Asset Allocation and Pension Liabilities in the Presence of a Downside Constraint

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Abstract

We revisit the question of a pension sponsor's optimal asset allocation in the presence of a downside constraint and the possibility for the pension sponsor to contribute money to the pension plan. When there is disutility associated with contributions, interestingly we find that the optimal portfolio decision often looks like a "risky gambling" strategy where the pension sponsor increases the pension plan's allocation to risky assets in bad states. This is very different from the traditional prediction, where in economy downturns the pension sponsor should fully switch to the risk-free portfolio. Our solution method involves a separation of the pension sponsor's problem into a utility maximization problem and a disutility minimization one.

1 Introduction

A large decline in pension plans' funding ratio motivated the creation of mandatory contribution rules and public insurance on defined benefit pension plans. For example, in the U.S. Employee Retirement Income Security Act (ERISA) in 1974 created the minimum funding contribution (MFC) and Pension Benefit Guaranty Corporation (PBGC).[1] Despite of these government's interventions to save underfunded pension plans, unfortunately large number of defined benefit pension plans are still underfunded.[2] Thus, we believe that it is important to understand how underfunded pension plans can end up with funded status through the optimal asset allocation and contribution policy in the first place.

To this end, we revisit the question of a defined-benefit pension sponsor's optimal asset allocation in the presence of a downside constraint. It is well-known (Grossman and Vila (1989)) that when markets are complete a put-based strategy is optimal by combining the unconstrained optimal portfolio and a put option on that unconstrained portfolio to hedge the downside. This analysis ignores, however, the possibility for the pension sponsor to contribute money to the pension plan over time. We analyze the joint problem of optimal investing and contribution decisions, when there is disutility associated with contributions.[3] Interestingly, we find that with the possibility of costly contributions to the pension plan in bad states to satisfy the downside constraint, the optimal portfolio decision often looks like a "risky gambling" strategy where the pension sponsor increases the pension plan's allocation to risky assets in bad states. This is very different from the traditional prediction, where in economy downturns the pension sponsor should fully switch to the risk-free portfolio that replicates the downside constraint.

Bad states of the economy affect the optimal portfolio weight in two different directions. First, the pension sponsor starts to contribute contemporaneously and keeps doing so as long as the economy is in bad states. Thus, the pension sponsor can invest more aggressively by increasing the equity weight as if the pension plan's asset is increased by the present value of

[1] MFC requirements specify that sponsors of underfunded pension plans must contribute an amount equal to any unfunded liabilities. PBGC has insurance obligations to pay defined benefits to employees when pension sponsors fail to fulfill due to firms' bankruptcy.

[2] In 2013 the largest 100 corporate defined benefits pension plans in the U.S. reported 1.78 trillion USD of liabilities guaranteed with only 1.48 trillion USD of asset, which represents underfunding of more than 15%. See Milliman 2014 Corporate Pension Funding Study, www.milliman.com.

[3] Rauh (2006) finds that mandatory contributions leads to a reduction in corporate investment. Thus, the disutility from contributions is a reduced form of costs of foregone investment opportunities due to a use of internal cash for contributions.

contemporaneous and future contributions. In other words, increased risky allocations will be hedged by contemporaneous and future contributions. Second, the pension sponsor decreases the equity weight to hedge the downside risk. If the former effect dominates the latter one, then a risky gambling behavior can be observed. However, note that this risk taking incentive is induced not by a moral hazard problem, but by a commitment to contributions.

We propose a separation approach to solve the optimal contribution and portfolio policy. The pension sponsor's problem is cast in two separate shadow price problems. The first problem solves for the shadow price of maximizing the utility over the terminal pension plan's asset. The second problem solves for the shadow price of minimizing the intermediate disutility from contributions. We interpret the shadow price of the utility maximization problem as the marginal benefit of increasing contributions. Similarly, the shadow price of the disutility minimization problem is the marginal cost of doing so. We show that the shadow prices for two problems are identical such that the marginal benefit and cost of increasing contributions are equal at the optimal solution.

Our approach allows us to characterize the optimal contribution, portfolio policy, and the value of put option in a simple way. Especially, the optimal contribution and the value of put option shed light on the level of minimum mandatory contributions and the premium that PBGC should charge to the pension sponsor. Also, by comparing with a case without a downside constraint, we can predict morally hazardous reactions of the pension sponsor in the presence of government insurance.

The investment behavior of pension plans has been studied by Sharpe (1976), Sundaresan and Zapatero (1997), Boulier, Trussant, and Florens (1995), and Van Binsbergen and Brandt (2007). Sharpe (1976) first recognized the value of implicit put option in pension plan's asset to insure shortfall at the maturity. Sundaresan and Zapatero (1997) consider the interaction of pension sponsors and their employees. Given the marginal productivity of workers, the retirement date is endogenously determined. Then, pension sponsors solve the investment problem of maximizing the utility over excess assets in liabilities. Our focus is to derive the optimal contribution and portfolio policy, we model the exogenous and deterministic benefits of the pension plan.[4]

Our paper is closely related to Boulier, Trussant, and Florens (1995). In their problem, the investment manager chooses his portfolio weights and contribution rates to minimize the

[4] As long as the market is complete, our model can be extended to incorporate a stochastic feature of liabilities, and the solution technique goes through.

quadratic disutility from contributions with the downside constraint. However, from the perspective of the pension sponsor the surplus at the end of the pension plan also matters since it is usually refunded to the pension sponsor and can be used to fund profitable projects. We model this motive as the utility over the terminal pension plan's asset. Van Binsbergen and Brandt (2007) solve for the optimal asset allocation of the pension sponsor under regulatory constraints. They assume time-varying investment opportunity sets, and explore the impact of regulatory constraints on asset allocations. However, a contribution is not a control variable and a downside constraint is not explicitly specified. Instead, we assume an absence of any government regulations and derive the optimal contribution and portfolio policy. By doing this, we can have policy implications on how minimum contribution rules and premium paid to PBGC should be decided.

Our methodology is based on Karatzas, Lehoczky, and Shreve (1987) and El Karoui, Jeanblanc, and Lacoste (2005). Karatzas, Lehoczky, and Shreve (1987) solve a consumption and portfolio choice problem. They find that the initial wealth can be allocated in two problems, maximizing the utility over intermediate consumption and maximizing the utility over the terminal wealth. The optimal allocation leads to the optimal solution to the original problem. In our model, a contribution is a counterpart of consumption, but it generates the disutility and the pension sponsor's objective is to minimize this disutility. Thus, the problem can be cast in a problem to decide how much to contribute to satisfy the downside constraint while minimizing the disutility. El Karoui, Jeanblanc, and Lacoste (2005) find a put option based solution to maximize the utility over the terminal wealth with the downside constraint. However, their solution can be applied to only initially overfunded pension plans. We allow initially underfunded pension plans to contribute in order to guarantee the terminal benefits.

There are at least three important aspects that we do not address explicitly. First, we do not incorporate time-varying investment opportunities. The expected returns of bonds and equities are predicted by macro variables, such as short rates, yield slopes, and dividend yields. This induces non-trivial hedging demands and liability risks, which drive a wedge between myopic and dynamic investment. Second, we do not consider the taxation issues. Drawing contributions from firm's internal resources is costly for sure, however there is also a benefit from tax deductions. Third, our model do not include inflation. Depending on whether the pension sponsor's preference is in real or nominal term, the allocation to real assets such as TIPS should be considered.

The paper is organized as follows. Section 2 describes the pension plan's benefits and asset return dynamics. Section 3 considers a constrained case in which there is the downside constraint, and the separation method for the optimal investment and contribution policy. Section 4 presents the pension sponsor's problem without the downside constraint as a benchmark case. Section 5 presents our results and Section 6 concludes.

2 Model

2.1 Liability

A defined benefit pension plan pays pre-defined benefits to employees on their retirement date. Usually, the benefits depend on the last 5-year average of salary and the number of years of employment. Let L_t be an index of the pension benefits, i.e. if employees retire right now, they receive L_t. It follows:

$$dL_t = gL_t dt.$$

The pension benefits grow with the rate of g. This reflects an increase in years of employment and growth of salary. The terminal date T is exogenously given. This can be thought as the average duration of employment. We define the downside constraint as

$$K = L_T = L_0 e^{gT}.$$

The pension sponsor should optimally manage the pension plan's asset and contribute to the pension plan's asset such that the terminal value of the pension plan's asset is greater than the amount of the benefits promised to retiring employees.

2.2 Investment Sets

The pension sponsor has two available assets, a risky stock and a risk-free money market account. Let r be the risk-free rate. We assume that r is constant. The stock price follows

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where μ is the expected return of the stock, σ is the volatility parameter, and Z is a standard Brownian motion. Hence, in our model there is only one shock and one risky asset, and the market is complete. This implies that there exists a unique pricing kernel or stochastic discount factor. We have following dynamics of the pricing kernel:

$$\frac{dM_t}{M_t} = -rdt - \eta dZ_t,$$

where $\eta = \frac{\mu - r}{\sigma}$ is the market price of risk. Without loss of generality, we assume that the initial value of the pricing kernel is normalized to one, $M_0 = 1$. Now, the pension plan's asset value follows

$$dW_t = \left[\left(r + \pi_t(\mu - r) \right) W_t + Y_t \right] dt + \pi_t \sigma W_t dZ_t,$$

where π is a fraction of the asset invested in the risky stock, and Y_t is the pension sponsor's contribution to the pension plan's asset.

2.3 Pension Sponsor's Problem

The pension sponsor's problem is

$$\max_{\pi, Y} \mathbb{E} \left[e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi\left(Y_t\right) dt \right] \tag{1}$$
$$\text{s.t. } W_T \geq K,$$

where $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $\phi(x) = k\frac{x^\theta}{\theta}$. The first term in equation (1) is a standard power utility with a relative risk aversion of γ over the final pension plan's asset. The utility over the final pension plan's asset can be justified since at the maturity the pension sponsor receives any pension plan's surplus, which is valuable when internal financing is scarce or external financing is too costly.[5]

The second term in equation (1) represents the pension sponsor's disutility from contributing to the pension plan. The pension sponsor has limited internal resources for profitable projects which might be foregone if the pension sponsor uses the internal cash to contribute to the pension plan. We capture the cost of foregone projects due to contributions as the separable disutility function. A parameter θ will capture a desire to smooth contributions over time. To have convex disutility, we assume that $\theta > 1$. A parameter k captures the importance of the disutility from contributing relative to the utility over the final pension plan's asset. For example, if the pension sponsor is financially healthy (sufficiently high internal resources), the impact of contributing to the pension plan is relatively small and thus the disutility function have low k. Finally, β is the subjective discount rate of the pension sponsor.

