{i,\alpha}$'s are given in (4),

$$d_k(\alpha, \theta) = \prod_{j=1}^{n_u} \left(\frac{\eta_j - \beta_{k,\alpha}}{\eta_j - \theta} \right) \times \prod_{j=1, j\neq k}^{n_u+1} \left(\frac{\beta_{j,\alpha} - \theta}{\beta_{j,\alpha} - \beta_{k,\alpha}} \right) \tag{8}$$

for $k = 1, 2, \cdots, n_u + 1$, and $\sum_{k=1}^{n_u+1} d_k(\alpha, \theta) = 1$.

In the application of barrier-option pricing, the event of interest is like $\{\tau \le T\}$; that is, whether the underlying process X_t hits the barrier level b during the contract life $[0, T]$. However, for sake of feasibility, the monitoring of such event is conducted only discretely as specified in most of real contracts. Therefore, the associated first-passage time for real cases should be alternatively described as

$$\tau_m = \tau_m(b, X) = \inf\{k \in \mathbb{N} : X_{k\Delta t} \ge b\}, \tag{9}$$

where $\Delta t = T/m$ with m being the monitoring frequency. Then, the problem of calculating $E\left[e^{-\alpha(\tau_m \Delta t)+\theta X_{\tau_m \Delta t}}\right]$ thus follows. We will sketch our general approach in the next subsection.

Before that, let us introduce an auxiliary model of (1) under the discretization scheme behind (9). Consider $X_{m,k} = \sigma\sqrt{\Delta t}\, W_{m,k}$, where $W_{m,0} = 0$ and

$$W_{m,k} = \sum_{j=1}^{k} \left(Z_j + \frac{\mu}{\sigma}\sqrt{\Delta t} + \frac{M_{m,j}}{\sigma\sqrt{\Delta t}} \right) := \sum_{j=1}^{k} V_{m,j} \tag{10}$$

for $k \in \mathbb{N}$. Here $Z_j \overset{i.i.d.}{\sim} Normal(0, 1)$ and $M_{m,j} = \sum_{l=1}^{N_{m,j}} \Upsilon_{j,l}$ with $N_{m,j} \overset{i.i.d.}{\sim} Poisson(\lambda\Delta t)$ and $\Upsilon_{j,l} \overset{i.i.d.}{\sim} \Upsilon_1$. All random variables in this discrete setting are also independent. Note that in this way $X_{k\Delta t}$ and $X_{m,k}$ will have same distribution. We will utilize (10) to achieve some asymptotic properties about τ_m and X_{m,τ_m}.

2.3. PIDE approach

It is known that $(\tau_m\Delta t, X_{m,\tau_m})$ weakly converges to (τ, X_τ). Hence, for any bounded continuous function f, we have

$$E\left[f(\tau_m\Delta t, X_{m,\tau_m})\right] = E\left[f(\tau, X_\tau)\right] + o(1) \quad \text{as } m \to \infty. \tag{11}$$

However, the rate of convergence is not sufficiently rapid enough for practical use. Later, we will show you the evidence via numerical examples. Fortunately, if the function f additionally meets the "Principle of Smooth-Fit" (see, e.g., Shepp and Shiryaev, 1993), then we could have $E\left[f(\tau_m\Delta t, X_{m,\tau_m})\right] = E\left[f(\tau, X_\tau)\right] + o(1/\sqrt{m})$. This is exactly the insight of Keener (2013) to improve the convergence for general f.

We now expound on the meaning of *smooth-fit*. One typical way to derive $E\left[f(\tau, X_\tau)\right]$ is to solve the associated PIDE satisfied by $v(t, x) = E\left[f(\tilde{\tau}, Y_{\tilde{\tau}})\right]$ for $t \ge 0$ and $x < b$. Here

$Y_s = Y_s(t,x) = x + X_{s-t}$ is a process starting at time t and position x for $s \geq t$ and

$$\tilde{\tau} = \tilde{\tau}(t,x) = \inf\{s \geq t : \ Y_s \geq b\}.$$

In this way, $E[f(\tau, X_\tau)] = v(0,0)$. Then, by the celebrated Feynman-Kăc formula,

$$\partial_t v(t,x) + (\mathcal{L}v)(t,x) = 0 \tag{12}$$

subject to the boundary condition $v(t,b) = f(t,b)$. Here $\mathcal{L}$ denotes the infinitesimal generator associated with X_t in (1), which satisfies

$$(\mathcal{L}h)(x) = \frac{1}{2}\sigma^2 h''(x) + \mu h'(x) + \lambda \int_{-\infty}^{\infty} [h(x+y) - h(x)] f_\Upsilon(y) dy$$

for any twice continuously differentiable function $h(\cdot)$, and $f_\Upsilon(y)$ is given by (2). In other words, we have

$$v(t,x) = \begin{cases} u(t,x), & x < b; \\ f(t,x), & x \geq b \end{cases}$$

for some function u such that $u(t,b) = f(t,b)$. Smooth-fit, associated with v, means that partial derivatives (respective to x to some order) of $u(t,x)$ and $f(t,x)$ also coincide upon the boundary b, which is not always the case actually.

To capture the boundary effect brought by the non-smoothness of v, Keener suggests to decompose f into the sum of $f^0 + f^1$, in which f^0 is the smoothing part and f^1 is the remainder. To account for the existence of jumps, we here set

$$\begin{aligned} f^0(t,x) &= f(t,x) - \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{D}_b^{(n)}(t)\,(x-b)^n \quad \text{and} \\ f^1(t,x) &= \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{D}_b^{(n)}(t)\,(x-b)^n, \end{aligned} \tag{13}$$

where

$$\mathcal{D}_b^{(n)}(t) = \frac{\partial^n}{\partial x^n} f(t,x)\Big|_{x=b} - \frac{\partial^n}{\partial x^n} u(t,x)\Big|_{x=b^-} \tag{14}$$

measures the "n-th order" non-smoothness of v upon the boundary. In this way, we have both $u(t,b) = f^0(t,b)$ and

$$\frac{\partial^k}{\partial x^k} u(t,x)\big|_{x=b^-} = \frac{\partial^k}{\partial x^k} f^0(t,x)\big|_{x=b} \tag{15}$$

for each $k \in \mathbb{N}$.[2] Note that in Keener (2013) f^0 is only required to be smooth of first-order, but here we require infinite-order smoothness.

Since also $f^0(t,b) = f(t,b)$, we can achieve

$$E\left[f^0(\tau_m \Delta t, X_{m,\tau_m})\right] = E\left[f^0(\tau, X_\tau)\right] + o(1/\sqrt{m}) = v(0,0) + o(1/\sqrt{m}); \tag{16}$$

that is, there is no need to call for continuity correction for such a boundary function f^0 (see Appendix A.2). Since $Ef = Ef^0 + Ef^1$, to improve the convergence between $Ef(\tau_m \Delta t, X_{m,\tau_m})$ and $Ef(\tau, X_\tau)$ for general f, the key thus hinges on the estimate of $Ef^1(\tau_m \Delta t, X_{m,\tau_m})$, which requires the investigation on limiting behavior of τ_m and the "overshoot"

$$X_{m,\tau_m} - b = \sigma\sqrt{\Delta t}\,(W_{m,\tau_m} - b_m) := \sigma\sqrt{\Delta t}\,R_m, \tag{17}$$

where $b_m = b/(\sigma\sqrt{\Delta t})$. The study of $\lim_{m\to\infty} Ef^1(\tau_m \Delta t, X_{m,\tau_m})$ is exactly the channel how we establish our correction result with

$$f(t,x) = \exp(-\alpha t + \theta x). \tag{18}$$

What mainly distinguishes our efforts here from Keener's is the equation (12). His work is under pure diffusion model where PIDE (12) reduces to a classic PDE—the heat equation; and (16) is feasible conditional on the existence of fourth moment. Instead, here we have exponential moments so that we can adopt Taylor's expansion (with respect to x) of any order we want. In this way, we can approximate the smoothing part $f^0(t,x)$ by the Taylor series of $u(t,x)$ upon the boundary b. This is exactly the intuition behind Lemma 2. Another difficulty we face here is the algebraic manipulation when working out the expression of $\mathcal{D}_b^{(1)}(t)$ to get the final correction form with our choice of f in (18) (see Lemma 7).

3. Continuity Correction

3.1. Main results

Recall $\bar{\mu}$ in (5), β_i in (4), and d_k in (8); the next theorem extends Keener's result to our jump-diffusion model:

[2] Since $\frac{\partial^k}{\partial x^k}(x-b)^n = 0$ for each integer $n < k$, we have $\frac{\partial^k}{\partial x^k} f^0(t,x) = \frac{\partial^k}{\partial x^k} f(t,x) - \mathcal{D}_b^{(k)}(t) - \sum_{n=k+1}^{\infty} \frac{1}{n!}\mathcal{D}_b^{(n)}(t)\frac{\partial^k}{\partial x^k}(x-b)^n$. Also, $\frac{\partial^k}{\partial x^k}(x-b)^n = \frac{n!}{k!}(x-b)^{n-k}$ for $n > k$; we hence finally get $\frac{\partial^k}{\partial x^k} f^0(t,x)\big|_{x=b} = \frac{\partial^k}{\partial x^k} f(t,x)\big|_{x=b} - \mathcal{D}_b^{(k)}(t) = \frac{\partial^k}{\partial x^k} u(t,x)\big|_{x=b-}$ thanks to (14).

Theorem 1. *Suppose $\bar{\mu} > 0$ in the model (1). For any feasible function f, as $m \to \infty$, we have*

$$E\left[f(\tau_m \Delta t, X_{m,\tau_m})\right] = E\left[f(\tau, X_\tau)\right] + E\left[\mathcal{D}_b^{(1)}(\tau)\right]\left(\rho\,\sigma\sqrt{\Delta t}\right) + o\left(1/\sqrt{m}\right), \tag{19}$$

in which $\rho = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$ with $\zeta(.)$ being the Riemann zeta function. In particular, with the choice (18), we hence get

$$E\left[e^{-\alpha(\tau_m \Delta t)+\theta X_{m,\tau_m}}\right] = \sum_{k=1}^{n_u+1} d_k(\alpha,\theta)\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha}}{\bar{\eta}}\right)\left(\beta_{k,\alpha}-\theta\right)} + o\left(1/\sqrt{m}\right) \tag{20}$$

for any sufficiently large $\alpha > 0$ and real $\theta < \eta_1$. Here

$$\bar{\eta} = \prod_{i=1}^{n_u} \eta_i \quad \textit{and} \quad \bar{\beta}_{(k),\alpha} = \left(\prod_{i=1}^{n_u+1} \beta_{i,\alpha}\right) \Big/ \beta_{k,\alpha}, \quad \textit{for } k = 1, \cdots, n_u+1.$$

Comparing (7) and (20), it is clear to see that one can approximate the distribution of τ_m and X_{m,τ_m} by its continuous correspondent but with adjustments on the boundary parameter. Unlike the previous literature, the amounts of adjustment $\rho\sigma\sqrt{\Delta t}\left(\bar{\beta}_{(k),\alpha}/\bar{\eta}\right)$ are now different across distinct exponential terms. We will discuss more on the meaning of each exponential term and the interpretation of discordant correction in the next subsection.

Remark 1. *For DEM case, the equation (20) becomes*

$$\begin{aligned} E\left[e^{-\alpha(\tau_m \Delta t)+\theta X_{m,\tau_m}}\right] =& \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1 - \theta}\frac{\beta_{2,\alpha} - \theta}{\beta_{2,\alpha} - \beta_{1,\alpha}}\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\,\frac{\beta_{2,\alpha}}{\eta_1}\right)(\beta_{1,\alpha}-\theta)} \\ &+ \frac{\eta_1 - \beta_{2,\alpha}}{\eta_1 - \theta}\frac{\beta_{1,\alpha} - \theta}{\beta_{1,\alpha} - \beta_{2,\alpha}}\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\,\frac{\beta_{1,\alpha}}{\eta_1}\right)(\beta_{2,\alpha}-\theta)} + o\left(1/\sqrt{m}\right). \end{aligned} \tag{21}$$

On the other hand, Fuh et al. (2013) gets the following result:

$$\begin{aligned} E\left[e^{-\alpha(\tau_m \Delta t)+\theta X_{m,\tau_m}}\right] =& \frac{\eta_1 - \beta_{1,\alpha}}{\eta_1 - \theta}\frac{\beta_{2,\alpha} - \theta}{\beta_{2,\alpha} - \beta_{1,\alpha}}\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\right)(\beta_{1,\alpha}-\theta)} \\ &+ \frac{\eta_1 - \beta_{2,\alpha}}{\eta_1 - \theta}\frac{\beta_{1,\alpha} - \theta}{\beta_{1,\alpha} - \beta_{2,\alpha}}\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\right)(\beta_{2,\alpha}-\theta)} + o\left(1/\sqrt{m}\right). \end{aligned} \tag{22}$$

Clearly, the above two approximation results cannot hold concurrently because the first items on the right hand side of (21) and (22) differ by a term of order $O(1/\sqrt{m})$. What went wrong in Fuh et al. (2013, pp. 2710–2711) is their construction of approximating functions regarding function u. They do not specify the exact smoothness construction upon the boundary, not to mention that they only require 3rd-order smoothness. Accordingly, their convergence rate between $EM_t^{(n)}$ and $EM_k^{(n)}$ should be $O(1/\sqrt{m})$ instead of $o(1/\sqrt{m})$ as claimed. However, intriguingly, the numerical difference of this two approximations is quite small; see the discussion in Section 4.

With the approximation (20), we then can analytically analyze the joint probability $P(\tau_m \Delta t \le T, X_T \le a)$, which is the key in determining the value of a plain barrier option, via a two-dimensional Laplace transform by analogy with Cai and Kou (2011). Specifically, let

$$L_m(\alpha,\theta) = \int_0^\infty \int_{-\infty}^\infty e^{-\alpha T - \theta a} P(\tau_m \Delta t \le T, X_T \le a)\, da\, dT,$$

and recall (3); then

Corollary 1. *For $0 < \theta < \xi_1$ and sufficiently large $\alpha > \max(0, G(-\theta))$, if $\bar{\mu} > 0$, we have*

$$L_m(\alpha,\theta) = \frac{1}{\theta\left(\alpha - G(-\theta)\right)} \sum_{k=1}^{n_u+1} d_k(\alpha,-\theta)\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha}}{\bar{\eta}}\right)\left(\beta_{k,\alpha}+\theta\right)} + o\left(1/\sqrt{m}\right)$$

as $m \to \infty$.

For numerical applications, in this paper we will adopt the procedure proposed in Cai and Kou (2012) to invert the Laplace transforms in Corollaries 1 and 2. Generally speaking, this procedure is a combination of two-sided Euler inversion and Euler transformation for alternating series. The inversion parameters we choose are $A_1 = A_2 = 20$, $n_1 = 30$, and $n_2 = 50$. Actually, to meet the criteria about transform parameters α and θ, we have to modify the transform a bit before implementing the inversion. The details will be given in next section.

3.2. Application to option pricing

Now, let us formally address the issues of pricing and hedging of a discrete barrier option and a discrete lookback option.

3.2.1. Barrier options

We will take up-and-in puts (UIP) as the illustrative case in our main derivation. Note that from now on, whenever we talk about option pricing, we are assuming the model of log-return (1) is built in a risk-neutral world so that $\mu = r - \sigma^2/2 - \lambda\delta$. Here r stands for the constant risk-free rate and

$$\delta = E\left[e^{\Upsilon_1}\right] - 1 = p_u \sum_{i=1}^{n_u} \frac{p_i \eta_i}{\eta_i - 1} + q_d \sum_{j=1}^{n_d} \frac{q_j \xi_j}{\xi_j + 1} - 1.$$

Suppose S_0, K, and H are the initial stock price, the strike price, and the barrier level, respectively. Define $k = \log(K)$ and take $b = \log(H/S_0)$; then the price of a discrete UIP, with time to maturity T, can be expressed as

$$UIP_m(k,T) = e^{-rT} E\left[\left(e^k - S_0 e^{X_T}\right)^+ 1_{\{\tau_m \Delta t \le T\}}\right].$$

Accordingly, we can get its *delta* by taking the first-order partial derivative of $UIP_m(k,T)$, which is denoted by

$$\Delta_{UIP_m}(k,T) = \frac{\partial}{\partial S_0} UIP_m(k,T).$$

Like Corollary 1, UIP_m and Δ_{UIP_m} can be worked out via two-dimensional Laplace transforms, which are defined by

$$\widehat{UIP}_m(\alpha,\theta) = \int_0^\infty \int_{-\infty}^\infty e^{-\alpha T - \theta k}\, UIP_m(k,T)\, dk\, dT$$

and

$$\widehat{\Delta}_{UIP_m}(\alpha,\theta) = \int_0^\infty \int_{-\infty}^\infty e^{-\alpha T - \theta k}\, \Delta_{UIP_m}(k,T)\, dk\, dT,$$

respectively. Recall (8); then, we have

Corollary 2. *For $1 < \theta < 1 + \xi_1$ and sufficiently large $\alpha > \max\left(G(1-\theta) - r, 0\right)$, if $\bar{\mu} > 0$, we have*

$$\widehat{UIP}_m(\alpha,\theta) = \frac{(S_0)^{1-\theta} \sum_{k=1}^{n_u+1} d_k(r+\alpha, 1-\theta)\, e^{-(b+\rho\sigma\sqrt{\Delta t}\frac{\bar{\beta}_{(k),r+\alpha}}{\bar{\eta}})\left(\beta_{k,r+\alpha}-1+\theta\right)}}{\theta(\theta-1)\,(r+\alpha-G(1-\theta))} + o\,(\frac{1}{\sqrt{m}})$$

and

$$\widehat{\Delta}_{UIP_m}(\alpha,\theta) = \frac{(S_0)^{-\theta} \sum_{k=1}^{n_u+1} d_k(r+\alpha, 1-\theta)\, \beta_{k,r+\alpha}\, e^{-(b+\rho\sigma\sqrt{\Delta t}\frac{\bar{\beta}_{(k),r+\alpha}}{\bar{\eta}})\left(\beta_{k,r+\alpha}-1+\theta\right)}}{\theta(\theta-1)\,(r+\alpha-G(1-\theta))} + o\,(\frac{1}{\sqrt{m}})$$

as $m \to \infty$.

Note that, Corollary 2 can also be applied for pricing up-and-out puts (UOP). To see this, note that

$$\widehat{UOP}_m(\alpha,\theta) = \lim_{H\to S_0} \widehat{UIP}_m(\alpha,\theta) - \widehat{UIP}_m(\alpha,\theta). \tag{23}$$

Actually, we will only report the results of UOP cases in the numerical analysis next section for the sake of both compactness and completeness.

3.2.2. Lookback options

We here only consider a lookback put because a looback call can be obtained by symmetry. Under the risk-neutral measure, the price of a discrete lookback put, with a pre-maximum M ($\geq S_0$), can be expressed as

$$\begin{aligned} LP_m(T) &= E\left[e^{-rT}\left(\max\left\{M, \max_{0\leq k\Delta t\leq T} S_{k\Delta t}\right\} - S_T\right)\right] \\ &= E\left[e^{-rT}\max\left\{M, \max_{0\leq k\Delta t\leq T} S_{k\Delta t}\right\}\right] - S_0. \end{aligned}$$

where $k \in \mathbb{N}$. With the introduction of a prescribed maximum, one can easily determine the price of a lookback option at any time during the contract life.

Similarly, $LP_m(T)$ can be worked out via a Laplace transform (one-dimensional), which is defined by

$$\widehat{LP}_m(\alpha) = \int_0^\infty e^{-\alpha T} LP_m(T)\, dT.$$

Then,

Corollary 3. *For all sufficiently large $\alpha > 0$, we have the following approximation for the Laplace transform of the price of a discrete lookback put:*

$$\begin{aligned} \widehat{LP}_m(\alpha) =& \frac{S_0}{\alpha + r} \sum_{k=1}^{n_u+1} \frac{d_k(\alpha + r, 0) e^{-\left(\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha+r}}{\bar{\eta}}\right)\left(\beta_{k,\alpha+r}\right)}}{\beta_{k,\alpha+r} - 1} \left(\frac{S_0}{M}\right)^{\beta_{k,\alpha+r}-1} \\ &+ \frac{M}{\alpha + r} - \frac{S_0}{\alpha} + o\left(1/\sqrt{m}\right) \end{aligned}$$

as $m \to \infty$.

3.3. Special case: DEM

The purpose of the upcoming discussion is to explore the meaning behind the correction scheme (20). We will do this by considering a special case of model (1) for ease of presentation. Specifically, we choose a DEM in which $n_u = n_d = 1$.[3] Also, for brevity, throughout the subsection we will use the following notational simplification: $\beta_i = \beta_{i,\alpha}$, for $i = 1, \cdots, n_u + 1$, and $\gamma_j = \gamma_{j,\alpha}$, for $j = 1, \cdots, n_d + 1$ in (4).

In a DEM, the cumulant function in (3) becomes $G(\theta) = \sigma^2\theta^2/2 + \mu\theta + \lambda\left(p_u\eta_1/(\eta_1 - \theta) + q_d\xi_1/(\xi_1 + \theta) - 1\right)$. A typical graph of $G(\cdot)$ can be found in Panel (a) of Figure 1. Now,

[3] Hence $p_1 = q_1 = 1$ automatically.

it is easy to see $G(\theta) = \alpha$ has four distinct roots for any $\alpha > 0$. Actually, this time we have

$$-\infty < \gamma_2 < -\xi_1 < \gamma_1 < 0 < \beta_1 < \eta_1 < \beta_2 < +\infty. \tag{24}$$

Also, the approximation (20) becomes

$$E\left[e^{-\alpha(\tau_m \Delta t)+\theta X_{m,\tau_m}}\right] = d_1(\alpha,\theta)\, e^{-(b+\rho\,\sigma\sqrt{\Delta t}\,\frac{\beta_2}{\eta_1})(\beta_1-\theta)} + d_2(\alpha,\theta)\, e^{-(b+\rho\,\sigma\sqrt{\Delta t}\,\frac{\beta_1}{\eta_1})(\beta_2-\theta)} + o\left(1/\sqrt{m}\right) \tag{25}$$

for any $\alpha > 0$ and $\theta < \eta_1$, where $d_1(\alpha,\theta) = \frac{\eta_1-\beta_1}{\eta_1-\theta}\,\frac{\beta_2-\theta}{\beta_2-\beta_1}$ and $d_2(\alpha,\theta) = 1 - d_1(\alpha,\theta)$.

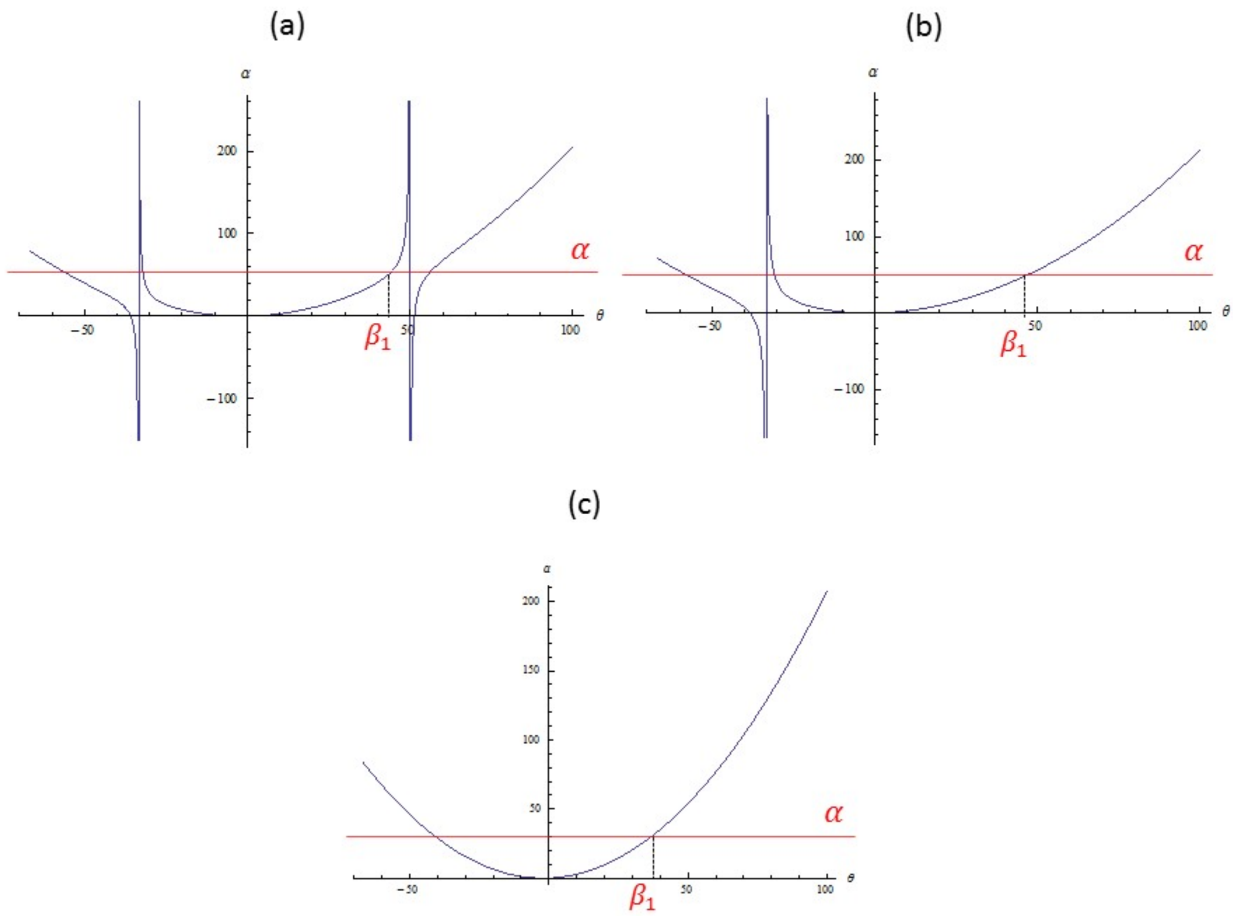

Figure 1. $\mathbf{G(\boldsymbol{\theta})}$ **in (3) with** $\boldsymbol{\mu} = \mathbf{r} - \boldsymbol{\sigma}^2/2 - \boldsymbol{\lambda\delta}$ **and** $\mathbf{n_u} = \mathbf{n_d} = \mathbf{1}$**. Universal parameter setting is** $\mathbf{r} = 0.1$**,** $\boldsymbol{\sigma} = 0.2$**,** $\mathbf{q_d} = 1 - \mathbf{p_u}$**, and** $\boldsymbol{\xi_1} = 100/3$**. The other parameters are (a)** $\boldsymbol{\lambda} = 3$**,** $\mathbf{p_u} = 0.5$**,** $\boldsymbol{\eta_1} = 100/2$**; (b)** $\boldsymbol{\lambda} = 3$**,** $\mathbf{p_u} = 0.0$**,** $\boldsymbol{\eta_1} = 100/2$**; and (c)** $\boldsymbol{\lambda} = 0$**,** $\mathbf{p_u} = 0.5$**,** $\boldsymbol{\eta_1} = 100/2$**.**

In Figure 1, Panels (b) and (c) further depict $G(\cdot)$ when there is no overshooting even monitored continuously. The former case stands for "no positive jumps" ($p_u = 0$) while the latter further considers "no jumps" ($\lambda = 0$). Clearly, in this two special cases, there is only one positive root left, which is determined by the curve passing the origin. In view of the graph in Panel (a), we thus tend to call this only positive root β_1 as well. In other words, the appearance of other β_i, $i \neq 1$, is due to the existence of positive jumps.

On the other hand, as $p_u \to 0$ or $\lambda \to 0$, we could also deem $\beta_2 \to \eta_1$; that is, the positive root accompanied by positive jumps degenerates to η_1. To see this, note that $G(\theta) = 0$ under DEM is equivalent to solving $g(\theta; \alpha) = 0$, where

$$g(\theta;\alpha) = \frac{1}{2}\sigma^2\,\theta^2\,(\eta_1-\theta)(\xi_1+\theta) + \mu\,\theta\,(\eta_1-\theta)(\xi_1+\theta) - (\lambda+\alpha)(\eta_1-\theta)(\xi_1+\theta)$$
$$+\,\lambda\,p_u\,\eta_1\,(\xi_1+\theta) + \lambda\,q_d\,\xi_1\,(\eta_1-\theta)$$

is a polynomial of θ with degree 4. Then, $g(\cdot)$ will reduce to $(\eta_1-\theta)\,[\frac{1}{2}\sigma^2\theta^2(\xi_1+\theta)+\mu\theta(\xi_1+\theta)-(\lambda+\alpha)(\xi_1+\theta)+\lambda\xi_1]$ if $p_u = 0$, and to $(\eta_1-\theta)\,(\xi_1+\theta)\,(\frac{1}{2}\sigma^2\theta^2+\mu\theta-\alpha)$ if $\lambda = 0$. Both suggests there is one positive root η_1 in this two special cases. Since such degeneracy comes from the fact that there is no positive jumps, we deem it as a degenerate β_2.

Accordingly, as $\beta_2 \to \eta_1$, we have $d_1(\alpha,\theta) \to 1$ and $d_2(\alpha,\theta) \to 0$; hence the correction result (25) degenerates to

$$E\left[e^{-\alpha(\tau_m\Delta t)+\theta X_{m,\tau_m}}\right] = e^{-(b+\rho\,\sigma\sqrt{\Delta t})(\beta_1-\theta)} + o\left(1/\sqrt{m}\right) \tag{26}$$

when $p_u = 0$ or $\lambda = 0$. Beware that this is exactly the correction form in a BSM, and the shifted amount $\Delta B := \rho\,\sigma\sqrt{\Delta t}$ just measures the overshoot effect due to discretization. The computation about ρ can be found in Chernoff (1965) or Siegmund (1985). Moreover, the form (26) is still true when there is only downward jumping, which is also intuitively.