The disutility from contributions is a counterpart of adjustment costs in the investment literature.[6] The key difference is that the disutility shows up as a separable objective while ad-

[5] Petersen (1992) uses plan-level data to find evidence in support of the financing motives.

[6] An investment can increase firm's capital, but also incur adjustment costs. See Caballero (1999) for summaries on this literature.

justment costs decrease firm's liquidity. This motivates us to decompose the pension sponsor's problem into two separate ones:

- **Utility Maximization Problem**

  The pension sponsor cannot contribute over time. The pension sponsor manages the initial endowment W_0^u to maximize the expected utility over the final pension plan's asset given the downside constraint:

$$\begin{aligned} &\max_{\pi^u} \mathbb{E}\left[e^{\beta T} u\left(W_T^u\right)\right] \\ &\text{s.t } W_0^u \geq \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} W_T^u\right] \\ &W_T^u \geq K, \end{aligned} \tag{2}$$

  where π^u is a fraction of W^u invested in the risky stock, $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is an expectation under the risk-neutral measure $\mathbb{Q}$.

- **Disutility Minimization Problem**

  The pension sponsor minimizes the expected disutility from contributions while satisfying that the present value of contributions is at least X_0:

$$\begin{aligned} &\min_{Y} \mathbb{E}\left[\int_0^T e^{-\beta t} \phi\left(Y_t\right) dt\right] \\ &\text{s.t } X_0 \leq \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} Y_t dt\right]. \end{aligned} \tag{3}$$

- **Budget Constraint**

  The sum of the pension plan's original initial asset value and the lower bound for the present value of contributions should be equal to the initial endowment W_0^u for the utility maximization problem:

$$W_0 + X_0 = W_0^u. \tag{4}$$

Whenever $K > 0$, we consider the problem as a constrained case. When $K = 0$, there is no downside constraint and it serves as a benchmark case. We will show that solving two problems separately and satisfying the budget constraint (4) will lead us to the solution to the original problem (1).

3 Constrained Case

3.1 Utility Maximization Problem

First, we solve the utility maximization problem. The budget constraint (4) implies that the initial endowment for the first problem W_0^u is greater than the original endowment, W_0, and that the difference $W_0^u - W_0$ is the present value of the contribution stream. That is, the pension sponsor expects the contribution stream in the future and thus, at time zero the pension sponsor can behave as if the pension sponsor borrows the present value of the contribution stream. The optimal amount of borrowing will be determined later by taking into account both the utility over the final asset and the disutility from the contribution. If the initial endowment for the first problem is less than the present value of the downside constraint, $W_0^u \leq Ke^{-rT}$, there is no solution that guarantees the benefits for sure at the maturity. This implies that the present value of the contribution $X_0 = W_0^u - W_0$ should be greater than the (if any) deficit $\max(Ke^{-rT} - W_0, 0)$. For example, if the pension plan is initially underfunded, the present value of the contribution should be greater than the initial shortfall, $Ke^{-rT} - W_0$. The dynamic budget constraint for the first problem is

$$dW_t^u = (r + \pi_t^u(\mu - r))W_t^u dt + \pi_t^u W_t^u \sigma dZ_t. \tag{5}$$

Note that there's no contribution process since it's already reflected in the increased initial endowment W_0^u.

Put-based Strategy

It is well-known (Grossman and Vila (1989)) that when the market is complete the optimal strategy of the first problem consists in investing a fraction of asset in the unconstrained optimal portfolio and using the remaining fraction of asset to purchase a put option on that unconstrained portfolio to hedge the downside. We call this strategy a pub-based strategy. To decide the optimal fraction in the unconstrained optimal portfolio, we define the following functions for any $0 < y < \infty$:

$$\mathcal{W}_u(y) = \underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-rT} I_u\left(y\xi_T\right)\right]}_{\text{Unconstraind optimal portfolio}} + \underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(K - I_u\left(y\xi_T\right)\right)^+\right]}_{\text{Put option}},$$

where $I_u(\cdot)$ is the inverse function of marginal utility $u'(\cdot)$, $\xi_t = M_t e^{\beta t}$ is (subjective) marginal rate of substitution, and $(x)^+ = \max(x, 0)$ is max operator. This function calculates the cost of constructing the put-based strategy when the terminal asset value is random variable $I_u(y\xi_T)$ and the put option's strike price is K. The terminal asset value is chosen such that the marginal utility is proportional to the marginal rate of substitution of the economy at the terminal date. It will be shown that the parameter y is a shadow price, i.e. a marginal increase in the utility when the initial endowment W_0^u for the first problem is marginally increased or the present value of the contribution stream is marginally increased. Proposition 1 explicitly computes this function.

Proposition 1. *The function $\mathcal{W}_u(y)$ is given by*

$$\mathcal{W}_u(y) = y^{-\frac{1}{\gamma}} e^{-\alpha_u T} N\left(\delta_1(y,T)\right) + K e^{-rT} N\left(-\delta_2(y,T)\right), \tag{6}$$

where $\alpha_u = \frac{\beta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\eta^2}{2\gamma}\right)$. δ_1 and δ_2 can be found in Appendix **??**. *Also, the first derivative of $\mathcal{W}_u(y)$ is given by*

$$\mathcal{W}_u'(y) = -\frac{1}{\gamma} y^{-\frac{1}{\gamma}-1} e^{-\alpha_u T} N\left(\delta_1(y,T)\right) < 0. \tag{7}$$

Since the market is complete and the put-based strategy consists in the underlying asset and the put option, the expression for $\mathcal{W}_u(y)$ looks like Black and Scholes (1973) option pricing formula. The first part is the present value of the terminal unconstrained optimal portfolio value multiplied by the probability that the downside constraint is met at the maturity under the forward measure. Note that the final unconstrained optimal portfolio value is discounted with a rate of α_u which is a weighted average of the pension sponsor's subjective discount rate and subjective risk-adjusted expected return. Suppose that the pension sponsor is extremely risk averse. Then, the pension sponsor will allocate all pension plan's asset in the risk-free asset, and thus the terminal unconstrained optimal portfolio value can be discounted with the risk-free rate: $\lim_{\gamma\to\infty} \alpha_u = r$. The second part is the present value of the benefits multiplied by the probability that the put option is in-the-money under the risk-neutral measure.

Since we have the concave utility function, a higher shadow price implies a lower cost of constructing the put-based strategy. Thus, we can see that $\mathcal{W}_u(y)$ is decreasing in the shadow price y, which implies that $\mathcal{W}_u(y)$ is invertible. Let $\mathcal{Y}_u$ denote the inverse of this function. For a fixed initial endowment for the first problem, $W_0^u \geq K e^{-rT}$, we introduce the following random variable

$$W_T^u = I_u\left(\mathcal{Y}_u\left(W_0^u\right)\xi_T\right) + \left(K - I_u\left(\mathcal{Y}_u\left(W_0^u\right)\xi_T\right)\right)^+.$$

The following Theorem 2 states that the constructed terminal asset value is optimal for the problem (2).

Theorem 2. *For any $W_0^u \geq Ke^{-rT}$, W_T^u is optimal for the problem (2), and the optimal portfolio weight is given by*

$$\pi_t^u = \frac{\eta}{\gamma\sigma}\left(1 - \varphi_t\right), \tag{8}$$

where $\varphi_t = \frac{Ke^{-r\tau}}{W_t^u} N\left(-\delta_2\left(y_t, \tau\right)\right) < 1$, $\tau = T - t$, and $y_t = \mathcal{Y}_u(W_0^u)\xi_t$.

Since the put-based strategy is constructed by combining the underlying unconstrained optimal portfolio and its put option, the downside constraint is always satisfied not only at the terminal date, but also along the horizon. Now, the question is how much the pension sponsor should hold the underlying unconstrained optimal portfolio to achieve the maximum utility. Theorem 2 states that the optimal shadow price should be $\mathcal{Y}_u(W_0^u)$ such that the cost of constructing the put-based strategy is exactly same as the initial endowment for the first problem, W_0^u. Then, the optimal portfolio weight is a weighted average of the mean-variance efficient portfolio and zero investment in the equity. The weight on the mean-variance efficient portfolio is $1 - \varphi_t$. The parameter φ_t measures how far away the current asset value is from the present value of the benefits. The closer the asset is to the present value of the benefits, the less fraction of the asset is invested in the equity.

Now, we compute the value function of the first problem and relate its first derivative to the shadow price. Let $J\left(W_0^u\right)$ be the value function of the first problem and define the following function $G(y)$ for $0 < y < \infty$:

$$G(y) = \mathbb{E}\left[e^{-\beta T} u\left(I_u\left(y\xi_T\right) + \left(K - I_u\left(y\xi_T\right)\right)^+\right)\right]. \tag{9}$$

This function computes the expected utility when the put-based strategy is used with the shadow price of y. At the optimal solution, we choose the shadow price satisfying the budget constraint with equality, $y = \mathcal{Y}_u(W_0^u)$, so that we can obtain the value function $J(W_0^u)$ by substituting y in $G(y)$ with $\mathcal{Y}_u(W_0^u)$. Proposition 3 states that the first derivative of the value function, i.e. the shadow price is indeed $\mathcal{Y}_u(W_0^u)$.