However, when upward jumping is possible, the correction amount in (25) is large than ΔB for β_1-exponent term while it is smaller for β_2-exponent term, thanks to (24). From all the discussion above, we can deem that β_1-exponent term captures the situation where there is no jumping overshoot, while β_1-exponent term does. With this aspect, the discordant way to correct in (25) then is also intuitively convincing. Let $\lceil\cdot\rceil$ denote the ceiling function. Then, as we mentioned in the introduction section, the probability $P(X_{m,\lceil\tau/\Delta t\rceil} \geq b|X_\tau = b)$ is less than $P(X_{m,\lceil\tau/\Delta t\rceil} \geq b|X_\tau > b)$. As a result, the shift amount should be large in the previous scenario (i.e., when there is no jumping overshoot observed continuously) to account for further less probability of boundary-crossing monitored discretely. The relationship $\beta_2/\eta_1 > \beta_1/\eta_1$ about multipliers related to ΔB just reflects this truth. This is also why we call the result (20) a *scenario-dependent correction.*

Finally note that we also have $\beta_2/\eta_1 > 1 > \beta_1/\eta_1$, which can be understood via the following. First, conditional on $X_\tau > b$ (jump occurring), it is very likely there will be no more jumps during time interval $((\lceil \tau/\Delta t \rceil - 1)\Delta t, (\lceil \tau/\Delta t \rceil)\Delta t)$; hence the underlying process behaves like a pure diffusion process after time τ in a short future. Nevertheless, it still has an advantage of higher initial value at $X_\tau > b$, relatively to a real diffusion process whose $X_\tau = b$ always. Accordingly, the occurrence of event $\{X_{m,\lceil \tau/\Delta t \rceil} \geq b\}$ is more likely in the former situation than in the latter. This is one aspect why we see the correction amount now is less than ΔB for the β_2-exponent term. On the contrary, when $X_\tau = b$ (i.e., no jumping recently), the jump-diffusion model possesses more uncertainty than a pure diffusion model, especially that there is a possibility of downward jumping in a short future. Thereby, we have to correct more for the β_1-exponent term than that in a BSM. These observations also help to explain why the new correction form we propose additionally involves the jump parameters.

4. Numerical Analyses

All the examples in this section are set under a risk-neutral framework for option pricing; that is, $\mu = r - \sigma^2/2 - \lambda\delta$. The baseline parameters are $r = 0.1$, $\sigma = 0.2$, $p_u = q_d = 0.5$, $n_u = n_d = 1$, and $\eta_1 = 100/2$, $\xi_1 = 100/3$, where the setting about jumps is taken from Kou and Wang (2003). Additionally, $S_0 = 90$, $K = 96$, $a = log(K/S_0)$, and $b = log(H/S_0)$ with varying barrier level H. Also, we will use the values obtained by Monte-Carlo (MC) simulation as the benchmark of those expectations associated with discrete random variables. The number of trials in each simulation is one million.

The main purpose of this section is to see the numerical performance of our scenario dependent correction (20) (SDC) in each application. We will also compare the outcomes with the method by Fuh *et al.* (2013), which indicates that the correction form is identical in each exponent term of (25) and the common amount is just ΔB. We thus call their method a scenario independent correction (SIC). We will also show what if one does not consider continuity correction (NC) when applying (7). The basis of comparison is relative error (R.E.) defined by (“X” − MC)/MC, where X can be NC, SDC, or SIC.

4.1. MGF

This subsection presents the approximation performance of (20), or say, (25) indeed. The outcomes are shown in Tables 1 to 3. Generally, we see the SDC method systematically outperforms the SIC method in the cases we considered, except the cases with $\lambda = 0.01$ where the two correction methods coincide with each other, thanks to (26). Most importantly, the tables also demonstrate that the weak convergence (11) is really not sufficiently good, even with daily monitoring (roughly corresponding to case $\Delta t = 1/250$). This evidence again shows the need of continuity correction.

TABLE 1.

Approximation Performance:

$$\mathbf{E}\left[\mathbf{e}^{-\boldsymbol{\alpha}(\boldsymbol{\tau}_{\mathbf{m}}\boldsymbol{\Delta}\mathbf{t})+\boldsymbol{\theta}\,\mathbf{X}_{\mathbf{m},\boldsymbol{\tau}_{\mathbf{m}}}}\right] \textbf{ with } \mathbf{H} = \mathbf{90.5}$$

λ	Δt	MC	S.E.	(1) NC	(2) SIC	(3) SDC	(1) R.E.	(2) R.E.	(3) R.E.
Panel A: $\alpha = 1$, $\theta = 1$.									
0.01	1/10	0.7756503	0.03 %	0.976197	0.831681	0.831673	25.86 %	7.22 %	7.22 %
	1/50	0.8655288	0.02 %	0.976197	0.908698	0.908695	12.79 %	4.99 %	4.99 %
	1/250	0.9420316	0.03 %	0.976197	0.945412	0.945411	3.63 %	0.36 %	0.36 %
3	1/10	0.7751203	0.03 %	0.976369	0.834661	0.832386	25.96 %	7.68 %	7.39 %
	1/50	0.8939539	0.02 %	0.976369	0.909939	0.909132	9.22 %	1.79 %	1.70 %
	1/250	0.9423967	0.02 %	0.976369	0.945987	0.945706	3.60 %	0.38 %	0.35 %
6	1/10	0.7747749	0.03 %	0.976538	0.837492	0.833093	26.04 %	8.09 %	7.53 %
	1/50	0.8940878	0.02 %	0.976538	0.91113	0.909564	9.22 %	1.91 %	1.73 %
	1/250	0.9430291	0.02 %	0.976538	0.946543	0.945997	3.55 %	0.37 %	0.31 %
Panel B: $\alpha = 1$, $\theta = 0$.									
0.01	1/10	0.7346433	0.03 %	0.970803	0.797164	0.797155	32.15 %	8.51 %	8.51 %
	1/50	0.8418132	0.02 %	0.970803	0.888907	0.888904	15.32 %	5.59 %	5.59 %
	1/250	0.929304	0.02 %	0.970803	0.933285	0.933283	4.47 %	0.43 %	0.43 %
3	1/10	0.7328537	0.03 %	0.970855	0.799645	0.796982	32.48 %	9.11 %	8.75 %
	1/50	0.8710412	0.02 %	0.970855	0.889811	0.888844	11.46 %	2.15 %	2.04 %
	1/250	0.9293613	0.02 %	0.970855	0.933623	0.933283	4.46 %	0.46 %	0.42 %
6	1/10	0.7312164	0.03 %	0.970907	0.802006	0.796832	32.78 %	9.68 %	8.97 %
	1/50	0.8706003	0.02 %	0.970907	0.890682	0.888796	11.52 %	2.31 %	2.09 %
	1/250	0.9297108	0.02 %	0.970907	0.933954	0.933288	4.43 %	0.46 %	0.38 %

Model parameters: $r = 0.1$, $\sigma = 0.2$, $p_u = 0.5$, $\eta_1 = 100/2$, $\xi_1 = 100/3$, and $b = \log(90.5/90)$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

TABLE 2.

Approximation Performance:

$\mathbf{E}\left[\mathbf{e}^{-\boldsymbol{\alpha}(\boldsymbol{\tau}_{\mathbf{m}}\boldsymbol{\Delta}\mathbf{t})+\boldsymbol{\theta}\,\mathbf{X}_{\mathbf{m},\boldsymbol{\tau}_{\mathbf{m}}}}\right]$ **with** $\mathbf{H}=\mathbf{92}$

λ	Δt	MC	S.E.	(1) NC	(2) SIC	(3) SDC	(1) R.E.	(2) R.E.	(3) R.E.
Panel A: $\alpha=1$, $\theta=1$.									
0.01	1/10	0.737	0.03 %	0.909	0.774	0.774	23.37 %	5.11 %	5.10 %
	1/50	0.840	0.03 %	0.909	0.846	0.846	8.19 %	0.71 %	0.71 %
	1/250	0.879	0.02 %	0.909	0.880	0.880	3.37 %	0.11 %	0.11 %
3	1/10	0.738	0.03 %	0.910	0.779	0.776	23.37 %	5.56 %	5.18 %
	1/50	0.842	0.03 %	0.910	0.849	0.847	8.10 %	0.81 %	0.66 %
	1/250	0.881	0.02 %	0.910	0.882	0.882	3.30 %	0.11 %	0.05 %
6	1/10	0.738	0.03 %	0.911	0.783	0.777	23.44 %	6.03 %	5.31 %
	1/50	0.843	0.03 %	0.911	0.851	0.849	8.08 %	0.95 %	0.67 %
	1/250	0.882	0.02 %	0.911	0.884	0.883	3.28 %	0.17 %	0.06 %
Panel B: $\alpha=1$, $\theta=0$.									
0.01	1/10	0.689	0.03 %	0.889	0.730	0.730	28.99 %	5.92 %	5.92 %
	1/50	0.808	0.03 %	0.889	0.814	0.814	10.10 %	0.81 %	0.81 %
	1/250	0.854	0.02 %	0.889	0.855	0.855	4.14 %	0.12 %	0.12 %
3	1/10	0.689	0.03 %	0.890	0.734	0.731	29.17 %	6.51 %	6.04 %
	1/50	0.809	0.03 %	0.890	0.816	0.815	10.05 %	0.93 %	0.76 %
	1/250	0.855	0.02 %	0.890	0.856	0.856	4.08 %	0.13 %	0.05 %
6	1/10	0.688	0.03 %	0.891	0.737	0.731	29.42 %	7.12 %	6.22 %
	1/50	0.809	0.03 %	0.891	0.818	0.816	10.07 %	1.11 %	0.76 %
	1/250	0.856	0.02 %	0.891	0.858	0.856	4.08 %	0.19 %	0.05 %

Model parameters: $r=0.1$, $\sigma=0.2$, $p_u=0.5$, $\eta_1=100/2$, $\xi_1=100/3$, and $b=\log(92/90)$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

4.2. Joint probability

This subsection presents the approximation performance of Corollary 1. However, before directly applying the approximation formula, we need to make some adjustment first to meet the required conditions about α and θ. To this end, we introduce a shifting factor x such that

$$P(\tau_m\Delta t\leq T,X_T\leq a)=P(\tau_m\Delta t\leq T,X_T-x\leq\tilde{a})$$

where $\tilde{a}=a-x$, and reconsider the Laplace transform as

$$\tilde{L}_m(\alpha,\theta)=\int_0^\infty\int_{-\infty}^\infty e^{-\alpha T-\theta\tilde{a}}P(\tau_m\Delta t\leq T,X_T-x\leq\tilde{a})\,d\tilde{a}\,dT.$$

TABLE 3.

Approximation Performance:
$\mathbf{E}\left[\mathbf{e}^{-\boldsymbol{\alpha}(\boldsymbol{\tau}_{\mathbf{m}}\boldsymbol{\Delta}\mathbf{t})+\boldsymbol{\theta}\,\mathbf{X}_{\mathbf{m},\boldsymbol{\tau}_{\mathbf{m}}}}\right]$ **with** $\mathbf{H = 95}$

λ	Δt	MC	S.E.	(1) NC	(2) SIC	(3) SDC	(1) R.E.	(2) R.E.	(3) R.E.
Panel A: $\alpha = 1$, $\theta = 1$.									
0.01	1/10	0.655	0.03 %	0.790	0.673	0.673	20.71 %	2.84 %	2.84 %
	1/50	0.733	0.03 %	0.790	0.736	0.736	7.91 %	0.45 %	0.45 %
	1/250	0.765	0.03 %	0.790	0.766	0.766	3.38 %	0.12 %	0.12 %
3	1/10	0.658	0.03 %	0.794	0.680	0.677	20.71 %	3.34 %	2.91 %
	1/50	0.737	0.03 %	0.794	0.741	0.740	7.82 %	0.58 %	0.40 %
	1/250	0.769	0.03 %	0.794	0.770	0.770	3.32 %	0.16 %	0.08 %
6	1/10	0.662	0.03 %	0.798	0.686	0.681	20.66 %	3.76 %	2.95 %
	1/50	0.741	0.03 %	0.798	0.746	0.744	7.71 %	0.68 %	0.33 %
	1/250	0.773	0.03 %	0.798	0.774	0.773	3.24 %	0.17 %	0.02 %
Panel B: $\alpha = 1$, $\theta = 0$.									
0.01	1/10	0.596	0.03 %	0.749	0.615	0.615	25.65 %	3.17 %	3.17 %
	1/50	0.682	0.03 %	0.749	0.686	0.686	9.75 %	0.49 %	0.49 %
	1/250	0.719	0.03 %	0.749	0.720	0.720	4.15 %	0.13 %	0.13 %
3	1/10	0.598	0.03 %	0.752	0.621	0.618	25.79 %	3.79 %	3.26 %
	1/50	0.686	0.03 %	0.752	0.690	0.689	9.69 %	0.65 %	0.43 %
	1/250	0.723	0.03 %	0.752	0.724	0.723	4.11 %	0.18 %	0.08 %
6	1/10	0.600	0.03 %	0.755	0.626	0.620	25.88 %	4.33 %	3.31 %
	1/50	0.689	0.03 %	0.755	0.695	0.692	9.61 %	0.77 %	0.34 %
	1/250	0.726	0.03 %	0.755	0.728	0.726	4.03 %	0.19 %	0.00 %

Model parameters: $r = 0.1$, $\sigma = 0.2$, $p_u = 0.5$, $\eta_1 = 100/2$, $\xi_1 = 100/3$, and $b = \log(95/90)$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

Then, we can get

$$\tilde{L}_m(\alpha,\theta) = \frac{e^{\theta x}}{\theta\left(\alpha - G(-\theta)\right)} \sum_{k=1}^{n_u+1} d_k(\alpha,-\theta)\, e^{-\left(b+\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha}}{\bar{\eta}}\right)\left(\beta_{k,\alpha}+\theta\right)} + o\left(1/\sqrt{m}\right)$$

for $0 < \theta < \xi_1$ and $\alpha > \max(0, G(-\theta))$, provided that $\bar{\mu} > 0$.

The beauty here is that the parameter constraint is not affected by the introduction of factor x, and hence we can arbitrarily choose it to meet our need. In our inversion algorithm, we choose

$$x = a - 4\,\frac{A_2}{2\,c_p} \quad \text{with} \quad c_p = \min\left\{\frac{1}{\bar{\sigma}^2}\left(\bar{\mu} + \sqrt{\bar{\mu}^2 + 2\bar{\sigma}^2\left(\frac{A_1}{2T}\right)}\right), \eta_2\right\},$$

in which $\bar{\mu}$ & $\bar{\sigma}$ are defined in (5); and A_1 & A_2 are inversion parameters in the algorithm which we set both to be 20. The final outcomes are shown in Tables 4 and 5.

TABLE 4.