Proposition 3. *The function $G(y)$ is given by*

$$G(y) = \frac{y^{1-\frac{1}{\gamma}}}{1-\gamma} e^{-\alpha_u T} N\left(\delta_1(y, T)\right) + e^{-\beta T} \frac{K^{1-\gamma}}{1-\gamma} N\left(-\delta_3(y, T)\right), \tag{10}$$

where δ_3 can be found in Appendix **??**. *Also, $G(y)$ satisfies*

$$\begin{aligned} J\left(W_0^u\right) &= G\left(\mathcal{Y}_u\left(W_0^u\right)\right) \\ J'\left(W_0^u\right) &= \mathcal{Y}_u\left(W_0^u\right). \end{aligned} \tag{11}$$

3.2 Disutility Minimization Problem

The second problem is to decide how to contribute along the horizon to minimize the expected disutility while satisfying the minimum present value of the contribution. Alternatively, the problem can also be stated that the pension sponsor has the initial endowment X_0 in its internal cash account to fund the future contribution stream and decides how to manage this internal resource. The assumption is that the pension sponsor considers only self-financing strategies. Let X_t be the time t value of this account. Then, the dynamic budget constraint of the second problem is

$$dX_t = \left[\left(r + \pi_t^\phi(\mu - r)\right) X_t - Y_t\right] dt + \pi_t^\phi \sigma X_t dZ_t, \tag{12}$$

where π_t^ϕ is a fraction of this account invested in the equity and the rest of it is invested in the risk-free asset. The contribution to the pension plan decreases the account balance.

Now, the problem becomes a standard portfolio choice problem with intermediate outflow (contribution) and no bequest objective. However, there are two important differences. First, contribution does not increase the pension sponsor's utility, but increase the disutility. Second, the static budget constraint states that the present value of the contribution should be greater than the initial endowment. At the optimal solution, the static budget constraint is binding and thus the terminal value of the internal cash account is zero, $X_T = 0$.

We define the following function for any $0 < y < \infty$:

$$\mathcal{W}_\phi(y) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} I_\phi\left(y\xi_t\right) dt\right],$$

where $I_\phi(\cdot)$ be the inverse function of $\phi'(\cdot)$. The function $\mathcal{W}_\phi(y)$ computes the present value of the contribution stream from time zero to the terminal date when an intermediate contribution is set to be $I_\phi(y\xi_t)$, i.e. the marginal disutility is proportional to the marginal rate of substitution of the economy at each time. As the first problem, it will be shown that the parameter y is a shadow price, i.e. a marginal increase in the disutility when the minimum present value of the contribution X_0 is marginally increased. Proposition 4 explicitly computes this function.

Proposition 4. *The function $\mathcal{W}_\phi(y)$ is given by*

$$\mathcal{W}_\phi(y) = \left(\frac{y}{k}\right)^{\frac{1}{\theta-1}} \frac{1 - e^{-\alpha_\phi T}}{\alpha_\phi}, \tag{13}$$

where $\alpha_\phi = \frac{\theta}{\theta-1}\left(r - \frac{\eta^2}{2(\theta-1)}\right) - \frac{\beta}{\theta-1}$. Also, the first derivative of $\mathcal{W}_\phi(y)$ is given by

$$\mathcal{W}_\phi'(y) = \frac{1}{y(\theta-1)}\mathcal{W}_\phi(y) > 0. \tag{14}$$

The present value of the contribution stream has a form of annuity with a rate of return α_ϕ, which is a weighted average of the pension sponsor's subjective discount rate and the subjective risk adjusted expected return. An incentive to smooth contributions over time (high θ) implies that the contribution stream can be discounted with a rate r: $\lim_{\theta\to\infty}\alpha_\phi = r$. Since we have the convex disutility function, the present value of the contribution would be higher if a marginal disutility (shadow price) is higher. Thus, we can see that $\mathcal{W}_\phi(y)$ is increasing, which implies that $\mathcal{W}_\phi(y)$ is invertible. Let us denote $\mathcal{Y}_\phi$ be the inverse of the function $\mathcal{W}_\phi$. For the minimum contribution requirement $X_0 > 0$, we introduce the contribution process

$$Y_t = I_\phi\left(\mathcal{Y}_\phi\left(X_0\right)\xi_t\right).$$

Theorem 5 states that the above contribution policy is optimal for the problem (3).

Theorem 5. *For any $X_0 > 0$, Y_t constructed above is optimal for the problem (3), and the optimal hedging policy is*

$$\pi^\phi = -\frac{\eta}{(\theta-1)\sigma}.$$

By setting the marginal disutility of the contribution to be proportional to the marginal rate of substitution of the economy at each time, the minimum disutility can be achieved. The shadow price is determined such that the present value of the contribution stream is identical with the minimum contribution requirement X_0. The optimal hedging policy is to short the equity, since the contribution is increasing in the marginal rate of substitution or decreasing in the stock return. Whenever the stock price decreases, the pension sponsor should increase the contribution which can be funded with profits from short positions in the equity. If the pension sponsor has a strong desire to smooth the contribution (higher θ), the pension sponsor would decrease short positions in the equity since the contribution stream is stable.

Finally, we compute the value function of the second problem. Let $L\left(X_0\right)$ be the value function of the second problem and define the following function $C(y)$ for $0 < y < \infty$:

$$C(y) = \mathbb{E}\left[\int_0^T e^{-\beta t}\phi\left(I_\phi\left(y\xi_t\right)\right)dt\right]. \tag{15}$$

This function computes the expected disutility when the contribution is set to be $I_\phi(y\xi_t)$ as a function of y. At the optimal solution, we choose $y = \mathcal{Y}_\phi(X_0)$ so that we can obtain the value function $L(X_0)$ by substituting y in $C(y)$ with $\mathcal{Y}_\phi(X_0)$. Proposition 6 states that the first derivative of $L(X_0)$ (shadow price) is indeed $\mathcal{Y}_\phi(X_0)$.

Proposition 6. *The function $C(y)$ is given by*

$$C(y) = \frac{k}{\theta}\left(\frac{y}{k}\right)^{\frac{\theta}{\theta-1}} \frac{1-e^{-\alpha_\phi T}}{\alpha_\phi},$$

and satisfies

$$\begin{aligned} L\left(X_0\right) &= C\left(\mathcal{Y}_\phi\left(X_0\right)\right) \\ L^{'}\left(X_0\right) &= \mathcal{Y}_\phi\left(X_0\right). \end{aligned} \tag{16}$$

3.3 Optimality of Separation

So far, we derive the solutions for the utility maximization problem and the disutility minimization problem while taking the present value of the contribution as given. We now show that the optimal choice of the present value of the contribution X_0 leads that separately solved solutions are indeed the solution for the original problem. The pension sponsor behaves as if taking a leverage at time zero, $W_0^u = W_0 + X_0$. With W_0^u, the pension sponsor solves the utility maximization problem with the downside constraint. Then, the pension sponsor solves the disutility minimization problem to pay back the borrowing X_0 through the contributions. The next theorem shows that how the initial borrowing amount X_0 is decided to achieve the optimality of the original problem.

Theorem 7. *Consider an arbitrary portfolio and contribution policy pair $(\tilde{\pi}, \tilde{Y})$ satisfying the downside constraint. Then, there exists a pair (π, Y) dominating $(\tilde{\pi}, \tilde{Y})$. In particular, the value function of the original problem $V\left(W_0\right)$ is*

$$V\left(W_0\right) = \max_{X_0} J\left(W_0 + X_0\right) - L\left(X_0\right) = \max_{\mathcal{W}_u(y_u) - \mathcal{W}_\phi\left(y_\phi\right) = W_0} G\left(y_u\right) - C\left(y_\phi\right). \tag{17}$$

For an arbitrary portfolio and contribution policy pair, we can take the present value of that contribution stream, $X_0 = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\tilde{Y}_t dt\right]$. Then, for X_0, $\tilde{\pi}$ is a feasible strategy to the utility maximization problem (2), and $\tilde{Y}$ is a feasible strategy to the disutility minimization problem (3). We can find the optimal solutions to each problem and they will (weakly) dominate $(\tilde{\pi}, \tilde{Y})$. Thus, finding the optimal solution to the original problem (1) can be translated into the problem to find the optimal present value of the contribution X_0 to maximize the difference between two value functions of (2) and (3), $J(W_0 + X_0) - L(X_0)$.

Suppose that (17) has an interior solution. This implies that the FOC with respect to X_0 equals zero:

$$J'\left(W_0 + X_0\right) = L'\left(X_0\right).$$

This condition states that at the optimal solution, the marginal increase in the value function of the utility maximization problem should be identical with the marginal increase in the value function of the disutility minimization problem. Thus, we can interpret LHS as the marginal benefit of increasing the present value of the contribution, and RHS as the marginal cost of increasing the present value of the contribution. Recall that the shadow prices of both problems are satisfying the static budget constraints with equality. Hence, we have $y = \mathcal{Y}_u(W_0 + X_0) = \mathcal{Y}_\phi(X_0)$, which is determined by the budget constraint:

$$\mathcal{W}_u\left(y\right) - \mathcal{W}_\phi\left(y\right) = W_0. \tag{18}$$

Define the following function for $0 < y < \infty$:

$$\mathcal{W}(y) = \mathcal{W}_u\left(y\right) - \mathcal{W}_\phi\left(y\right).$$

This function computes the initial pension fund's asset required to have the shadow price of y for both problems. Proposition 8 shows that there exists a unique y solving $\mathcal{W}(y) = W_0$, and thus we obtain the optimal solution to the original problem.

Proposition 8. *The function $\mathcal{W}(y)$ is decreasing in y and $\lim_{y\to 0}\mathcal{W}(y) = \infty$ and $\lim_{y\to\infty}\mathcal{W}(y) = -\infty$. Hence, there exists a unique y satisfying $\mathcal{W}(y) = W_0$.*

Suppose that we find y solving (18). Then, the time t pension plan's asset can be expressed as $W_t = W_t^u - X_t$. This implies that the present value of the terminal pension plan's asset ($W_T = W_T^u$ since $X_T = 0$) is the sum of the current pension plan's asset and the pension sponsor's the internal fund for hedging the contributions. Proposition 9 describes the optimal portfolio weight and contribution policy to the original problem.

Proposition 9. *The optimal portfolio weight is given by*

$$\pi_t = \pi_t^u \rho_t + \pi^\phi\left(1 - \rho_t\right),$$

and the optimal contribution rate is given by

$$\frac{Y_t}{W_t} = \left(\rho_t - 1\right)\frac{\alpha_\phi}{1 - e^{-\alpha_\phi(T-t)}},$$

where $\rho_t = \frac{W_t^u}{W_t} = 1 + \frac{X_t}{W_t}$.