Approximation Performance:

$$\mathbf{P}\left(\tau_{\mathbf{m}}\Delta\mathbf{t} \le \mathbf{T}, \mathbf{X}_{\mathbf{T}} \le \mathbf{a}\right) \text{ with } \mathbf{T} = \mathbf{0.1}$$

λ	m	MC	S.E.	(1) NC	(2) SIC	(3) SDC	(1) R.E.	(2) R.E.	(3) R.E.
Panel A: $H = 90.5$.									
0.01	2	0.440949	0.05 %	0.754849	0.468371	0.468371	71.19 %	6.22 %	6.22 %
	5	0.553458	0.05 %	0.754849	0.573481	0.573473	36.39 %	3.62 %	3.62 %
	25	0.669234	0.05 %	0.754849	0.674253	0.67425	12.79 %	0.75 %	0.75 %
3	2	0.43366	0.05 %	0.747047	0.466501	0.462443	72.27 %	7.57 %	6.64 %
	5	0.54663	0.05 %	0.747047	0.569135	0.566902	36.66 %	4.12 %	3.71 %
	25	0.661645	0.05 %	0.747047	0.667772	0.667007	12.91 %	0.93 %	0.81 %
6	2	0.426403	0.05 %	0.739646	0.464711	0.456849	73.46 %	8.98 %	7.14 %
	5	0.541266	0.05 %	0.739646	0.564998	0.560668	36.65 %	4.38 %	3.58 %
	25	0.654239	0.05 %	0.739646	0.661617	0.660131	13.05 %	1.13 %	0.90 %
Panel B: $H = 91$.									
0.01	2	0.400	0.05 %	0.694	0.413	0.413	73.52 %	3.23 %	3.23 %
	5	0.506	0.05 %	0.694	0.514	0.514	37.03 %	1.43 %	1.42 %
	25	0.613	0.05 %	0.694	0.613	0.613	13.14 %	-0.08 %	-0.08 %
3	2	0.393	0.05 %	0.687	0.412	0.408	74.90 %	4.92 %	3.79 %
	5	0.500	0.05 %	0.687	0.511	0.508	37.32 %	2.06 %	1.53 %
	25	0.607	0.05 %	0.687	0.608	0.607	13.21 %	0.12 %	-0.06 %
6	2	0.388	0.05 %	0.681	0.412	0.403	75.63 %	6.21 %	3.98 %
	5	0.495	0.05 %	0.681	0.508	0.503	37.54 %	2.62 %	1.58 %
	25	0.602	0.05 %	0.681	0.603	0.601	13.10 %	0.16 %	-0.18 %
Panel C: $H = 92$.									
0.01	2	0.313	0.05 %	0.574	0.306	0.305	83.11 %	-2.52 %	-2.52 %
	5	0.404	0.05 %	0.574	0.399	0.399	42.12 %	-1.16 %	-1.17 %
	25	0.495	0.05 %	0.574	0.494	0.494	15.87 %	-0.23 %	-0.24 %
3	2	0.312	0.05 %	0.570	0.307	0.302	82.77 %	-1.44 %	-3.03 %
	5	0.403	0.05 %	0.570	0.399	0.396	41.35 %	-1.04 %	-1.83 %
	25	0.494	0.05 %	0.570	0.492	0.490	15.36 %	-0.42 %	-0.70 %
6	2	0.308	0.05 %	0.565	0.309	0.299	83.85 %	0.36 %	-2.77 %
	5	0.399	0.05 %	0.565	0.398	0.392	41.62 %	-0.21 %	-1.76 %
	25	0.490	0.05 %	0.565	0.489	0.487	15.49 %	-0.07 %	-0.63 %

Model parameters: $r = 0.1$, $\sigma = 0.2$, $p_u = 0.5$, $\eta_1 = 100/2$, $\xi_1 = 100/3$, $a = \log(96/90)$, $b = \log(H/90)$, and $\Delta t = T/m$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

As we transfer the concern of distributions from $(\tau_m\Delta t, X_{m,\tau_m})$ to $(\tau_m\Delta t, X_{m,m})$,[4] we see from the tables that SDC method does not totally dominate any more. SIC method is sufficiently of use especially when monitoring frequency is at least daily ($m = 250$) and the boundary is far above from the initial process level ($H = 92$ for $T = 0.1$ & $H = 95$ for $T = 1.$). This is because the effect of asymptotic independence between τ_m and $X_{m,\tau}$ significantly shows up, leading to numerical equivalence between SIC and SDC in these cases.

4.3. Option pricing

Finally let see the numerical performance of our approximation method in option pricing, with the aid of (23) primarily. Here, we merely focus on pricing. Again, we have to make some adjustment on the Laplace transform first. Still, introduce a scaling factor X such that

$$e^{-rT}E\left[\left(K - S_0e^{X_T}\right)^+ 1_{\{\tau_m\Delta t\le T\}}\right] = Xe^{-rT}E\left[\left(e^{\tilde{k}} - \frac{S_0}{X}e^{X_T}\right)^+ 1_{\{\tau_m\Delta t\le T\}}\right] := U\tilde{I}P_m(\tilde{k}, T)$$

where $\tilde{k} = \log(K/X)$, and consider

$$\widehat{U\tilde{I}P}_m(\alpha, \theta) = \int_0^\infty \int_{-\infty}^\infty e^{-\alpha T - \theta\tilde{k}}\, U\tilde{I}P_m(\tilde{k}, T)\, d\tilde{k}\, dT.$$

In this way, if $\bar{\mu} > 0$, we can get

$$\begin{aligned}\widehat{U\tilde{I}P}_m(\alpha, \theta) &= \frac{X\left(S_0/X\right)^{1-\theta}}{\theta(\theta-1)} \times \frac{1}{r + \alpha - G(1-\theta)} \\ &\quad \times \sum_{k=1}^{n_u+1} d_k(r+\alpha, 1-\theta)\, e^{-(b+\rho\sigma\sqrt{\Delta t}\frac{\bar{\beta}_{(k),r+\alpha}}{\bar{\eta}})\left(\beta_{k,r+\alpha}-1+\theta\right)} + o\,(\frac{1}{\sqrt{m}}).\end{aligned}$$

for the same parameter constraint that $1 < \theta < 1 + \xi_1$ and $\alpha > \max\left(G(1-\theta) - r, 0\right)$. Our practice is to take

$$X = \frac{K}{\exp(x_p)} \quad \text{with} \quad x_p = \frac{A_2}{4}\left(\frac{2 + c_p}{1 + c_p}\right)$$

with same c_p defined in the last subsection. Associated numerical results can be found in Tables 6 to 7.

Overall, the approximation performance for UOP pricing here is similar to that in the previous subsection. This is expected since the price of an UOP can be expressed as a linear combination of two joint probabilities as considered in Corollary 1. The bad performance of SIC method for small m and close H has been pointed out in Fuh *et al.* (2013). Here,

[4]That is, we narrow down the state space with additional concern about the behavior of terminal level.

TABLE 5.

Approximation Performance:

$\mathbf{P}\left(\boldsymbol{\tau_m \Delta t} \leq \mathbf{T}, \mathbf{X_T} \leq \mathbf{a}\right)$ **with** $\mathbf{T = 1}$

				(1)	(2)	(3)	(1)	(2)	(3)
λ	m	MC	S.E.	NC	SIC	SDC	R.E.	R.E.	R.E.
Panel A: $H = 90.5$.									
0.01	10	0.349316	0.05 %	0.456307	0.363872	0.363866	30.63 %	4.17 %	4.17 %
	50	0.411467	0.05 %	0.456307	0.416337	0.416336	10.90 %	1.18 %	1.18 %
	250	0.437809	0.05 %	0.456307	0.438726	0.438725	4.23 %	0.21 %	0.21 %
3	10	0.350461	0.05 %	0.459955	0.368168	0.366423	31.24 %	5.05 %	4.55 %
	50	0.41383	0.05 %	0.459955	0.419962	0.419403	11.15 %	1.48 %	1.35 %
	250	0.440867	0.05 %	0.459955	0.442282	0.442096	4.33 %	0.32 %	0.28 %
6	10	0.351632	0.05 %	0.46344	0.372292	0.368901	31.80 %	5.88 %	4.91 %
	50	0.416573	0.05 %	0.46344	0.423447	0.422352	11.25 %	1.65 %	1.39 %
	250	0.443922	0.05 %	0.46344	0.44569	0.445324	4.40 %	0.40 %	0.32 %
Panel B: $H = 92$.									
0.01	10	0.313	0.05 %	0.416	0.320	0.320	32.90 %	2.07 %	2.06 %
	50	0.373	0.05 %	0.416	0.374	0.374	11.54 %	0.27 %	0.27 %
	250	0.397	0.05 %	0.416	0.398	0.398	4.78 %	0.11 %	0.11 %
3	10	0.316	0.05 %	0.420	0.325	0.323	32.81 %	2.75 %	2.03 %
	50	0.378	0.05 %	0.420	0.378	0.378	11.23 %	0.22 %	-0.02 %
	250	0.402	0.05 %	0.420	0.402	0.401	4.46 %	-0.12 %	-0.21 %
6	10	0.317	0.05 %	0.424	0.330	0.325	33.51 %	4.00 %	2.60 %
	50	0.380	0.05 %	0.424	0.382	0.381	11.33 %	0.52 %	0.06 %
	250	0.405	0.05 %	0.424	0.405	0.405	4.55 %	0.05 %	-0.13 %
Panel C: $H = 95$.									
0.01	10	0.232	0.04 %	0.333	0.233	0.233	43.20 %	0.27 %	0.26 %
	50	0.289	0.05 %	0.333	0.288	0.288	15.34 %	-0.30 %	-0.31 %
	250	0.313	0.05 %	0.333	0.313	0.313	6.33 %	-0.10 %	-0.10 %
3	10	0.237	0.04 %	0.338	0.240	0.238	42.45 %	1.34 %	0.30 %
	50	0.293	0.05 %	0.338	0.294	0.292	15.21 %	0.17 %	-0.22 %
	250	0.318	0.05 %	0.338	0.318	0.317	6.24 %	0.05 %	-0.11 %
6	10	0.242	0.04 %	0.342	0.247	0.242	41.73 %	2.26 %	0.31 %
	50	0.299	0.05 %	0.342	0.299	0.297	14.54 %	0.11 %	-0.63 %
	250	0.323	0.05 %	0.342	0.323	0.322	5.85 %	-0.11 %	-0.41 %

Model parameters: $r = 0.1$, $\sigma = 0.2$, $p_u = 0.5$, $\eta_1 = 100/2$, $\xi_1 = 100/3$, $a = \log(96/90)$, $b = \log(H/90)$, and $\Delta t = T/m$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

the SDC method still maintains good performance over those cases. Hence, we would like to say that our endeavors here complement the existing literature on discrete option pricing as considered in Fuh *et al.* (2013) or, more generally, Dia and Lamberton (2011).

TABLE 6.

Approximation Performance:

$\mathbf{UOP_m(k, T)}$ **with** $\mathbf{T = 0.1}$

λ	m	MC	S.E.	(1) NC	(2) SIC	(3) SDC	(1) R.E.	(2) R.E.	(3) R.E.
Panel A: $H = 90.5$.									
0.01	2	3.8289	0.53%	0.7169	3.4420	3.4421	-81.28%	-10.10%	-10.10%
	5	2.8155	0.50%	0.7169	2.5767	2.5767	-74.54%	-8.48%	-8.48%
	25	1.6633	0.42%	0.7169	1.6029	1.6029	-56.90%	-3.64%	-3.64%
3	2	3.8927	0.55%	0.7237	3.4526	3.4783	-81.41%	-11.31%	-10.65%
	5	2.8453	0.51%	0.7237	2.5874	2.6046	-74.57%	-9.06%	-8.46%
	25	1.6756	0.43%	0.7237	1.6132	1.6203	-56.81%	-3.72%	-3.30%
6	2	3.9573	0.56%	0.7304	3.4631	3.5142	-81.54%	-12.49%	-11.20%
	5	2.8914	0.53%	0.7304	2.5980	2.6323	-74.74%	-10.15%	-8.96%
	25	1.6953	0.44%	0.7304	1.6235	1.6376	-56.91%	-4.24%	-3.40%
Panel B: $H = 91$.									
0.01	2	4.1279	0.53%	1.3871	3.8637	3.8638	-66.40%	-6.40%	-6.40%
	5	3.2285	0.52%	1.3871	3.0950	3.0951	-57.04%	-4.14%	-4.13%
	25	2.2210	0.47%	1.3871	2.2088	2.2089	-37.55%	-0.55%	-0.55%
3	2	4.1800	0.55%	1.3970	3.8750	3.9030	-66.58%	-7.30%	-6.63%
	5	3.2569	0.53%	1.3970	3.1055	3.1259	-57.11%	-4.65%	-4.02%
	25	2.2328	0.48%	1.3970	2.2196	2.2292	-37.44%	-0.59%	-0.16%
6	2	4.2361	0.56%	1.4067	3.8861	3.9418	-66.79%	-8.26%	-6.95%
	5	3.2884	0.54%	1.4067	3.1159	3.1563	-57.22%	-5.24%	-4.02%
	25	2.2423	0.48%	1.4067	2.2304	2.2493	-37.26%	-0.53%	0.31%
Panel B: $H = 92$.									
0.01	2	4.6244	0.52%	2.5727	4.5386	4.5387	-44.37%	-1.86%	-1.86%
	5	3.9643	0.53%	2.5727	3.9588	3.9588	-35.10%	-0.14%	-0.14%
	25	3.2418	0.51%	2.5727	3.2541	3.2541	-20.64%	0.38%	0.38%
3	2	4.6671	0.54%	2.5834	4.5538	4.5804	-44.65%	-2.43%	-1.86%
	5	3.9874	0.54%	2.5834	3.9703	3.9916	-35.21%	-0.43%	0.11%
	25	3.2513	0.52%	2.5834	3.2645	3.2756	-20.54%	0.41%	0.75%
6	2	4.7255	0.55%	2.5941	4.5689	4.6218	-45.10%	-3.31%	-2.19%
	5	4.0257	0.55%	2.5941	3.9818	4.0239	-35.56%	-1.09%	-0.04%
	25	3.2696	0.54%	2.5941	3.2750	3.2971	-20.66%	0.16%	0.84%

Model parameters: $r = 0.1$, $\sigma = 0.2$, $p_u = 0.5$, $\eta_1 = 100/2$, $\xi_1 = 100/3$, $S_0 = 90$, $k = \log(96)$, $b = \log(H/90)$, and $\Delta t = T/m$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

4.4. Discussion

All numerical outcomes so far suggest that SIC approximation seems to be a convenient SDC substitute, for its comparable numerical performance especially with small Δt. Here

TABLE 7.

Approximation Performance:

$\mathbf{UOP_m(k, T)}$ with $\mathbf{T = 1}$

λ	m	MC	S.E.	(1) NC	(2) SIC	(3) SDC	(1) R.E.	(2) R.E.	(3) R.E.
Panel A: $H = 90.5$.									
0.01	10	2.2147	0.66%	0.2663	1.9180	1.9181	-87.98%	-13.40%	-13.39%
	50	1.1458	0.50%	0.2663	1.0339	1.0340	-76.76%	-9.76%	-9.76%
	250	0.6436	0.38%	0.2663	0.6145	0.6145	-58.62%	-4.52%	-4.52%
3	10	2.3518	0.69%	0.2797	1.9884	2.0147	-88.11%	-15.45%	-14.34%
	50	1.2049	0.52%	0.2797	1.0774	1.0872	-76.79%	-10.58%	-9.77%
	250	0.6698	0.40%	0.2797	0.6428	0.6463	-58.24%	-4.03%	-3.51%
6	10	2.4773	0.72%	0.2925	2.0543	2.1075	-88.19%	-17.07%	-14.92%
	50	1.2600	0.54%	0.2925	1.1184	1.1384	-76.79%	-11.24%	-9.65%
	250	0.6976	0.41%	0.2925	0.6696	0.6768	-58.07%	-4.01%	-2.99%
Panel B: $H = 92$.									
0.01	10	2.7206	0.71%	1.0321	2.5662	2.5663	-62.06%	-5.68%	-5.67%
	50	1.7755	0.60%	1.0321	1.7539	1.7539	-41.87%	-1.22%	-1.22%
	250	1.3638	0.54%	1.0321	1.3613	1.3613	-24.32%	-0.19%	-0.19%
3	10	2.8698	0.74%	1.0756	2.6553	2.6872	-62.52%	-7.47%	-6.36%
	50	1.8548	0.62%	1.0756	1.8196	1.8344	-42.01%	-1.90%	-1.10%
	250	1.4253	0.56%	1.0756	1.4153	1.4218	-24.54%	-0.71%	-0.25%
6	10	2.9995	0.77%	1.1165	2.7384	2.8029	-62.78%	-8.70%	-6.55%
	50	1.9331	0.65%	1.1165	1.8811	1.9111	-42.24%	-2.69%	-1.14%
	250	1.4776	0.58%	1.1165	1.4660	1.4792	-24.44%	-0.79%	0.11%
Panel B: $H = 95$.									
0.01	10	3.6584	0.77%	2.3852	3.6312	3.6313	-34.80%	-0.75%	-0.74%
	50	2.9898	0.73%	2.3852	2.9873	2.9874	-20.22%	-0.08%	-0.08%
	250	2.6689	0.70%	2.3852	2.6631	2.6631	-10.63%	-0.22%	-0.22%
3	10	3.8348	0.81%	2.4690	3.7565	3.7853	-35.62%	-2.04%	-1.29%
	50	3.1142	0.76%	2.4690	3.0895	3.1046	-20.72%	-0.79%	-0.31%
	250	2.7674	0.73%	2.4690	2.7551	2.7623	-10.78%	-0.44%	-0.19%
6	10	3.9788	0.84%	2.5472	3.8732	3.9317	-35.98%	-2.65%	-1.18%
	50	3.2119	0.79%	2.5472	3.1846	3.2151	-20.69%	-0.85%	0.10%
	250	2.8583	0.76%	2.5472	2.8409	2.8553	-10.88%	-0.61%	-0.10%

Model parameters: $r = 0.1$, $\sigma = 0.2$, $p_u = 0.5$, $\eta_1 = 100/2$, $\xi_1 = 100/3$, $S_0 = 90$, $k = \log(96)$, $b = \log(H/90)$, and $\Delta t = T/m$; MC serves as the benchmark (true value) obtained by simulation with one million replications. We also report associated standard errors (S.E.) in the fourth column.

we further explore the difference between (21) and (22) to get a better understanding why this is the case.

First, we take Taylor expansion respective to $\sqrt{\Delta t}$ for both exponential terms in RHS of

(21) and (22). In this way, we can easily get the leading term in the difference between (21) and (22) has the form

$$A(\alpha,\theta)\left(e^{-b(\beta_{1,\alpha}-\theta)}-e^{-b(\beta_{2,\alpha}-\theta)}\right)\rho\sigma\sqrt{\Delta t}, \tag{27}$$

where

$$A(\alpha,\theta)=\frac{\eta_1-\beta_{1,\alpha}}{\eta_1-\theta}\frac{\beta_{2,\alpha}-\theta}{\beta_{2,\alpha}-\beta_{1,\alpha}}\left(\frac{\beta_{2,\alpha}}{\eta_1}-1\right)(\beta_{1,\alpha}-\theta)$$

Now, we will show you the numerical output with regard to (27) to see how small it is. From Panels (d) and (e) of Figure 2, it is easy to see that the two correction terms about MGF do differ little at the third digit after the decimal point, even taking out the rate of $\sqrt{\Delta t}$.

5. Conclusion

We study some quantities related to boundary problems from PIDE (partial integro-differential equation) point of view. In particular, the distribution of a first passage time can be analyzed through the idea of infinitesimal generators in continuous-time. In this paper, for mixed-exponential jump diffusion models, we provide a new continuity correction to approximate some related distributions under discretely modeling by its continuous counterpart. Unlike the existing results as suggested in Dia and Lamberton (2011), the correction term now depends not only the diffusion but also the jump parts, which leads to smaller numerical errors especially when the discretized time interval is not small and the boundary level is not far away. Such improvement can be observed in the pricing of discrete barrier options as well.

The idea of continuity correction is not only useful to approximate some discrete quantity by its continuous correspondent, but also vice versa. For example, continuity correction suggests a way to improve Monte Carlo estimators for default probabilities in structural credit models. Since now the adjustment in barrier level is no longer simple as solely shifting a uniform amount, how we can carry out this idea of improved Monte Carlo estimators is an interesting topic for future studies.

Another possible direction of future studies is to see if the current PIDE approach can be extended to other processes such as Ornstein-Uhlenbeck processes. We would also like to see if the current method works for other hitting times like two-boundary problems associated with double-barrier options. To further demonstrate the practical value of our new result,

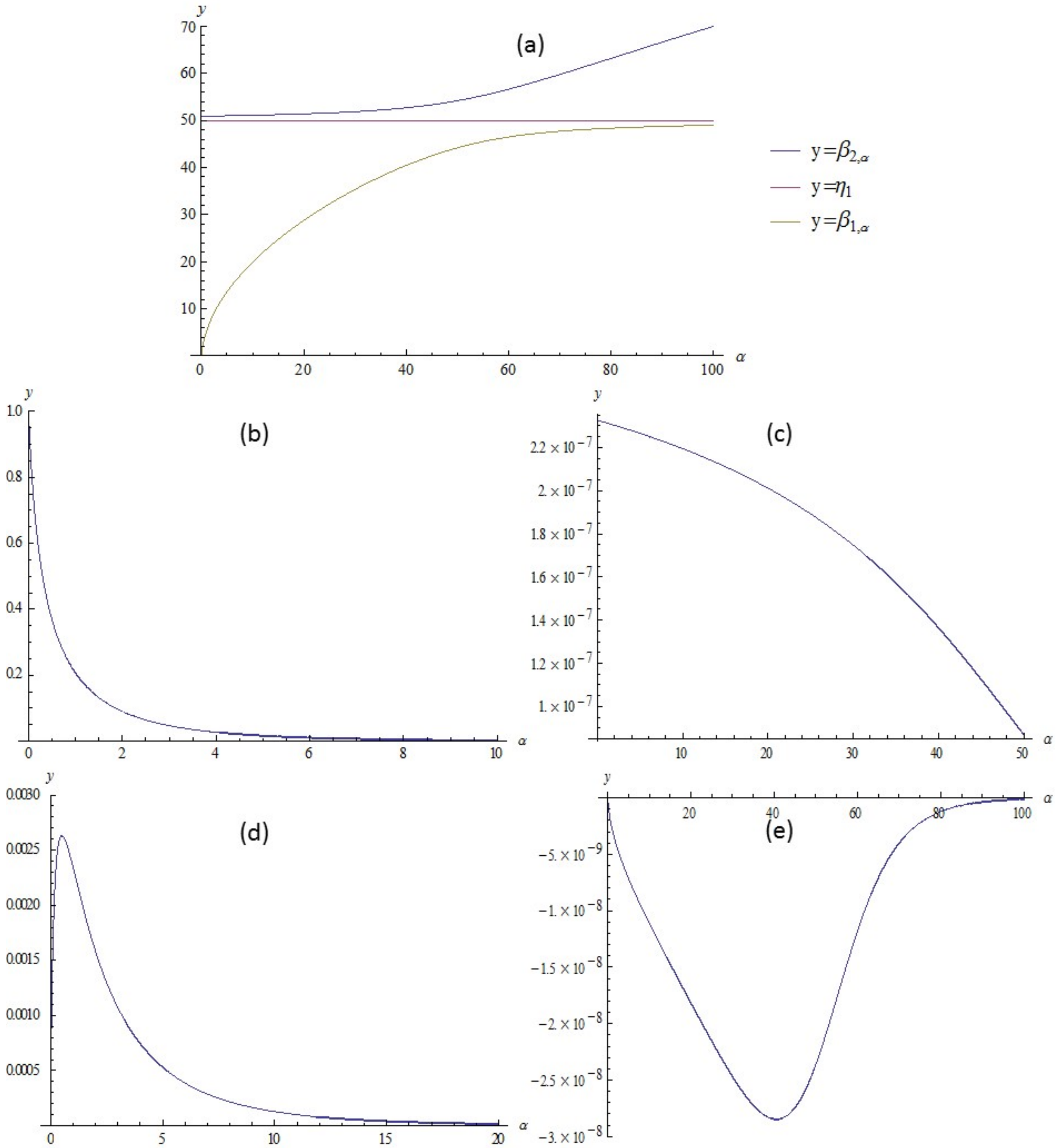

Figure 2. $\mathbf{G(\boldsymbol{\theta}) = \alpha}$ **has two positive roots,** $\beta_1(\alpha)$, $\beta_2(\alpha)$, **with** $\boldsymbol{\mu} = \mathbf{r} - \boldsymbol{\sigma}^2/2 - \boldsymbol{\lambda\delta}$ **and** $\mathbf{n_u = n_d = 1}$. **Set** $\beta_i(\alpha)$ **is a function of** α. **The parameter setting is** $\mathbf{r = 0.1}$, $\boldsymbol{\sigma} = \mathbf{0.2}$, $\boldsymbol{\lambda} = \mathbf{2}$, $\mathbf{q_d = 1 - p_u}$, $\mathbf{p_u = 0.5}$, $\boldsymbol{\eta}_\mathbf{1} = \mathbf{100/2}$, **and** $\boldsymbol{\xi}_\mathbf{1} = \mathbf{100/3}$. **And** $b = 3/10$. **Panel (a): From top to bottom** $y = \beta_{2,\alpha}$, $y = \eta_1$, **and** $y = \beta_{1,\alpha}$, **Panel (b):** $y = e^{-b\,\beta_{1,\alpha}}$, **Panel (c):** $y = e^{-b\,\beta_{2,\alpha}}$, **Panel (d):** $y = A(\alpha, 0)\left(e^{-b\,\beta_{1,\alpha}}\right)\rho\sigma$, **Panel (e):** $y = -A(\alpha, 0)\left(e^{-b\,\beta_{2,\alpha}}\right)\rho\sigma$.

in the future we might also consider pricing contingent claims with endogenous default like Chen and Kou (2009). In a such framework, default event is modeled as boundary crossing and can be triggered only at, say, coupon payment dates. Typically, the coupons will not be paid frequently so that the concern of moderate Δt arises naturally. We want to see if the current correction method can help a lot in the problems of such type.

Appendix A. Preliminary Results

We here independently introduce three lemmas which are highly of great help to establish Theorem 1. The first one is related to the scaled random walk (10), the second one is relevant to the result (16), and the last is the key to final correction form (20).

A.1. Moments of increments

We here estimate the the nth moment of $V_{m,1}$ in (10) first, which are crucial in the pursuit of (16). We will handle this by analyze its cumulant function $G_m(\theta) = \log\left(E[\exp(\theta V_{m,1})]\right)$, which is given by

$$G_m(\theta) = \frac{1}{2}\,\theta^2 + \frac{\mu}{\sigma}\sqrt{\Delta t}\,\theta + \lambda\Delta t\left[\Phi_\Upsilon\left(-i\theta/(\sigma\sqrt{\Delta t})\right) - 1\right]$$

for $\theta \in (-\sigma\sqrt{\Delta t}\,\xi_1,\, \sigma\sqrt{\Delta t}\,\eta_1)$ according to (3). The results are summarized in the next lemma:

Lemma 1. *For the incremental random variables in the process (10), we have*

$$EV_{m,1} = \frac{\sqrt{\Delta t}}{\sigma}\left[\mu + \lambda\left(E\Upsilon_1\right)\right];$$
$$EV_{m,1}^2 = 1 + \frac{\lambda}{\sigma^2}\left(E\Upsilon_1^2\right) + \frac{\Delta t}{\sigma^2}\left[\mu + \lambda\left(E\Upsilon_1\right)\right]^2;$$
$$EV_{m,1}^3 = \frac{\lambda}{\sigma^3\sqrt{\Delta t}}\left(E\Upsilon_1^3\right) + \frac{3\sqrt{\Delta t}}{\sigma}\left[\mu + \lambda\left(E\Upsilon_1\right)\right]\left[1 + \frac{\lambda}{\sigma^2}\left(E\Upsilon_1^2\right)\right] + \frac{\Delta t\sqrt{\Delta t}}{\sigma^3}\left[\mu + \lambda\left(E\Upsilon_1\right)\right]^3,$$

and for each $n \geq 4$,

$$E\left(\sqrt{\Delta t}\,V_{m,1}\right)^n = \Delta t \cdot \frac{\lambda}{\sigma^n}\left(E\Upsilon_1^n\right) + O\left(1/m^2\right).$$

The proof of Lemma 1 is somewhat straightforward but tedious. One only need to carefully examine the moment relationship between $V_{m,1}$ and Υ by taking nth differentiation on $\exp\{G_m(\cdot)\}$ and $\Phi_\Upsilon(\cdot)$ respectively. For the sake of space, we just skip the details here.

A.2. Error estimation

We now start to deal with the no-correction approximation (16). First, let us define

$$\bar{v}(t,x) = \begin{cases} u(t,x), & x < b; \\ f^0(t,x), & x \geq b. \end{cases} \tag{28}$$

Note that $\bar{v}$ is C-infinity thanks to (15). Next, for $t = k\Delta t$ with some integer k, consider functions

$$e_m(t,x) = \begin{cases} E_t \bar{v}(t+\Delta t, x + \sigma\sqrt{\Delta t}\, V_{m,1}) - \bar{v}(t,x), & x < b; \\ 0, & x \geq b, \end{cases} \tag{29}$$

in which E_t denotes the expectation conditional on $\mathcal{F}_t$, and $V_{m,1}$ is the increment defined in (10). Then, the definition (28) leads to

$$\begin{aligned} & Ef^0(\tau_m \Delta t, X_{m,\tau_m}) = E\bar{v}(\tau_m \Delta t, X_{m,\tau_m}) \\ =& \bar{v}(0,0) + E\sum_{k=0}^{\tau_m - 1} \left[\bar{v}\left((k+1)\,\Delta t, \sigma\sqrt{\Delta t}\, W_{m,k+1} \right) - \bar{u}\left(k\,\Delta t, \sigma\sqrt{\Delta t}\, W_{m,k} \right) \right] \\ =& \bar{v}(0,0) + E\sum_{k=0}^{\tau_m - 1} e_m(k\,\Delta t, \sigma\sqrt{\Delta t}\, W_{m,k}), \end{aligned} \tag{30}$$

where we have utilized the tower property of conditional expectations for each summand in the last equality.

Therefore, to achieve (16), the decomposition (30) clearly indicates that we only need to assess the impact of each e_m. Next lemma provides the desired result:

Lemma 2. *For integer k with $0 \leq k < \tau_m$,*

$$e_m(k\,\Delta t, x) = O\left(1/m^2\right) \quad \text{as } m \to \infty,$$

uniformly for all $x < b$.

Proof of Lemma 2. The proof is adapted from Keener (2013, P. 141). The success of extension from PDE to PIDE was due to different moment condition we have. Here, the $V_{m,1}$ in (10) admits exponential moments so that we can get the Taylor expansions to the infinite order. This is also why we can get more concise form as in the current lemma.