The optimal portfolio weight is a weighted average of two weights, π_t^u and π^ϕ. The weight is the ratio of the present value of the terminal pension plan's asset over the current pension plan's asset. We call ρ_t the pension plan's leverage ratio. Note that because a possibility of future contributions, this ratio is generally not equal to one. When the state of the economy is good and the expected contribution is small, then the weight ρ is close to one. Also, π_t^u becomes the mean-variance efficient portfolio ($\frac{\eta}{\gamma\sigma}$) since it is more likely that the downside constraint is not binding. Thus, the optimal portfolio weight, π_t is close to the mean-variance efficient portfolio.

As the economy gets worse (the equity price drops), the pension plan's asset gets close to the downside constraint. There are two effects of bad states of the economy in the optimal portfolio weight. First, the pension sponsor will hold large internal resources X_t to hedge large contemporaneous and future contributions, which indicates an increase in ρ_t. Thus, the pension sponsor will increases the equity weight, which is hedged by contemporaneous and future contributions. Second, the optimal equity weight for the utility maximization problem π_t^u will decrease, since the present value of the terminal pension plan's asset, W_t^u approaches to the present value of the benefits. If the latter effect dominates the former one, then a risk management behavior can be observed, i.e. a decrease in the equity weight as the economy gets worse. On the other hand, if the former effect dominates, we can see a risk taking behavior. However, note that this risk taking incentive is induced not by a moral hazard problem, but by a commitment to contributions in the future.

The optimal contribution policy as a fraction of the current pension plan's asset also depends on the pension plan's leverage ratio ρ_t and time-to-maturity $T - t$. The pension sponsor contributes more when the pension plan's asset return is low so that the leverage ratio is high. For the same pension plan's leverage ratio ρ_t, the ratio of the contribution to the pension plan's asset is higher when time-to-maturity is short. Since the pension plan's objective is to minimize the expected disutility, the pension plan would defer a contribution as much as it can.

4 Benchmark Case

Now, we consider the benchmark case. There's no downside constraint and the pension sponsor contributes purely for maximizing the terminal pension plan's asset while taking into account the disutility from contributions. The pension sponsor's problem becomes

$$\max_{\pi, Y} \mathbb{E}\left[e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi\left(Y_t\right) dt\right].$$

Everything we derive for the constrained case goes through, except for the first problem. Now, let $\mathcal{W}_u^{BC}(y)$ be the counterpart of $\mathcal{W}_u(y)$ in the constrained case:

$$\mathcal{W}_u^{BC}(y) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} I_u\left(y\xi_T\right)\right].$$

Note that the pension plan holds just the unconstrained optimal portfolio since there's no downside constraint. Similarly, we can consider $\mathcal{Y}_u^{BC}(W_0^u)$, $J^{BC}(W_0^u)$, and $G^{BC}(y)$ as the benchmark version of $\mathcal{Y}_u(W_0^u)$, $J(W_0^u)$, and $G(y)$. Proposition 10 summarizes the results for the first problem in the benchmark case.

Proposition 10. *The function $\mathcal{W}_u^{BC}(y)$ is given by*

$$\mathcal{W}_u^{BC}(y) = y^{-\frac{1}{\gamma}} e^{-\alpha_u T}. \tag{19}$$

Also, the first derivative is given by

$$\mathcal{W}_u^{BC}(y)' = -\frac{1}{\gamma} y^{-\frac{1}{\gamma}-1} e^{-\alpha_u T} < 0. \tag{20}$$

For a given y, we have $\mathcal{W}_u^{BC}(y) < \mathcal{W}_u(y)$. For any W_0^u, $W_T^u = I_u\left(\mathcal{Y}_u^{BC}\left(W_0^u\right)\xi_T\right)$ is optimal for the utility maximization problem, and the optimal portfolio weight is given by $\pi_{BC}^u = \frac{\eta}{\gamma\sigma}$. The function $G^{BC}(y)$ is given by

$$G^{BC}(y) = \frac{y^{1-\frac{1}{\gamma}}}{1-\gamma} e^{-\alpha_u T},$$

and satisfies

$$\begin{aligned} J^{BC}\left(W_0^u\right) &= G^{BC}\left(\mathcal{Y}_u^{BC}\left(W_0^u\right)\right) \\ J^{BC}\left(W_0^u\right)' &= \mathcal{Y}_u^{BC}\left(W_0^u\right). \end{aligned} \tag{21}$$

Without the downside constraint, the present value of the terminal pension plan's asset is smaller than the constrained case. To achieve the same level of marginal utility, the benchmark case requires smaller initial asset since the put option doesn't have to be purchased. As we expect, the optimal portfolio weight is the mean-variance efficient portfolio. Now, Theorem 7, Proposition 8 and 9 can be stated for the benchmark case by substituting corresponding counterparts with $J^{BC}(W_0^u)$, $G^{BC}(y)$, $\mathcal{W}_u^{BC}(y)$, $\mathcal{Y}_u^{BC}(W_0^u)$, and π_{BC}^u.

Table 1: Summary of key variables and parameters

Variable	Symbol	Parameters	Symbol	Value
Terminal benefits	K	Pension plan's investment horizon	T	10-year
Pension's asset	W	Price of Risk	η	0.4
Present value of the terminal asset	W^u	Risk-free rate	r	2%
Pension sponsor's internal resources for hedging contribution	X	Pension sponsor's subjective discount rate	β	1%
Shadow price	y	Pension sponsor's risk aversion	γ	5
Pension sponsor's marginal rate of substitution	ξ	Pension sponsor's elasticity of disutility	θ	2
Portfolio weight of equity	π	Relative importance of disutility	k	100
Contribution flow	Y	Initial funding ratio (underfunded)	λ_0	80%
		Initial funding ratio (overfunded)	λ_0	120%

This table summarizes the symbols for the key variables used in the model and the parameter values in the baseline case.

5 Quantitative Analysis

We now turn to quantitative analysis of the model. For a baseline case, we use 10-year for the pension plan's maturity T. According to Bureau of Labor Statistics, as of 2014 the median years of tenure with current employer for workers with age over 65 years is 10.3-year. Also, we use $\eta = 0.4$ for the market price of risk, $\sigma = 20\%$ for the volatility of the equity, $r = 2\%$ for the risk-free rate, and $\beta = 1\%$ for the pension sponsor's subjective discount rate. These numbers are standard assumptions in the literature. The expected excess return of the equity is $\mu - r = \sigma\eta = 8\%$. We use $\gamma = 5$, which implies the equity weight of the mean-variance efficient portfolio is $\frac{\eta}{\gamma\sigma} = 40\%$. For the disutility function, we use $k = 100$ and $\theta = 2$. The quadratic disutility function is common in the investment literature, in which a firm is assumed to be risk-neutral and faces quadratic costs of investment adjustment.[7] Finally, we use two values for the initial funding ratio, $\lambda_0 = \frac{W_0}{Ke^{-rT}} = 80\%$ or 120%. We will vary preference parameters, (γ, k, θ), and the price of risk to see the impacts on the optimal present value of the contribution, portfolio and contribution policy. Table 1 summarizes all the key variables and parameters in the model.

[7] See Gould (1968); more recently Bolton, Chen, and Wang (2011); among others.

5.1 Present Value of Contribution

Figure 1 plots the determination of X_0 by equating the shadow prices of the first and second problem: $\mathcal{Y}_u(W_0 + X_0) = \mathcal{Y}_\phi(X_0)$. The initial pension plan's asset is normalized to one $W_0 = 1$ and thus the present value of the contribution can be interpreted as a fraction of the initial pension plan's asset. Panel A is a case when the pension plan is initially underfunded, $\lambda_0 = 80\%$, and Panel B is a case when overfunded, $\lambda_0 = 120\%$. We also plot the benchmark case. Since we assume the quadratic disutility function, the shadow price of the disutility minimization problem is linear in the present value of the contribution. The shadow price of the utility maximization problem is decreasing in the present value of the contribution, since the utility function is concave. Also, the shadow price of the first problem for the constrained case is always above that of the benchmark case since for the same shadow price, the pub-based strategy costs more.

We can see that the optimal present value of the contribution is $X_0 = 3.68\%$ and the shadow price is $y = 0.18$ for the benchmark case. This indicates that along the horizon the pension sponsor contributes 3.68% even though there is no downside constraint. This is because the marginal benefit of contributing is greater than the marginal cost of doing so when $X_0 < 3.68\%$ as we can see in Figure 1.

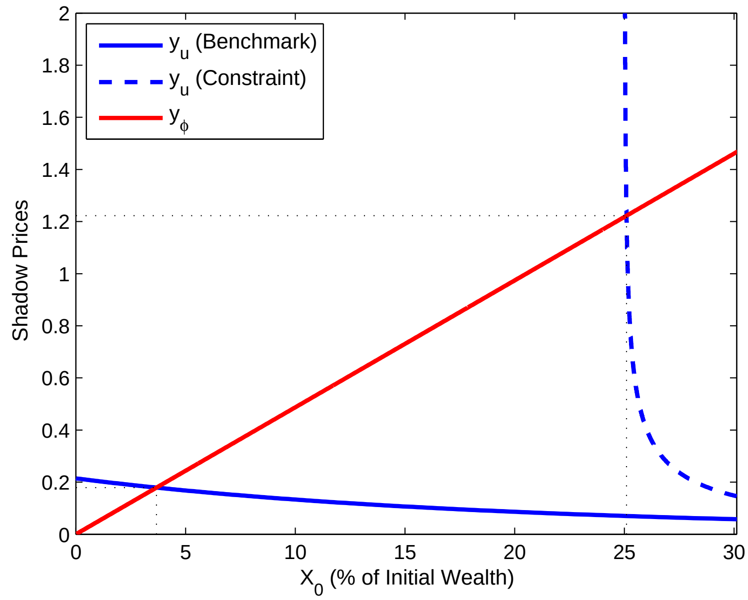
For the underfunded pension plan, the marginal benefit curve $\mathcal{Y}_u(W_0 + X_0)$ has the left asymptote line at $X_0 = Ke^{-rT} - W_0 = 25\%$, at which the solution for the utility maximization problem is zero investment in the equity. The optimal present value of the contribution is $X_0 = 25.10\%$ and the shadow price is $y = 1.22$. Compared to the benchmark case, the pension sponsor contributes more to make the pension plan overfunded at the maturity. For the overfunded pension plan, the present value of the contribution is $X_0 = 4.15\%$ and the shadow price is $y = 0.20$. This implies that relative to the benchmark case the additional contributions of 0.47% are required to guarantee the benefits for the initially overfunded pension plan.