To start with, take the infinite-order Taylor expansion of $\bar{v}$ at x on $\{x+\sigma\sqrt{\Delta t}\cdot V_{m,1} < b\}$ to get

$$\begin{aligned}u(t, x+\sigma\sqrt{\Delta t}\cdot V_{m,1}) =& \bar{v}(t,x) + \sigma\sqrt{\Delta t}\cdot \bar{v}_x(t,x)\cdot V_{m,1} \\ &+ (\sigma\sqrt{\Delta t})^2\cdot \bar{v}_{xx}(t,x)\cdot \frac{V_{m,1}^2}{2} \\ &+ (\sigma\sqrt{\Delta t})^3\cdot \bar{v}_{xxx}(t,x)\cdot \frac{V_{m,1}^3}{6} + \sum_{n=4}^{\infty}(\sigma\sqrt{\Delta t})^n\cdot \bar{v}^{(n)}(t,x)\cdot \frac{V_{m,1}^n}{n!}, \end{aligned} \tag{31}$$

where $\bar{v}^{(n)}(t,x) := \frac{\partial^n}{\partial x^n}\bar{v}(t,x)$. On the other hand, the expansion of $\bar{v}$ on $\{x+\sigma\sqrt{\Delta t}\cdot V_{m,1} \geq b\}$ is,

$$\begin{aligned}f^0(t, x+\sigma\sqrt{\Delta t}\cdot V_{m,1}) =& \bar{v}(t,x) + \sigma\sqrt{\Delta t}\cdot \bar{v}_x(t,x)\cdot V_{m,1} \\ &+ (\sigma\sqrt{\Delta t})^2\cdot \bar{v}_{xx}(t,x)\cdot \frac{V_{m,1}^2}{2} \\ &+ (\sigma\sqrt{\Delta t})^3\cdot \bar{v}_{xxx}(t,x)\cdot \frac{V_{m,1}^3}{6} + \sum_{n=4}^{\infty}(\sigma\sqrt{\Delta t})^n\cdot \bar{v}^{(n)}(t,x)\cdot \frac{V_{m,1}^n}{n!}. \end{aligned} \tag{32}$$

Notice that both RHS of (33) and (32) are identical, and we thus can combine then to obtain

$$\begin{aligned}\bar{v}(t, x+\sigma\sqrt{\Delta t}\cdot V_{m,1}) =& u(t,x) + \sigma\sqrt{\Delta t}\cdot u_x(t,x)\cdot V_{m,1} \\ &+ (\sigma\sqrt{\Delta t})^2\cdot u_{xx}(t,x)\cdot \frac{V_{m,1}^2}{2} \\ &+ (\sigma\sqrt{\Delta t})^3\cdot u_{xxx}(t,x)\cdot \frac{V_{m,1}^3}{6} + \sum_{n=4}^{\infty}(\sigma\sqrt{\Delta t})^n\cdot u^{(n)}(t,x)\cdot \frac{V_{m,1}^n}{n!} \end{aligned} \tag{33}$$

for all $x < b$.

Then, plugging (33) into (29) to get

$$\begin{aligned}e_m\left(k\Delta t, x\right) &= \bar{v}\left((k+1)\Delta t, x\right) - \bar{v}\left(k\Delta t, x\right) \\ &+ \sigma\sqrt{\Delta t}\cdot \bar{v}_x((k+1)\Delta t, x)\cdot EV_{m,1} + (\sigma\sqrt{\Delta t})^2\cdot \bar{v}_{xx}((k+1)\Delta t, x)\cdot \frac{EV_{m,1}^2}{2} \\ &+ (\sigma\sqrt{\Delta t})^3\cdot \bar{v}_{xxx}((k+1)\Delta t, x)\cdot \frac{EV_{m,1}^3}{6} + \sum_{n=4}^{\infty}(\sigma\sqrt{\Delta t})^n\cdot \bar{v}^{(n)}((k+1)\Delta t, x)\cdot \frac{EV_{m,1}^n}{n!}\end{aligned}$$

$$
\begin{aligned}
&= \bar{v}\left(\Delta t\cdot(k+1),x\right) - \bar{v}\left(\Delta t\cdot k,x\right)\\
&\quad + \Delta t\cdot\mu\cdot\bar{v}_x((k+1)\Delta t,x) + \Delta t\cdot\bar{v}_x((k+1)\Delta t,x)\cdot\lambda(E\Upsilon_1)\\
&\quad + \Delta t\cdot\frac{\sigma^2}{2}\cdot\bar{v}_{xx}((k+1)\Delta t,x) + \Delta t\cdot\bar{v}_{xx}((k+1)\Delta t,x)\cdot\frac{\lambda(E\Upsilon_1^2)}{2} + O(1/m^2)\\
&\quad + \Delta t\cdot\bar{v}_{xxx}((k+1)\Delta t,x)\cdot\frac{\lambda(E\Upsilon_1^3)}{6} + O(1/m^2)\\
&\quad + \Delta t\cdot\sum_{n=4}^{\infty}\bar{v}^{(n)}((k+1)\Delta t,x)\cdot\frac{\lambda(E\Upsilon_1^n)}{n!} + O(1/m^2), \qquad (34)
\end{aligned}
$$

where we have borrowed the moment estimation of $EV_{m,1}^n$ from Lemma 1 for the second equality.

Now, for the first two terms on the RHS of (34), we again apply the Taylor expansion of $\bar{u}$ but only to the first order at $t = k\Delta t$; and obtain

$$
\begin{aligned}
\bar{v}((k+1)\Delta t,x) &= \bar{v}(k\Delta t,x) + \Delta t\cdot\bar{v}_t(k\Delta t,x) + O(1/m^2)\\
&= \bar{v}(k\Delta t,x) - \Delta t\cdot(\mathcal{L}\bar{v})(k\Delta t,x) + O(1/m^2)\\
&= \bar{v}(k\Delta t,x) - \Delta t\cdot\mu\cdot\bar{v}_x(k\Delta t,x) - \Delta t\cdot\frac{\sigma^2}{2}\cdot\bar{v}_{xx}(k\Delta t,x)\\
&\quad - \Delta t\cdot\lambda\int_{-\infty}^{\infty}\left(\bar{v}(k\Delta t,x+y) - \bar{v}(k\Delta t,x)\right)f_{\Upsilon}(y)dy + O(1/m^2), \qquad (35)
\end{aligned}
$$

where we utilize (12) for the second equality and work out the definition of infinitesimal generator $\mathcal{L}$ in the third equality. Then, combining (34) and (35), we thus achieve

$$
\begin{aligned}
e_m(k\,\Delta t,x) =& -\Delta t\cdot\lambda\int_{-\infty}^{\infty}\bar{v}(k\Delta t,x+y)f_{\Upsilon}(y)\,dy + \Delta t\cdot\lambda\,\bar{v}(k\Delta t,x)\\
&+ \Delta t\cdot\lambda\,(E\Upsilon_1)\bar{v}_x(k\Delta t,x) + \Delta t\cdot\lambda\left(\frac{E\Upsilon_1^2}{2!}\right)\bar{v}_{xx}(k\Delta t,x)\\
&+ \Delta t\cdot\lambda\left(\frac{E\Upsilon_1^3}{3!}\right)\bar{v}_{xxx}(k\Delta t,x)\\
&+ \Delta t\cdot\sum_{n=4}^{\infty}\lambda\left(\frac{E\Upsilon_1^n}{n!}\right)\bar{v}^{(n)}(k\Delta t,x) + O(1/m^2). \qquad (36)
\end{aligned}
$$

Finally, note that in the first term of (36)

$$
\int_{-\infty}^{\infty}\bar{v}(k\Delta t,x+y)f_{\Upsilon_1}(y)dy = E\left[\bar{v}(k\Delta t,x+\Upsilon_1)\right];
$$

so by the infinite-order Taylor expansion of $\bar{u}$ at x again,

$$
\begin{aligned}
E\left[\bar{v}(k\Delta t,x+\Upsilon_1)\right] =& \bar{v}(k\Delta t,x) + \bar{v}_x(k\Delta t,x)\,E\Upsilon_1 + \frac{\bar{v}_{xx}(k\Delta t,x)}{2!}\,E\Upsilon_1^2\\
&+ \frac{\bar{v}_{xxx}(k\Delta t,x)}{3!}\,E\Upsilon_1^3 + \sum_{n=4}^{\infty}\frac{\bar{v}^{(n)}(k\Delta t,x)}{n!}\,E\Upsilon_1^n
\end{aligned}
$$

which clearly shows that all Δt terms in (36) are canceled out by each other. The result thus follows. $\square$

A.3. Two estimates

The incoming two lemmas here will be used to facilitate the estimation of Ef^1 in the next coming subsection. Let $W'_{m,k} = W_{m,k} - k\,\frac{\sqrt{\Delta t}}{\sigma}(\mu + \lambda E\Upsilon_1)$, and $W'_{m,k}$ has zero mean. Define

$$N_d = N_d(m) = \#\{k < \tau_m : W'_{m,k} > b'_m - d\},$$

the number of times the walk is within distance d of the boundary before stopping.
The following lemma follows immediately from Lemma 2.2 of Keener (2013).

Lemma 3. *There exists a finite constant $K \geq 0$ such that*

$$E[N_d] = K\,(1 + d^2),$$

for all $n \geq 1$ and $d > 0$.

Proof of Lemma 3. Without loss of generality, let d be a positive integer, $\sigma = 1$, and $T = 1$. By central limit theorem,

$$P\left[W'_{m,m^2} > m\right] \geq P\left[W'_{m,m^2} > (\sqrt{m}(1 + \lambda\, E\Upsilon_1^2)\, m\right] \geq \gamma, \tag{37}$$

for all m sufficiently large, say $m \geq m_0$. Since the $\Delta t\, \tau_m$ are uniformly integrable and $N_d \leq \tau_m$, we can assume that $m_0 d^2 \leq m$. Define

$$N_{m',d} = \#\{k \leq m' : k < \tau_m,\, W'_{m,k} > b'_m - d\},$$

and let

$$\nu_{j,d} = \inf\{m' : N_{m',d} = j\},$$

so that jth time the walk is within d of the boundary happens on step $\nu_{j,d}$. Note that $N_d \geq j + m_0 d^2$ implies the walk is below the boundary at time $\nu_{j,d} + m_0 d^2$, that is, $W'_{m,\nu_{j,d}+m_0 d^2} < b_m$, which, in turn, implies

$$W'_{m,\nu_{j,d}+m_0 d^2} - W'_{m,\nu_{j,d}} < b'_m - b'_m + d < \sqrt{m_0}\, d.$$

But $W'_{m,\nu_{j,d}+m_0d^2} - W'_{m,\nu_{j,d}}$ is independent of $\{N_d \geq k\}$. So using this bound and (37),

$$P[N_d \geq j + m_0 d^2] \leq P[N_d \geq j](1-\gamma).$$

Iterating this,

$$\begin{aligned}
P[N_d \geq 1 + j\, m_0 d^2] &= P[N_d \geq 1 + (j-1)\, m_0 d^2 + m_0 d^2] \\
&\leq P[N_d \geq 1 + (j-1)\, m_0 d^2](1-\gamma) \\
&\leq \cdots \\
&\leq P[N_d \geq 1](1-\gamma)^j, \quad j = 0, 1, \ldots .
\end{aligned}$$

Hence

$$\begin{aligned}
E[N_d] &= \int_0^\infty P[N_d \geq x] dx \\
&\leq P[N_d \geq 1] \left[1 + \int_1^\infty (1-\gamma)^{\lfloor (x-1)/(m_0 d^2) \rfloor} dx \right] \\
&= P[N_d \geq 1] \left[1 + \frac{m_0 d^2}{\gamma} \right].
\end{aligned}$$

□

The following lemma follows immediately from Corollary 2.3 of Keener (2013).

Lemma 4. *Let $c_k = c_{k,m}$, $k \geq 0$, $m \geq 1$ be constant. Define*

$$\Lambda = \sup_{k,m} (b'_m - c_k),$$

and let g be a non-decreasing function. If $\Lambda < \infty$, and $g(x) \to 0$ as $x \to -\infty$,

$$E \sum_{k=0}^{\tau_m - 1} g(W'_{m,k} - c_k) \leq K \left[g(\Lambda) + 2 \int_{-\infty}^0 |x|\, g(x+\Lambda) dx \right],$$

where K is the constant in Lemma 3.

Proof of Lemma 4. By Fubini's theorem and Lemma 3,

$$\begin{aligned}
E \sum_{k=0}^{\tau_m - 1} g(W'_{m,k} - c_k) &\leq E \sum_{k=0}^{\tau_m - 1} g(W'_{m,k} - b'_m + \Lambda) \\
&\leq \int E \sum_{k \geq 0} \mathbf{1}_{\{x < W'_{m,k} - b'_m + \Lambda,\ k < \tau_m\}} dg(x) \\
&= \int E N_{\Lambda - x} dg(x) \\
&\leq K \int_{-\infty}^{\Lambda} [1 + (\Lambda - x)^2] dg(x) \\
&= K \left[g(\Lambda) + 2 \int_{\infty}^0 |x|\, g(x+\Lambda) dx \right].
\end{aligned}$$

□

A.4. An approximation for $Ef^1(\tau_m \Delta t, X_{m,\tau_m})$

We now consider the overshoot problem for discrete model (10). For any μ', let $V_{m,j} = Z_j + \mu' + \frac{M_{m,j}}{\sigma\sqrt{\Delta t}}$, $j = 1, 2, \cdots$, such that $V_{m,j} \xrightarrow{d} V_j \equiv Z_j + \mu'$. Define

$$f_m(\theta) = E[e^{i\theta V_{m,1}}] = \exp\left\{-\frac{1}{2}\theta^2 + \lambda\Delta t\left[\Phi\left(\frac{\theta}{\sigma\sqrt{\Delta t}} - 1\right) - 1\right] + i\mu'\theta\right\}$$

and

$$g(\theta) \equiv E[e^{i\theta V_1}] = \exp\left\{-\frac{1}{2}\theta^2 + i\mu'\theta\right\}$$

for $\theta \in \mathbb{R}$ as the associated characteristic function of $V_{m,1}$ and V_1, respectively. Note that

$$f_m(\theta) = g(\theta) + O\left(\frac{1}{m}\right).$$

Next define the renewal measures ν_m and ν as $\nu_m(A) = \sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n} V_{m,j} \in A\right\}$ and $\nu(A) = \sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n} V_j \in A\right\}$ for any $A \in \mathcal{B}(\mathbb{R})$; we thus get

Lemma 5.

$$\nu_m(A) = \nu(A) + O\left(\frac{1}{m}\right) \quad as \quad m \to \infty \quad for\ any\ A \in \mathcal{B}(\mathbb{R}).$$

Proof of Lemma 5. By using distribution theory and following the same argument as in Carlsson (1983, P. 147), the proof is somehow straightforward. Here we only summarize the basic idea. For all large enough m such that $EV_{m,1} > 0$, , by repeatedly using (B.1), we can achieve

$$\begin{aligned}\hat{\nu}_m(\theta) \equiv \int_{-\infty}^{\infty} e^{i\theta x}\nu_m(dx) &= \lim_{N\to\infty}\sum_{n=0}^{N-1}[f_m(\theta)]^n = \lim_{N\to\infty}\frac{1-[f_m(\theta)]^N}{1-f_m(\theta)} \\ &= \frac{1}{1-g(\theta)} + \frac{\pi}{\mu'}\delta(\theta) + O(1/m) = \hat{\nu}(\theta) + O(1/m).\end{aligned}$$

Here $\hat{\nu}(\theta) \equiv \int e^{i\theta x}\nu(dx)$ and $\delta(.)$ denotes the Dirac-delta function. Note that the function form $(1 - f_m(\theta))^{-1}$ has a singularity point at $\theta = 0$ since $f_m(0) = 1$. This is where the Dirac-delta function comes into the play to help us overcoming this difficulty. Then, according to the uniform renewal theorem for a family of distributions in Theorem 1 of Blanchet and Glynn (2007), we have the result.