Figure 2 plots the the cost of constructing the put-based strategy for the utility maximization problem with the downside constraint. Again, Panel A is a case when the pension plan is initially underfunded, and Panel B is a case when overfunded. We put a fraction of the initial endowment $W_0^u = W_0 + X_0$ invested in the unconstrained optimal portfolio or the mean-variance efficient portfolio on x-axis. The dashed line is a 45-degree line, i.e. the cost of the unconstrained optimal portfolio in the put-based strategy and the solid line represents the total cost of the put-based strategy. Thus, the difference between two lines represents the cost of purchasing the put option.

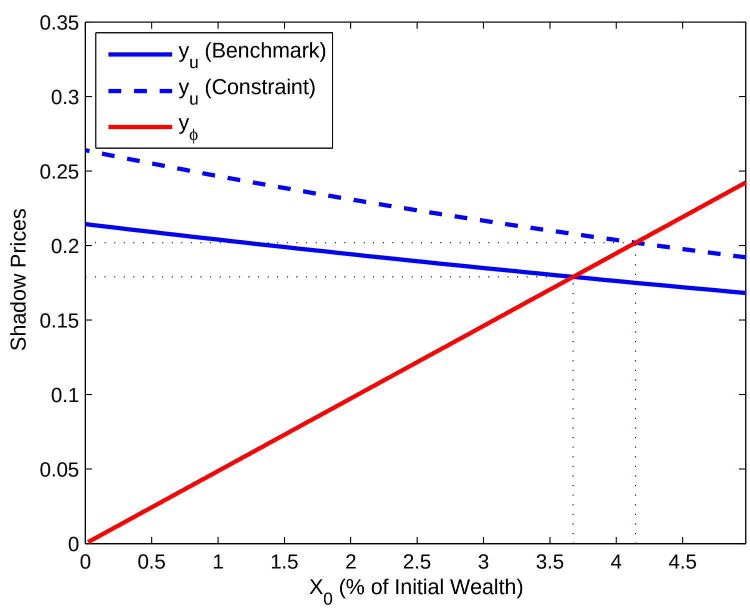
We can see that for the initially underfunded pension plan, without contributions there's no

Figure 1: Determination of Present Value of Contribution

Panel A: Initially Underfunded Pension



Panel B: Initially Overfunded Pension



This figure plots shadow prices of first and second problem as a function of present value of contribution. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially over-funded pension with 120% funding ratio.

feasible put-based strategy. That is, the cost of the put-based strategy is greater than $W_0^u = W_0 = 100\%$ when $X_0 = 0$. However, with the optimal present value of the contribution $X_0 = 25.10\%$, the initial endowment of the first problem is increased to $W_0^u = W_0 + X_0 = 125.10\%$ and thus there is the optimal put-based strategy which costs exactly $W_0^u = 125.10\%$. At the optimal put-based strategy, the effective allocation to the mean-variance efficient portfolio is 70.60%, and the rest 54.50% is used to replicate the put option on 70.70% of the mean-variance efficient portfolio.

On the other hand, for the initially overfunded pension plan, even with zero contribution there's a feasible put-based strategy, which costs exactly $W_0^u = W_0 = 100\%$ and consists in the mean-variance efficient portfolio of 95.92% and the put option of 4.08% on that portfolio. However, with contribution, the pension sponsor can be better off by allocating 101.21% to the mean-variance efficient portfolio and 2.94% to the put option on that portfolio. The total cost of this put-based strategy is $W_0^u = 104.15\%$ and the shortfall $X_0 = W_0^u - W_0 = 4.15\%$ will be funded through contributions.

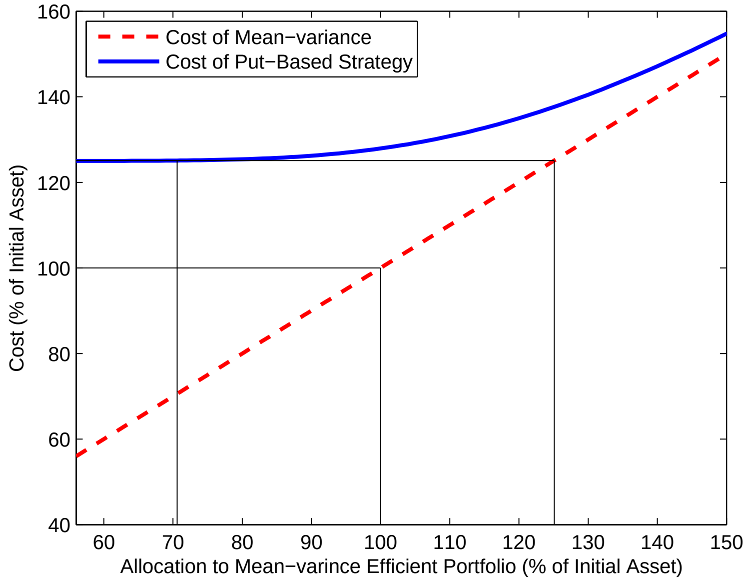
5.2 Portfolio Weight and Contribution Policy

Figure 3 plots the equity weight at time $t = 5$-year as a function of an annualized equity return over the last five years. We fix the initial pension plan's asset and vary the terminal benefits, K. We set $K = 153\%$ for Panel A, and $K = 102\%$ for Panel B such that the initial funding ratios are 80% and 120%, respectively. First, we can see that the equity weight of the benchmark case is decreasing in the past equity return. The low equity return over time zero to 5-year indicates that the state of economy is bad, i.e. the marginal rate of substitution is high. As we will see in Figure 4, the optimal contribution rule is to increase contributions in a such state. The pension sponsor expects that future contributions will be made, and thus can take more risks by increasing the equity weight. Put differently, when the state of economy is bad, future contributions can hedge positions in the equity, and thus the pension sponsor can take more risks. When the past equity return is higher, the equity weight of the benchmark case is approaching to the mean-variance efficient portfolio, which is $\frac{\eta}{\gamma\sigma} = 40\%$. We can see that the equity weight of the benchmark case is identical for the initially underfunded and overfunded pension plans. This is obvious since how far away from the present value of the benefits doesn't matter for the pension sponsor without the downside constraint.

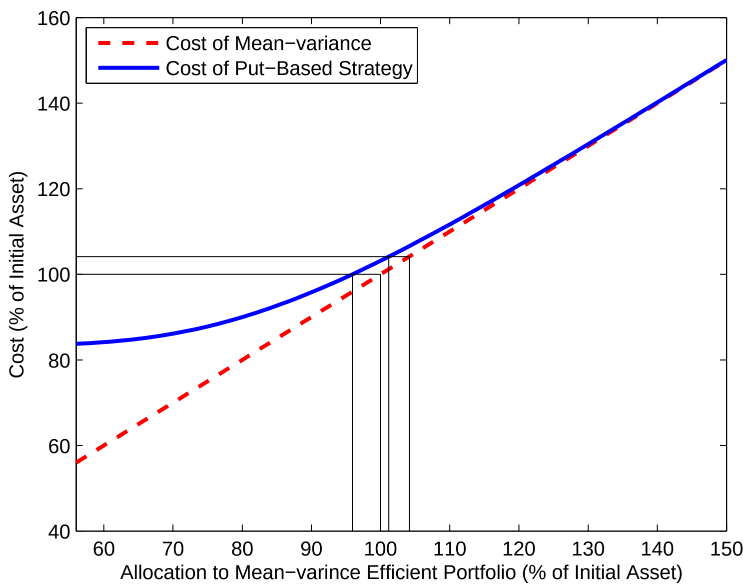
Next, the equity weight of the constrained case exhibits an U-shaped pattern. When the state of economy gets worse, the pension sponsor defers contributions and employs the risk manage-

Figure 2: Cost of Put-Based Strategy

Panel A: Initially Underfunded Pension



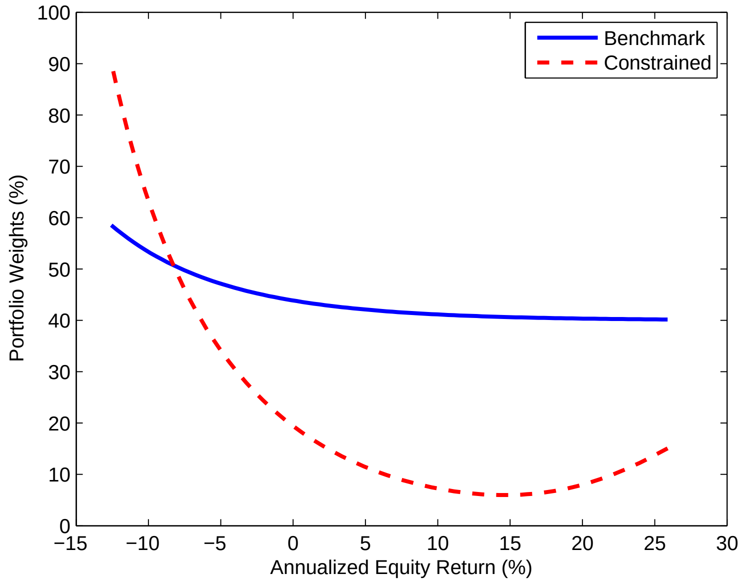
Panel B: Initially Overfunded Pension



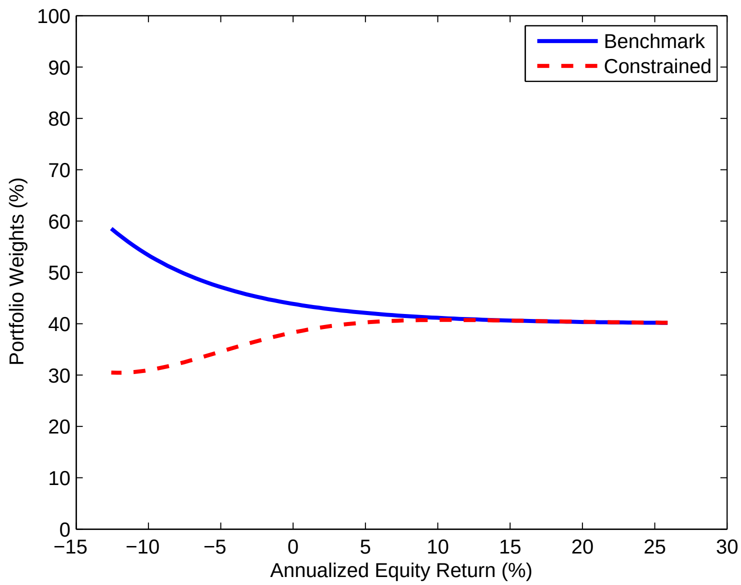
This figure plots costs of put-based strategy as a function of fraction of allocation to the mean-variance efficient portfolio. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially overfunded pension with 120% funding ratio.

Figure 3: Equity Weight

Panel A: Initially Underfunded



Panel B: Initially Overfunded Pension



This figure plots equity weights at time $t = 5$-year as a function of annualized equity return over the last five years. Panel A is for the initially underfunded pension with 80% funding ratio, and Panel B is for the initially overfunded pension with 120% funding ratio. We fix the initial asset value at one and vary the terminal downside constraint, K.

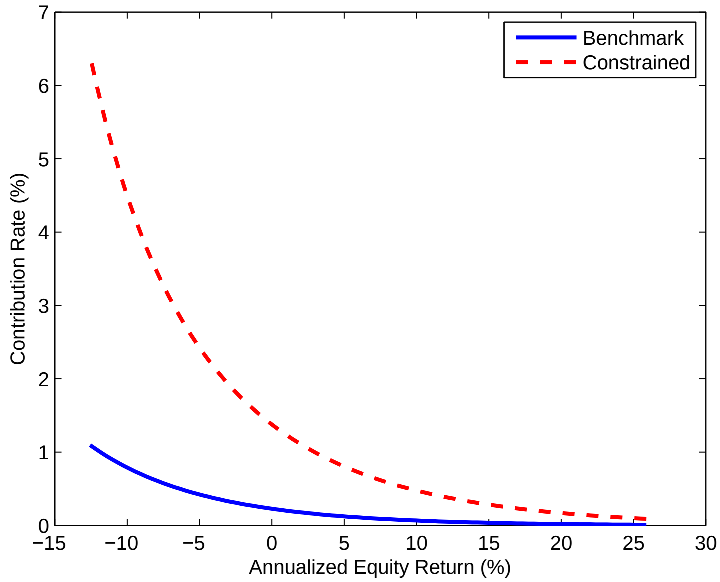
ment policy, i.e. decreases the equity weight, since the pension sponsor wants to avoid costly contributions as much as it can. On the other hand, when the economy downturn is significant, the pension sponsor starts to contribute. Since contemporaneous and future contributions can hedge high equity positions, the pension sponsor takes more risks. When the pension sponsor switches from risk management to risk taking depends on the initial funding status. For the initially underfunded pension plan, risk taking incentives dominate risk management incentives. The intuition is that for same negative shocks to the economy, the impact is greater for the initially underfunded pension plan so that the pension sponsor starts to contribute earlier, which yields risk taking incentives.

By comparing the benchmark case and the constrained case, we can predict a situation in which a government insurance exists. In the benchmark case, the pension sponsor has only risk taking incentives, which are hedged by contributions. Even if the pension plan ends up with underfunded, the government agency, such as PBGC will guarantee the benefits. Thus, as the economy gets worse the pension sponsor would take more risks. On the other hand, the pension sponsor without the government insurance would avoid large contributions as much as it can by managing risks, i.e. decreasing the equity weight. However, when the pension plan's asset is severely deteriorated, the pension sponsor will take more risks than the benchmark case since to save the pension plan the pension sponsor will contribute large amount, which can hedge high exposure to the equity shocks.

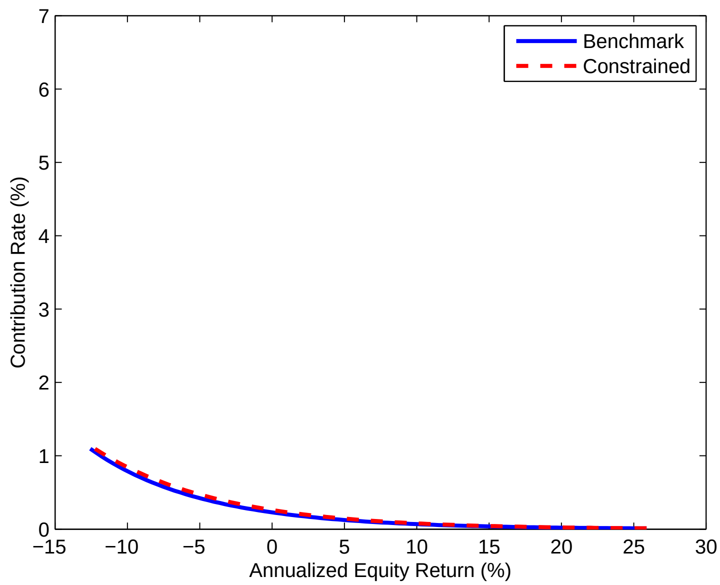
Figure 4 plots the contribution rate, Y_t/W_t as a function of an annualized equity return over time zero to 5-year. We can see that contribution rates of the benchmark case are decreasing in the state of economy and identical across initial funding status. The pension sponsor with the downside constraint behaves differently depending on the initial funding status. The initially underfunded pension sponsor contributes more than the benchmark case for the same state of the economy. The effect of negative shocks to the economy is greater to the initially underfunded pension plan, and thus to satisfy the downside constraint higher contributions should be made. Next, consider the initially overfunded pension plan in Panel B. Since the disutility from contributions is more important ($k = 100$) relative to the utility, the contributions are slightly higher than the benchmark case, which can explain why risk management incentives dominate risk taking incentives for the initially overfunded pension plan to guarantee the downside constraint.

Figure 4: Contribution Rate

Panel A: Initially Underfunded



Panel B: Initially Overfunded Pension



This figure plots contribution rates at time $t = 5$-year as a function of annualized equity return over the last five years. Panel A is for the initially underfunded pension with 80% funding ratio, and Panel B is for the initially overfunded pension with 120% funding ratio. We fix the initial asset value at one and vary the terminal downside constraint, K.

5.3 Effects of Relative Importance of Disutility

Figure 5 plots the optimal present value of the contribution and the put option value at time zero for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as we vary the relative importance of the disutility, k. We also plot the optimal present value of the contribution for the benchmark case. Since drawing contributions from the pension sponsor's internal resources is more costly when k is large, we can see that the optimal present value of the contribution decreases as k increases for both the benchmark and constrained cases. However, there is a key difference between the initially underfunded and overfunded pension plans. As we see in Figure 2, the initially overfunded pension plan has a put-based strategy even without a contribution. Hence, when k is sufficiently large, the pension sponsor won't contribute at all and just use the put-based strategy without any contribution. However, the initially underfunded pension plan can not construct a put-based strategy without a contribution. Thus, we can see that even if k is sufficiently large, the underfunded pension plan takes the present value of the contribution, which is equal to the time zero shortfall $Ke^{-rT} - W_0 = 25\%$.

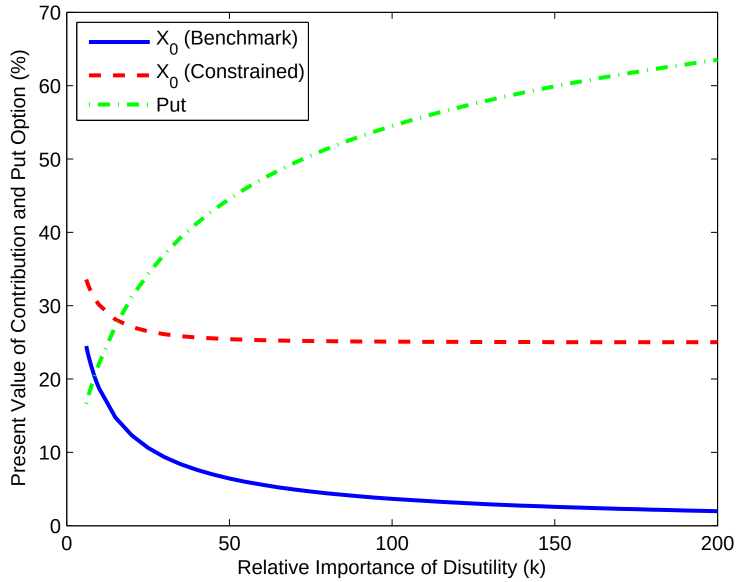
The put option value at time zero increases as k increases for both underfunded and overfunded pension plans. When contributing is more costly, the pension sponsor decreases the present value of the contribution and allocations in the unconstrained optimal portfolio, which makes the overall pension plan's asset less risky and increases the put option value. For low k, contributing more than the put option value is optimal since contributing is less costly and the pension sponsor can hold more unconstrained optimal portfolio. However, when k is high, the opposite happens. A fraction of the put option is funded by the initial pension plan's asset.