Now, denote $\tau(0) = \inf\{n > 0 : \sum_{j=1}^{n} V_j > 0\}$ and $\tau_m(b') = \inf\{n > 0 : \sum_{j=1}^{n} V_{m,j} > b'\}$. Using Lemma 5 along with Chapter VIII of Siegmund (1985), we can further get

$$P\left\{\sum_{j=1}^{\tau_m(b')} V_{m,j} - b' > y\right\} = \left(E\sum_{j=1}^{\tau(0)} V_j\right)^{-1}\int_{[y,\infty)} P\left\{\sum_{j=1}^{\tau(0)} V_j > u\right\} du + O\left(\frac{1}{m}\right)$$

as $b' \to \infty$. This equation indicates that the limiting distribution of the overshoot of the random walk $\sum V_{m,j}$ can be approximated by that of $\sum V_j$. However, our final goal is to obtain an uniform renewal theorem when both the drift μ' goes to zero and the boundary b' goes to infinity simultaneously while $\mu'\, b'$ remains a constant. Specifically, we will take $\mu' = \mu\sqrt{\Delta t}/\sigma$ and $b' = b_m$. Moreover,

$$E[R_m] = E\left(\sum_{j=1}^{\tau(0)} V_j\right)^2 \Bigg/ 2\,E\sum_{j=1}^{\tau(0)} V_j + o(1/\sqrt{m}). \tag{38}$$

We will denote by ρ as $E\left(\sum_{j=1}^{\tau(0)} V_j\right)^2 \Big/ 2\,E\sum_{j=1}^{\tau(0)} V_j$, which has been shown to be $-\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$; see Chernoff (1965). Here $\zeta(.)$ stands for the Riemann zeta function.

Because the convergence rate we adopt for Ef^0 is $O\left(\frac{1}{m}\right)$ actually (see the form (45)), we only examine the first Taylor term in f^1 here; specifically,

$$f^1(\tau_m\Delta t, X_{m,\tau_m}) = \sigma\sqrt{\Delta t}\,\mathcal{D}_b^{(1)}(\tau_m\Delta t)\,(R_m) + O\left(\frac{1}{m}\right)\sum_{n=2}^{\infty}\sigma^n(\sqrt{\Delta t})^{n-2}\frac{1}{n!}\,\mathcal{D}_b^{(n)}(\tau_m\Delta t)\,(R_m)^n\,.$$

Lemma 6. *As $m \to \infty$,*

$$E[\mathcal{D}_b^{(1)}(\tau_m\Delta t)\,R_m] \to E[\mathcal{D}_b^{(1)}(\tau)]\,\rho,$$

where ρ is the limiting mean.

Proof of Lemma 6. Let $W'_{m,k} = W_{m,k} - k\,\frac{\sqrt{\Delta t}}{\sigma}(\mu + \lambda E\Upsilon_1)$, and $W'_{m,k}$ has zero mean. For $x < 0$, define stopping times

$$T_x = \inf\{k \geq 1 : x + W'_{m,k} \geq 0\},$$

and define

$$H(x) = \begin{cases} x - \rho, & x \geq 0; \\ E[W'_{m,T_x} + x] - \rho, & x < 0. \end{cases}$$

Conditioning on $V_{m,1}$, for $x < 0$

$$H(x) = E\,H(x + V_m) - \frac{\sqrt{\Delta t}}{\sigma}(\mu + \lambda E\Upsilon_1).$$

On $\{W_{m,k} < b_m\}$,

$$E[H(W_{m,k+1} - b_m)|\mathcal{F}_k]) - \frac{\sqrt{\Delta t}}{\sigma}(\mu + \lambda E\Upsilon_1) = H(W_{m,k} - b_m). \tag{39}$$

Now

$$R_m \mathcal{D}_b^{(1)}(\tau_m \Delta t) = \rho \mathcal{D}_b^{(1)}(\tau_m \Delta t) + \mathcal{D}_b^{(1)}(\tau_m \Delta t) H(W_{m,\tau_m} - b_m),$$

and by a telescoping sum argument,

$$\begin{aligned}
&E\, \mathcal{D}_b^{(1)}(\tau_m \Delta t) H(W_{m,\tau_m} - b_m) - \mathcal{D}_b^{(1)}(0) H(-b_m) \\
=&E \sum_{k=0}^{\tau_m - 1} \left[\mathcal{D}_b^{(1)}((k+1)\Delta t) H(W_{m,k+1} - b_m) - \mathcal{D}_b^{(1)}(k\Delta t) H(W_{m,k} - b_m) \right] \\
=&E \sum_{k=0}^{\tau_m - 1} \left[\mathcal{D}_b^{(1)}((k+1)\Delta t) H(W_{m,k+1} - b_m) - \mathcal{D}_b^{(1)}(k\Delta t) H(W_{m,k+1} - b_m) + \mathcal{D}_b^{(1)}(k\Delta t) \frac{\sqrt{\Delta t}}{\sigma} (\mu + \lambda E \Upsilon_1) \right]
\end{aligned}$$

with the last equality from (39), since $\{k < \tau_m\} \in \mathcal{F}_k$. The final expectation is decomposed by the sum of

$$E \sum_{k=0}^{\tau_m - 1} [\mathcal{D}_b^{(1)}((k+1)\Delta t) - \mathcal{D}_b^{(1)}(k\Delta t)]\, H(W_{m,k+1} - b_m) \tag{40}$$

and

$$E \sum_{k=0}^{\tau_m - 1} \mathcal{D}_b^{(1)}(k\Delta t) \frac{\sqrt{\Delta t}}{\sigma} (\mu + \lambda E \Upsilon_1). \tag{41}$$

According to form (14) and Lemma 7, we have

$$\mathcal{D}_b^{(1)}(t) = -e^{-\alpha t + \theta b} \cdot \frac{(\beta_{1,\alpha} - \theta) \cdots (\beta_{n_u+1,\alpha} - \theta)}{(\eta_1 - \theta) \cdots (\eta_{n_u} - \theta)} =: e^{-\alpha t} \cdot \text{constant}.$$

The following proof is under this situation, fixed τ_m. We will utilize $W'_{m,k}$ to achieve asymptotic properties about form (40). Moreover, we must adjust the boundary from b_m to b'_m, where $b'_m = b_m - \tau_m \frac{\sqrt{\Delta t}}{\sigma}(\mu + \lambda E \Upsilon_1)$.

$$\begin{aligned}
&E_{\tau_m} \sum_{k=0}^{\tau_m - 1} [\mathcal{D}_b^{(1)}((k+1)\Delta t) - \mathcal{D}_b^{(1)}(k\Delta t)]\, H(W_{m,k+1} - b_m) \\
=&E_{\tau_m} \sum_{k=0}^{\tau_m - 1} [\mathcal{D}_b^{(1)}((k+1)\Delta t) - \mathcal{D}_b^{(1)}(k\Delta t)] \Big[H(W'_{m,k+1} - b'_m) \\
&\qquad\qquad - \tau_m \frac{\sqrt{\Delta t}}{\sigma} (\mu + \lambda E \Upsilon_1) + k \frac{\sqrt{\Delta t}}{\sigma} (\mu + \lambda E \Upsilon_1) \Big]
\end{aligned}$$

The form (40) is bounded by the sum of

$$E_{\tau_m} \sum_{k=0}^{\tau_m - 1} |\mathcal{D}_b^{(1)}((k+1)\Delta t) - \mathcal{D}_b^{(1)}(k\Delta t)|\, H(W'_{m,k+1} - b'_m) = O\left(\frac{1}{m}\right) E_{\tau_m} \sum_{k=0}^{\tau_m - 1} H(W'_{m,k+1} - b'_m) \tag{42}$$

and

$$(e^{-\alpha\Delta t}-1)\,\text{constant}\,E_{\tau_m}\sum_{k=0}^{\tau_m-1}e^{-\alpha k\Delta t}\,(k-\tau_m)\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1) \tag{43}$$

First of all, Using Lemma 4, it is easy to show that (42) tends to zero as $m\to\infty$.

And,

$$\begin{aligned}
&(e^{-\alpha\Delta t}-1)\,\text{constant}\,E_{\tau_m}\sum_{k=0}^{\tau_m-1}e^{-\alpha k\Delta t}\,(k-\tau_m)\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\\
=&(e^{-\alpha\Delta t}-1)\,\text{constant}\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\,E_{\tau_m}\sum_{k=0}^{\tau_m-1}(k-\tau_m)\,e^{-\alpha k\Delta t}\\
=&(e^{-\alpha\Delta t}-1)\,\text{constant}\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\\
&\qquad\times\left(\frac{e^{-\alpha\Delta t}(1-e^{-\alpha\tau_m\Delta t})}{(1-e^{-\alpha\Delta t})^2}-\frac{(\tau_m-1)\,e^{-\alpha\tau_m\Delta t}}{1-e^{-\alpha\Delta t}}-\frac{\tau_m\,(1-e^{-\alpha\tau_m\Delta t})}{1-e^{-\alpha\Delta t}}\right)\\
=&-\,\text{constant}\,\frac{\sqrt{\Delta t}\,e^{-\alpha\Delta t}}{\sigma(1-e^{-\alpha\Delta t})}(\mu+\lambda E\Upsilon_1)\left(1-e^{-\alpha\tau_m\Delta t}\right)\\
&\qquad+\text{constant}\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\,\left(\tau_m-\,e^{-\alpha\tau_m\Delta t}\right).
\end{aligned}$$

The next equation considers the form (41).

$$\begin{aligned}
E_{\tau_m}&\sum_{k=0}^{\tau_m-1}\mathcal{D}_b^{(1)}(k\Delta t)\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\\
&=E_{\tau_m}\sum_{k=0}^{\tau_m-1}e^{-\alpha k\Delta t}\cdot\text{constant}\cdot\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\\
&=\text{constant}\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\,(1+e^{-\alpha\Delta t}+\cdots+e^{-\alpha(\tau_m-1)\Delta t})\\
&=\text{constant}\,\frac{\sqrt{\Delta t}}{\sigma(1-e^{-\alpha\Delta t})}(\mu+\lambda E\Upsilon_1)\,\left(1-e^{-\alpha\tau_m\Delta t}\right)
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{D}_b^{(1)}(0)H(-b_m)&=\text{constant}\,\left[H(-b'_m)-\tau_m\,\frac{\sqrt{\Delta t}}{\sigma}(\mu+\lambda E\Upsilon_1)\right]\\
&=\text{constant}\,(E[W'_{m,T_x}-b'_m]-\rho)-\text{constant}\,(\mu+\lambda E\Upsilon_1)\,\frac{\sqrt{\Delta t}}{\sigma}\,\tau_m.
\end{aligned}$$

From the above statement and take off the fixed τ_m , we can easy to see that $E[\mathcal{D}_b^{(1)}(\tau_m\Delta t)$ $\cdot H(W_{m,\tau_m}-b_m)]$ tends to zero as $m\to\infty$, the results thus follows. □

A.5. Identities

The last lemma introduces a family of mathematical identities:

Lemma 7. *For each positive integer n, given arbitrary set of nonzero $\eta_1, \eta_2, \cdots, \eta_n$ and distinct, nonzero $\beta_1, \beta_2, \cdots, \beta_{n+1}$, we have*

$$\theta - \sum_{i=1}^{n+1} \beta_i \, d_i = -\frac{(\beta_1 - \theta)(\beta_2 - \theta) \cdots (\beta_{n+1} - \theta)}{(\eta_1 - \theta)(\eta_2 - \theta) \cdots (\eta_n - \theta)},$$

for any real $\theta \notin \{\eta_1, \eta_2, \cdots, \eta_n\}$. Here, similarly, $d_i = \prod_{j=1}^{n} \left(\frac{\eta_j - \beta_i}{\eta_j - \theta}\right) \times \prod_{j=1, j \neq i}^{n+1} \left(\frac{\beta_j - \theta}{\beta_j - \beta_i}\right)$.

Proof of Lemma 7. We will first establish the identity for $\theta = 0$ by mathematical induction, and then extend the result to any other feasible θ.

First note that when $\theta = 0$, the identity is equivalent to

$$\sum_{i=1}^{n+1} \frac{(\eta_1 - \beta_i)(\eta_2 - \beta_i) \cdots (\eta_n - \beta_i)}{\prod_{j=1, j \neq i}^{n+1} (\beta_j - \beta_i)} = 1. \tag{44}$$

Thus, if $n = 1$, it is easy to see

$$\text{RHS of (44)} = \frac{\eta_1 - \beta_1}{\beta_2 - \beta_1} + \frac{\eta_1 - \beta_2}{\beta_1 - \beta_2} = \frac{\eta_1 - \beta_1 + \beta_2 - \eta_1}{\beta_2 - \beta_1} = \frac{\beta_2 - \beta_1}{\beta_2 - \beta_1} = 1.$$

Namely, (44) holds for $n = 1$. Now, suppose (44) holds for $n = k$, and consider the function

$$g(x) = \sum_{i=1}^{k+2} \frac{(\eta_1' - \beta_i') \cdots (\eta_k' - \beta_i')(x - \beta_i')}{\prod_{j=1, j \neq i}^{k+2} \left(\beta_j' - \beta_i'\right)} - 1,$$

which is just linear in x. Then, we have

$$g(\beta_1') = \sum_{i=1}^{k+2} \frac{(\eta_1' - \beta_i') \cdots (\eta_k' - \beta_i')(\beta_1' - \beta_i')}{\prod_{j=1, j \neq i}^{k+2} \left(\beta_j' - \beta_i'\right)} - 1 = \sum_{i=2}^{k+2} \frac{(\eta_1' - \beta_i') \cdots (\eta_k' - \beta_i')}{\prod_{j=2, j \neq i}^{k+2} \left(\beta_j' - \beta_i'\right)} - 1$$

which is exactly zero by the presumption for $n = k$. Similarly, we also have $g(\beta_2') = g(\beta_3') = \cdots = g(\beta_{k+2}') = 0$. These observations imply that $g(x) = 0$ has more than 1 distinct root. But g is merely a polynomial of degree 1; thus $g(x)$ should be zero for all x. In particular, $g(\eta_{k+1}') = 0$, which means (44) also holds for $n = k + 1$. Hence, by mathematical induction, (44) is indeed an identity.

Next, let us turn back to the case with general θ. To start with, deem d_i as $d_i(\theta)$, a function of θ, and consider

$$\phi(\theta) = \theta - [\beta_1 d_1(\theta) + \beta_2 d_2(\theta) + \cdots + \beta_{n+1}(\theta)] + \frac{(\beta_1 - \theta)(\beta_2 - \theta) \cdots (\beta_{n+1} - \theta)}{(\eta_1 - \theta)(\eta_2 - \theta) \cdots (\eta_n - \theta)}.$$

Since $d_i(\beta_i) = 1$ and $d_i(\beta_j) = 0$ for any $j \neq i$ and $i = 1, 2, \cdots, n+1$, we have $\phi(\beta_i) = 0$ for all i. Also, $\phi(0) = 0$ by (44). Consequently, $\phi(\theta) = 0$ has at least $n+2$ distinct roots. However, $\phi(\theta)$ has the same roots as the polynomial function $\phi(\theta)\prod_{j=1}^{n}(\eta_j - \theta)$ with degree at most $n+1$. This implies $\phi(\theta) \equiv 0$ for any $\theta \in \mathbb{R} \setminus \{\eta_1, \ldots, \eta_n\}$. The proof is hence completed. □

Appendix B. Proofs

Proof of Theorem 1

First of all, it is clear from (13) that

$$E\left[f(\tau_m \Delta t, X_{m,\tau_m})\right] = E\left[f^0(\tau_m \Delta t, X_{m,\tau_m})\right] + E\left[f^1(\tau_m \Delta t, X_{m,\tau_m})\right].$$

Then, for the f^0 part, from (30) one can get

$$E\left[f^0(\tau_m \Delta t, X_{m,\tau_m})\right] = \bar{v}(0,0) + O\left(1/m^2\right) E\left(\tau_m\right). \tag{45}$$

according to Lemma 2. Note that $\bar{v}(0,0) = u(0,0)$, $E(\tau_m) < \infty$ under our assumption that $\bar{\mu} > 0$; (16) thus follows. As for the f^1 part, it follows Appendix A.4. Therefore,

$$\begin{aligned} E[f^1(\tau_m \Delta t, X_{m,\tau_m})] &= E\left[\mathcal{D}_b^{(1)}(\tau_m \Delta t)(\sigma\sqrt{\Delta t}\, R_m)\right] + O\left(\frac{1}{m}\right) \\ &= \sigma\sqrt{\Delta t}\left\{E[\mathcal{D}_b^{(1)}(\tau)] \times \rho + o\,(1)\right\} + O\left(\frac{1}{m}\right) \\ &= E\left[\mathcal{D}_b^{(1)}(\tau)\right]\left(\rho\,\sigma\sqrt{\Delta t}\right) + o\,(1/\sqrt{m}\,), \end{aligned}$$

in which we also borrow the weak convergence between $\tau_m \Delta t$ and τ for the third equality. Since $u(0,0) = E\left[f(\tau, X_\tau)\right]$, the result (19) hence follows.