5.4 Effects of Elasticity of Disutility

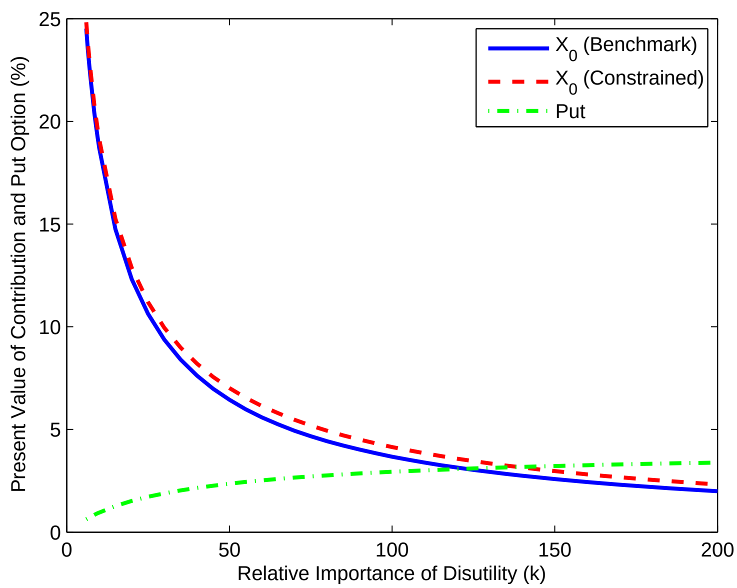
The elasticity of the disutility, θ has impacts on the determination of the optimal present value of the contribution. In Figure 6, we vary θ from 1.2 to 2 and see the optimal X_0 for the initially underfunded (Panel A), and overfunded (Panel B) pension plans. To focus on the shape of the optimal present value of the contribution, we omit the benchmark case and the put option value here. We can see that the present value of the contribution is U-shaped in θ for both cases, but it is clearer for the underfunded pension plan. Since the elasticity of the disutility only moves the marginal cost curve $\mathcal{W}_\phi(y)$ in Figure 1, given a contribution policy whether an increase in θ raises the present value of the contribution is our interest. If the present value of the contribution increases, the marginal cost curve moves downward and the optimal X_0 increases,

Figure 5: Effects of Relative Importance of Disutility

Panel A: Initially Underfunded



Panel B: Initially Overfunded Pension



This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the relative importance of the disutility (k). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.

and vice verse.

The optimal contribution policy, $Y_t = I_\phi(y\xi_t) = (\frac{y\xi_t}{k})^{\frac{1}{\theta-1}}$ is convex in ξ_t when $1 < \theta < 2$ and is concave in ξ_t when $\theta > 2$. Given that $y < k$ (this is the case for our parameter values), an increase in θ always increases the present value of the contribution since contributions become less convex when $1 < \theta < 2$ and more concave when $\theta > 2$. This can be seen clearly through the first derivative of $\mathcal{W}_\phi(y)$ with respect to θ:

$$\frac{\partial \mathcal{W}_\phi(y)}{\partial \theta} = -\mathcal{W}_\phi(y) \left[\frac{\log \frac{y}{k}}{(\theta-1)^2} + \frac{\partial \alpha_\phi}{\partial \theta} \frac{1-(1+\alpha_\phi T)e^{-\alpha_\phi T}}{\alpha_\phi} \right].$$

If the insides of the bracket are negative, the marginal cost curve moves downward and the optimal contribution increases, and vice verse. The first term in the bracket is negative since with our parameter values $y < k$.

The second part is the effect of θ on the annuity term. We find that for low θ the second term in the bracket is positive and dominates the first term, and thus the marginal cost curve moves upward and the optimal contribution decreases. On the other hand, for high θ, the opposite happens. The intuition is that to smooth contributions, the pension sponsor increases contributions in more likely states of the economy and decreases contributions in less likely states of the economy when $1 < \theta < 2$. The former action increases the present value of the contribution, while the latter decreases it. When θ is small, the latter effect is dominating.

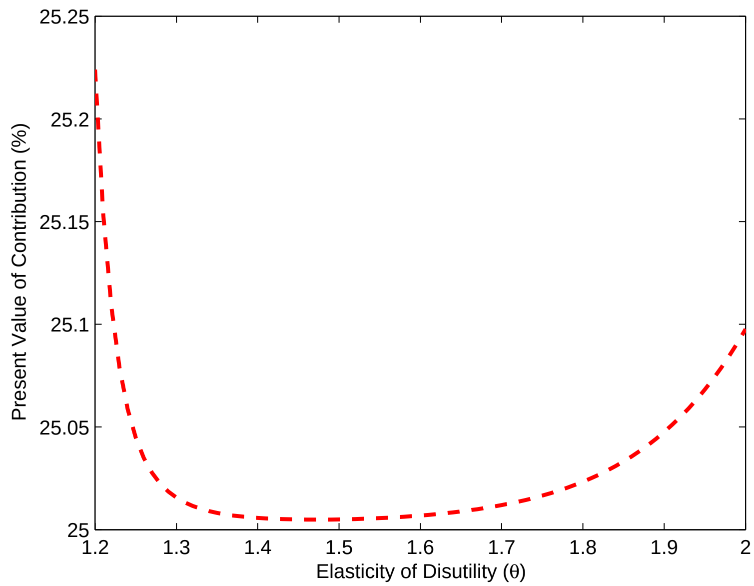
5.5 Effects of Risk Aversion and Price of Risk

Now, we investigate effects of the risk aversion and the price of risk. First, we report the optimal present value of the contribution and the put option value at time zero for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as we vary the relative risk aversion, γ in Figure 6. We also plot the optimal present value of the contribution for the benchmark case. When the risk aversion is high, the mean-variance efficient portfolio holds less equity. The risk of the underlying asset is reduced, and thus the put option value decreases, which implies that less contributions are required. For the initially overfunded pension plan, when the risk aversion is very high, the put option value becomes worthless since the pension sponsor holds zero equity position and the downside constraint is always satisfied (note that it is initially overfunded). Some contributions are still optimal since the pension sponsor can achieve higher utility even taking into account the disutility from contributions.

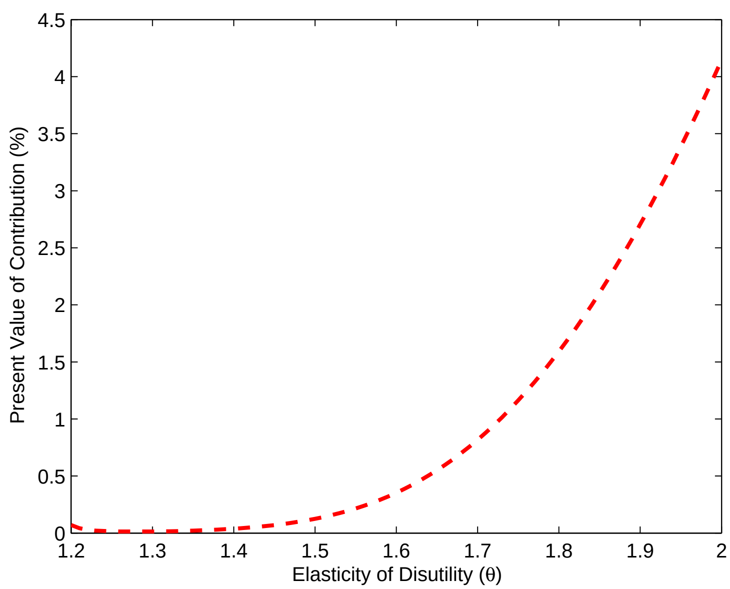
Next, we vary the price of risk while keeping the volatility of the equity returns at $\sigma = 20\%$. An increase in the price of risk moves both the marginal benefit curve $\mathcal{W}_u(y)$ and the marginal

Figure 6: Effects of Elasticity of Disutility

Panel A: Initially Underfunded



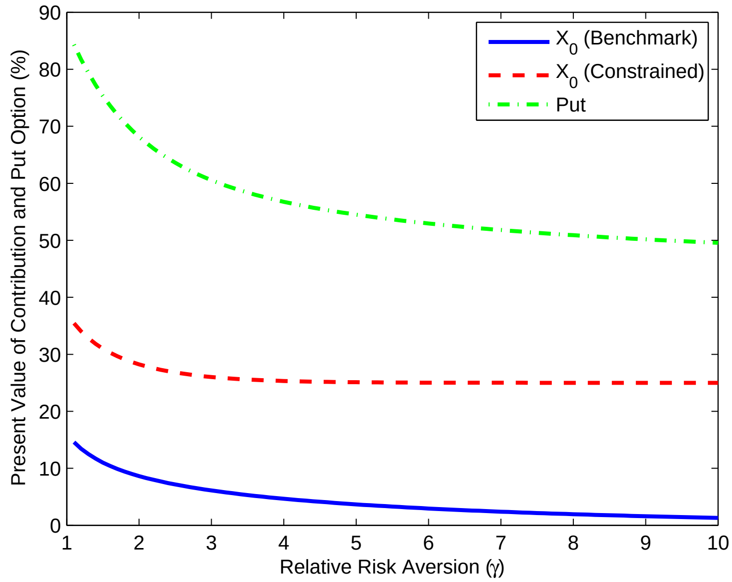
Panel B: Initially Overfunded Pension



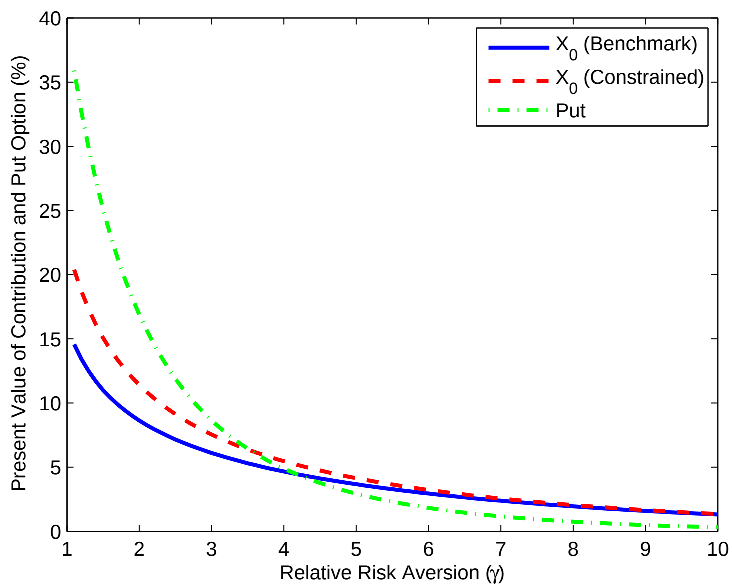
This figure plots the optimal present value of the contribution for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the elasticity of the disutility (θ). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.