We now go further under the choice $f(t,x) = e^{-\alpha t + \theta x}$. In this special case, we have $u(t,x) = e^{-\alpha t}\nu(x)$ with

$$\nu(x) = \begin{cases} e^{\theta x}, & x \geq b; \\ \sum_{k=1}^{n_u+1} d_k(\alpha,\theta) e^{\theta b} e^{-\beta_{k,\alpha}(b-x)}, & x < b, \end{cases}$$

according to Cai and Sun (2014, Eq. (3.16)). As a result,

$$\frac{\partial}{\partial x} f(t,x)|_{x=b} = \theta\, e^{-\alpha t + \theta b} \quad \text{and} \quad \frac{\partial}{\partial x} u(t,x)|_{x=b^-} = e^{-\alpha t + \theta b}\left(\sum_{k=1}^{n_u+1} \beta_{k,\alpha}\, d_k(\alpha,\theta)\right).$$

So, from (14) and Lemma 7,

$$\mathcal{D}_b^{(1)}(t) = e^{-\alpha t+\theta b}\left(\theta - \sum_{k=1}^{n_u+1} \beta_{k,\alpha}\, d_k(\alpha,\theta)\right) = -e^{-\alpha t+\theta b}\,\frac{(\beta_{1,\alpha}-\theta)\cdots(\beta_{n_u+1,\alpha}-\theta)}{(\eta_1-\theta)\cdots(\eta_{n_u}-\theta)}.$$

Now, note that $E\left[e^{-\alpha\tau}\right]$ can be obtained via (7) with $\theta = 0$, and by (8) we also have

$$d_k(\alpha,0)\,\frac{(\beta_{1,\alpha}-\theta)\cdots(\beta_{n_u+1,\alpha}-\theta)}{(\eta_1-\theta)\cdots(\eta_{n_u}-\theta)} = d_k(\alpha,\theta)\times\frac{\bar{\beta}_{(k),\alpha}}{\bar{\eta}}\,(\beta_{k,\alpha}-\theta)\,.$$

As a result, $E\left[\mathcal{D}_b^{(1)}(\tau)\right] = -\sum_{k=1}^{n_u+1} d_k(\alpha,\theta)e^{-b\,(\beta_{k,\alpha}-\theta)}\,(\bar{\beta}_{(k),\alpha}/\bar{\eta})\,(\beta_{k,\alpha}-\theta)$. Accordingly, by (19) and (7) again, we can achieve

$$\begin{aligned}
&E\left[e^{-\alpha(\tau_m\Delta t)+\theta\, X_{m,\tau_m}}\right]\\
&= \sum_{k=1}^{n_u+1} d_k(\alpha,\theta)e^{-b\,(\beta_{k,\alpha}-\theta)}\times\left(1-\rho\,\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha}}{\bar{\eta}}\,(\beta_{k,\alpha}-\theta)\right) + o\left(1/\sqrt{m}\right).
\end{aligned}$$

Finally, adopting the approximation $e^t = 1 - t + O(t^2)$ for each ρ-term in the last equation, the result (20) thus follows. □

Proof of Corollary 1

Note that for $\theta > 0$

$$\begin{aligned}
L_m(\alpha,\theta) =& \int_0^\infty\int_{-\infty}^\infty e^{-\alpha T-\theta a}E[\mathbf{1}_{\{\tau_m\Delta t\le T, X_T\le a\}}]\,da\,dT\\
=&E\left[\int_{\tau_m\Delta t}^\infty\left(\int_{X_T}^\infty e^{-\alpha T-\theta a}da\right)dT\right]\\
=&\frac{1}{\theta}\,E\left[\int_{\tau_m\Delta t}^\infty e^{-\alpha T-\theta a}\,dT\right] = \frac{1}{\theta}\,E\left[e^{-\alpha(\tau_m\Delta t)}\int_0^\infty e^{-\alpha t-\theta X_{t+\tau_m\Delta t}}\,dt\right]
\end{aligned}$$

On the other hand, the strong Markov property implies that for any $-\eta_1 < \theta < \xi_1$ and $\alpha > \max(0, G(-\theta))$,

$$\begin{aligned}
&E\left[e^{-\alpha(\tau_m\Delta t)}\int_0^\infty e^{-\alpha t-\theta X_{t+\tau_m\Delta t}}\,dt\middle|\mathcal{F}_{\tau_m\Delta t}\right]\\
&= e^{-\alpha(\tau_m\Delta t)-\theta X_{\tau_m\Delta t}}E\left[\int_0^\infty e^{-\alpha t-\theta X_t}\,dt\right]\\
&= e^{-\alpha(\tau_m\Delta t)-\theta X_{\tau_m\Delta t}}\int_0^\infty e^{(G(-\theta)-\alpha)t}\,dt = \frac{e^{-\alpha(\tau_m\Delta t)-\theta X_{m,\tau_m}}}{\alpha - G(-\theta)}.
\end{aligned}$$

Combining all the above together and applying (20), the result thus follows. □

Proof of Corollary 2

Using the standard argument as in Kou et al. (2005), for the parameter space specified in the Corollary, we have

$$\begin{aligned}
\widehat{UIP}_m(\alpha,\theta) &= E\left[\int_0^\infty \int_{-\infty}^\infty e^{-(r+\alpha)T-\theta k}\left(e^k - S_0 e^{X_T}\right)^+ 1_{\{\tau_m \Delta t \le T\}}\, dk\, dT\right] \\
&= E\left[\int_0^\infty e^{-(r+\alpha)T}\, 1_{\{\tau_m \Delta t \le T\}} \left(\int_{\log(S_0 e^{X_T})}^\infty e^{-\theta k}(e^k - S_0 e^{X_T})\, dk\right) dT\right] \\
&= E\left[\int_0^\infty e^{-(r+\alpha)T} 1_{\{\tau_m \Delta t \le T\}} \left(\frac{1}{\theta-1}\left(S_0 e^{X_T}\right)^{1-\theta} - \frac{1}{\theta}\left(S_0 e^{X_T}\right)^{1-\theta}\right) dT\right] \\
&= \frac{1}{\theta(\theta-1)}\, E\left[\int_{\tau_m \Delta t}^\infty e^{-(r+\alpha)T}\left(S_0 e^{X_T}\right)^{-(\theta-1)}\, dT\right] \\
&= \frac{1}{\theta(\theta-1)}\, E\left[e^{-(r+\alpha)\tau_m \Delta t} \int_0^\infty e^{-(r+\alpha)t}\left(S_0 e^{X_{t+\tau_m \Delta t}}\right)^{-(\theta-1)}\, dt\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\theta(\theta-1)}\, E\Bigg[e^{-(r+\alpha)\tau_m \Delta t}\left(S_0 e^{X_{\tau_m \Delta t}}\right)^{1-\theta} \\
&\qquad \times \int_0^\infty e^{-(r+\alpha)t}\, E\left[\left(e^{X_{t+\tau_m \Delta t} - X_{\tau_m \Delta t}}\right)^{-(\theta-1)} \Big| \mathcal{F}_{\tau_m \Delta t}\right] dt\Bigg] \\
&= \frac{1}{\theta(\theta-1)}\, E\left[e^{-(r+\alpha)\tau_m \Delta t}\left(S_0 e^{X_{m,\tau_m}}\right)^{1-\theta}\int_0^\infty e^{-(r+\alpha-G(1-\theta))t}\, dt\right] \\
&= \frac{1}{\theta(\theta-1)}\, \frac{1}{r+\alpha-G(1-\theta)}\, E\left[e^{-(r+\alpha)\tau_m \Delta t}\left(S_0 e^{X_{m,\tau_m}}\right)^{1-\theta}\right] \\
&= \frac{S_0^{1-\theta}}{\theta(\theta-1)}\, \frac{1}{r+\alpha-G(1-\theta)}\, E\left[e^{-(r+\alpha)\tau_m \Delta t + (1-\theta)X_{m,\tau_m}}\right].
\end{aligned}$$

Directly applying (20) to the last equality, the result for $\widehat{UIP}_m$ follows. Then, $\widehat{\Delta}_{UIP_m}$ can be obtained by interchanging derivatives and integrals in the derivation above, similarly to the argument in Cai and Kou (2011). The proof is hence completed. □

Proof of Corollary 3

Using the standard argument as in Kou et al. (2005), for the parameter space specified in the Corollary. Define $M_{X_m}(t) := \max_{0\le k\Delta t\le t} X_{m,k}$,

$$\begin{aligned} L_m(S_0, M, T) :=& E\left[e^{-rT} \max\left\{M, \max_{0\le k\Delta t\le T} S_{k\Delta t}\right\}\right] \\ =& E\left[e^{-rT} \max\left\{M, S_0 e^{M_{X_m}(T)}\right\}\right], \end{aligned}$$

and $z := \log(M/S_0) \ge 0$. Then we have

$$\begin{aligned} L_m(S_0, M, T) =& S_0 E\left[e^{-rT} \max\left\{e^z, e^{M_{X_m}(T)}\right\}\right] \\ =& S_0 e^{-rT} E\left[(e^{M_{X_m}(T)} - e^z)\mathbf{1}_{\{M_{X_m}(T)\ge z\}}\right] + S_0 e^{(z-r)T} \\ =& S_0 e^{-rT} E\left[(e^{M_{X_m}(T)} - e^z)\mathbf{1}_{\{M_{X_m}(T)\ge z\}}\right] + Me^{-rT}. \end{aligned}$$

On the other hand, we can obtain

$$\begin{aligned} E\left[(e^{M_{X_m}(T)} - e^z)\mathbf{1}_{\{M_{X_m}(T)\ge z\}}\right] &= \int_0^\infty (e^y - e^z)\mathbf{1}_{\{y\ge z\}} f_{M_{X_m}(T)}(y)dy \\ &= -\int_z^\infty (e^y - e^z) dP(M_{X_m}(T) \ge y) \\ &= \int_z^\infty e^y P(M_{X_m}(T) \ge y)\, dy, \end{aligned}$$

where $f_{M_{X_m}(T)}(y)$ is the pdf of $M_{X_m}(T)$. Therefore

$$L_m(S_0, M, T) = S_0 e^{-rT} \int_z^\infty e^y P(M_{X_m}(T) \ge y)\, dy + Me^{-rT}.$$

For any $\alpha > 0$, the Laplace transform of $L_m(S_0, M, T)$ w.r.t. T is given by

$$\int_0^\infty e^{-\alpha T} L_m(S_0, M, T) dT \tag{46}$$

$$= S_0 \int_0^\infty e^{-\alpha T} e^{-rT} \int_z^\infty e^y P(M_{X_m}(T) \ge y)\, dy\, dT + \frac{M}{\alpha + r} \tag{47}$$

$$= S_0 \int_z^\infty e^y \left[\int_0^\infty e^{-(\alpha+r)T} P(M_{X_m}(T) \ge y)\, dT\right] dy + \frac{M}{\alpha + r}. \tag{48}$$

Note that for any $y > z \ge 0$, integration by parts leads to

$$\begin{aligned} &\int_0^\infty e^{-(\alpha+r)T} P(M_{X_m}(T) \ge y)\, dT \\ &= \frac{1}{\alpha + r} \int_0^\infty e^{-(\alpha+r)T} dP(M_{X_m}(T) \ge y) \\ &= \frac{1}{\alpha + r} \int_0^\infty e^{-(\alpha+r)T} dP(\tau_m(y)\Delta t \le T) = \frac{1}{\alpha + r} E[e^{-(\alpha+r)\tau_m(y)\Delta t}]. \end{aligned}$$

Applying (20) with $\theta = 0$, we have that, for sufficiently large $\alpha > 0$,

$$\int_0^\infty e^{-(\alpha+r)T} P(M_{X_m}(T) \geq y)\, dT = \frac{1}{\alpha+r} \sum_{k=1}^{n_u+1} d_k(\alpha+r,0)\, e^{-\left(y+\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha+r}}{\bar{\eta}}\right)(\beta_{k,\alpha+r})} + o\left(1/\sqrt{m}\right) \tag{49}$$

Plugging (49) into (48) yields

$$\begin{aligned}
&\int_0^\infty e^{-\alpha T} L_m(S_0, M, T) dT \\
&\quad = S_0 \int_z^\infty e^y \left[\frac{1}{\alpha+r} \sum_{k=1}^{n_u+1} d_k(\alpha+r,0)\, e^{-\left(y+\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha+r}}{\bar{\eta}}\right)(\beta_{k,\alpha+r})}\right] dy + \frac{M}{\alpha+r} + o\left(1/\sqrt{m}\right) \\
&\quad = \frac{S_0}{\alpha+r} \sum_{k=1}^{n_u+1} d_k(\alpha+r,0) e^{-\left(\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha+r}}{\bar{\eta}}\right)(\beta_{k,\alpha+r})} \int_z^\infty e^{-(\beta_{k,\alpha+r}-1)y} dy + \frac{M}{\alpha+r} + o\left(1/\sqrt{m}\right).
\end{aligned}$$

Note that $\beta_{1,\alpha+r} > \beta_{1,r} = 1$ and $\beta_{i,\alpha+r} > \eta_1 > 1$ for any $i = 2, \ldots, n_u + 1$. So we have that

$$\begin{aligned}
&\int_0^\infty e^{-\alpha T} L_m(S_0, M, T) dT \\
&\quad = \frac{S_0}{\alpha+r} \sum_{k=1}^{n_u+1} \frac{d_k(\alpha+r,0) e^{-\left(\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha+r}}{\bar{\eta}}\right)(\beta_{k,\alpha+r})}}{\beta_{k,\alpha+r}-1} e^{-(\beta_{k,\alpha+r}-1)z} + \frac{M}{\alpha+r} + o\left(1/\sqrt{m}\right) \\
&\quad = \frac{S_0}{\alpha+r} \sum_{k=1}^{n_u+1} \frac{d_k(\alpha+r,0) e^{-\left(\rho\sigma\sqrt{\Delta t}\,\frac{\bar{\beta}_{(k),\alpha+r}}{\bar{\eta}}\right)(\beta_{k,\alpha+r})}}{\beta_{k,\alpha+r}-1} \left(\frac{S_0}{M}\right)^{\beta_{k,\alpha+r}-1} + \frac{M}{\alpha+r} + o\left(1/\sqrt{m}\right).
\end{aligned}$$

□

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