Figure 7: Effects of Relative Risk Aversion

Panel A: Initially Underfunded



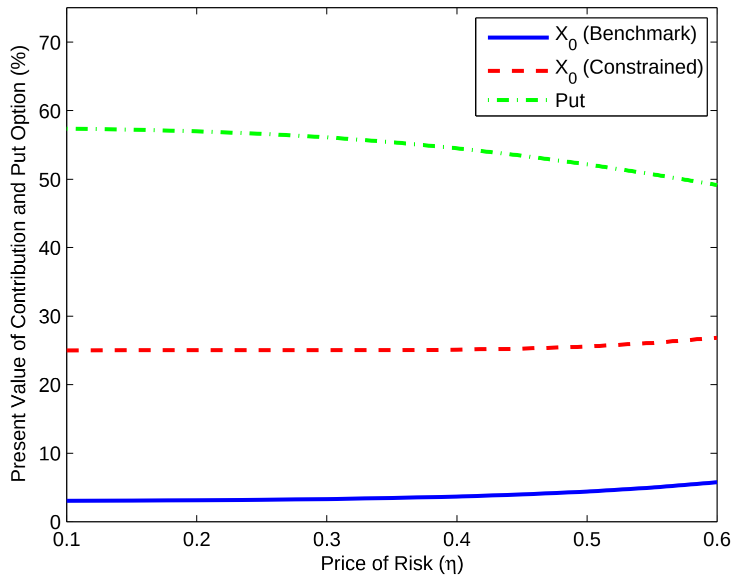
Panel B: Initially Overfunded Pension



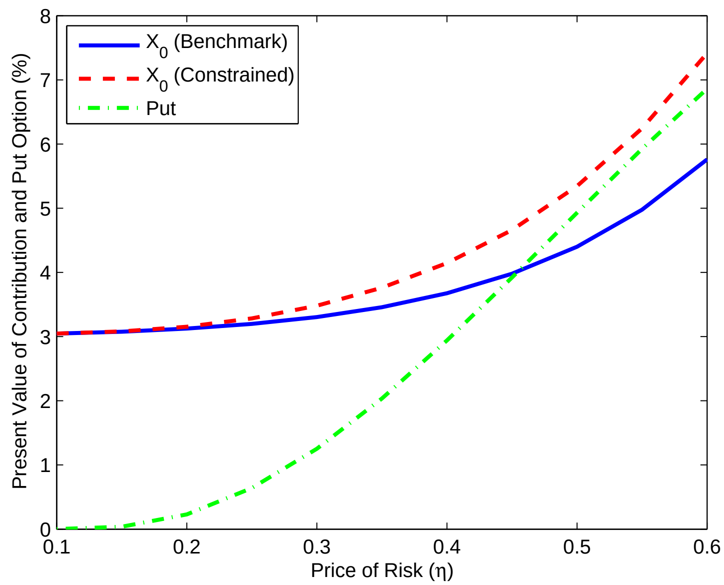
This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the relative risk aversion (γ). The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.

Figure 8: Effects of Price of Risk

Panel A: Initially Underfunded



Panel B: Initially Overfunded Pension



This figure plots the present value of the contribution and the put option value for the initially underfunded (Panel A) and overfunded (Panel B) pension plan as a function of the price of risk (η). We fix the volatility of the equity returns at $\sigma = 20\%$. The initially underfunded pension plan has 80% funding ratio, and the initially overfunded pension plan has 120% funding ratio.

cost curve $\mathcal{W}_\phi(y)$ in Figure 1. If the marginal benefit curve moves upward and the marginal cost curve moves downward as the price of risk increases, we will see that the optimal present value of the contribution also increases. This is the case for the initially overfunded pension plan (Panel B of Figure 8). Higher volatility of the mean-variance efficient portfolio ($\frac{\eta}{\gamma}$) increases the put option value, and thus higher initial endowment is required to have the same level of the marginal benefit, i.e. the marginal benefit curve moves upward. On the other hand, when the price of risk is higher, for a given contribution policy, the present value of the contribution is higher since a distribution of states of the economy are more positive skewed and it is more likely to contribute, i.e. the marginal cost curve moves downward. The net effect is an increase in the optimal present value of the contribution and the put option value.

For the initially underfunded pension plan, the movement of the marginal benefit curve is opposite. The same level of the marginal benefit can be financed with lower initial asset using higher expected equity return. However, if this downward movement of the marginal benefit curve is dominated by the downward movement of the marginal cost curve, we will still see an increase in the optimal present value of the contribution, but see a decrease in the put option value, which is due to a decrease in allocations to the unconstrained optimal portfolio. This is the case for the initially underfunded pension plan (Panel A of Figure 8).

6 Conclusion

We develop the separation approach to analyze the pension sponsor's contribution and portfolio policy in the presence of the downside constraint at the terminal date. The problem can be cast in two separate shadow price problems, the utility maximization problem and the disutility minimization problem. At the optimal solution, two shadow prices are identical. We show that while guaranteeing the benefits, both risk management and risk taking behaviors can emerge. When the pension plan's asset decreases, the pension sponsor first decreases the equity weight and defers contributions as much as it can to avoid costly contributions. Then, only when the pension plan's asset is significantly deteriorated, the pension sponsor starts to contribute and increases the equity weight, which is hedged by large contemporaneous and future contributions. In our model, the pension sponsor's risk taking behavior is induced not by a moral hazard problem, but by commitment to contributions. We hope to extend our analysis to include time-varying expected returns, and stochastic benefits in future research.

Appendix

Proof of Proposition 1. By Girsanov's Theorem, there exists a unique equivalent measure $\mathbb{Q}$ in which all traded assets earn the risk-free rate, and under $\mathbb{Q}$ measure the following stochastic process is a standard Brownian motion.

$$dZ_t^{\mathbb{Q}} = dZ_t + \eta dt.$$

To compute $\mathcal{W}_u(y)$, we can derive the dynamics of $\xi_t = M_t e^{\beta t}$ under $\mathbb{Q}$ measure.

$$\begin{aligned} \frac{d\xi_t}{\xi_t} &= (\beta - r)dt - \eta dZ_t \\ &= (\beta - r + \eta^2)dt - \eta dZ_t^{\mathbb{Q}}. \end{aligned}$$

The random variable $I_u(y\xi_T)$ can be expressed as

$$I_u(y\xi_T) = y^{-\frac{1}{\gamma}} \exp\left(-\frac{T}{\gamma}(\beta - r + \frac{1}{2}\eta^2) + \frac{\eta}{\gamma}(Z_T^{\mathbb{Q}} - Z_0^{\mathbb{Q}})\right),$$

given that $\xi_0 = 1$. Let A be the event in which $K > I_u(y\xi_T)$. The event A is equivalent to $x < -\delta_2(y, T)$, where x is a standard normal random variable, and $\delta_2(y, T)$ is given by

$$\delta_2(y, T) = \frac{\log \frac{y^{-\frac{1}{\gamma}}}{K} + \frac{T}{\gamma}(r - \beta - \frac{\eta^2}{2})}{\frac{\eta\sqrt{T}}{\gamma}},$$

since $Z_T^{\mathbb{Q}} - Z_0^{\mathbb{Q}}$ is normally distributed with zero mean and variance of T. Then, $\mathcal{W}_u(y)$ can be expressed as

$$\mathcal{W}_u(y) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} I_u(y\xi_T)(1 - 1(A))\right] + Ke^{-rT} N(-\delta_2(y, T)),$$

where $1(A)$ is an indicator function of event A and $N(\cdot)$ is a cumulative distribution function of standard normal random variable. The first part can be easily computed:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} I_u(y\xi_T)(1 - 1(A))\right] &= \exp\left(-\left(r + \frac{1}{\gamma}(\beta - r + \frac{1}{2}\eta^2)\right)T\right) y^{-\frac{1}{\gamma}} \\ &\quad \int_{-\delta_2(y,T)}^{\infty} \exp\left(\frac{\eta\sqrt{T}}{\gamma} x\right) n(x) dx \\ &= y^{-\frac{1}{\gamma}} e^{-\alpha_u T} \int_{-\delta_2(y,T)}^{\infty} n\left(x - \frac{\eta\sqrt{T}}{\gamma}\right) dx \\ &= y^{-\frac{1}{\gamma}} e^{-\alpha_u T} N(\delta_1(y, T)), \end{aligned}$$

where $n(\cdot)$ is a probability distribution function of a standard normal random variable, α_u and $\delta_1(y, T)$ are given by

$$\begin{aligned} \alpha_u &= \frac{\beta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\eta^2}{2\gamma}\right) \\ \delta_1(y, T) &= \delta_2(y, T) + \frac{\eta\sqrt{T}}{\gamma}. \end{aligned}$$

Now, the first derivative of $\mathcal{W}_u(y)$ can be computed as

$$\begin{aligned} \mathcal{W}_u'(y) &= -\frac{1}{\gamma} y^{-\frac{1}{\gamma}-1} e^{-\alpha_u T} N(\delta_1(y, T)) + y^{-\frac{1}{\gamma}} e^{-\alpha_u T} n(\delta_1(y, T)) \frac{\partial \delta_1(y, T)}{\partial y} \\ &\quad - Ke^{-rT} n(-\delta_2(y, T)) \frac{\partial \delta_2(y, T)}{\partial y}. \end{aligned}$$

Note that $\frac{\partial \delta_1(y,T)}{\partial y} = \frac{\partial \delta_2(y,T)}{\partial y}$ and

$$\begin{aligned}
y^{-\frac{1}{\gamma}} e^{-\alpha_u T} n(\delta_1(y,T)) &= y^{-\frac{1}{\gamma}} e^{-\alpha_u T} n\left(\delta_2(y,T) + \frac{\eta\sqrt{T}}{\gamma}\right) \\
&= y^{-\frac{1}{\gamma}} \exp\left(-\alpha_u T - \frac{\eta\sqrt{T}}{\gamma}\delta_2(y,T) - \frac{\eta^2 T}{2\gamma^2}\right) n(\delta_2(y,T)) \\
&= K e^{-rT} n(-\delta_2(y,T)).
\end{aligned}$$

Hence, last two terms cancel out. □

Proof of Theorem 2. Consider any random variable $\widetilde{W}_T^u \geq K$, which is feasible by a self financing trading strategy and the initial endowment W_0^u. This implies that

$$W_0^u \geq \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\widetilde{W}_T^u\right].$$

Then, we want to show that

$$\mathbb{E}\left[e^{-\beta T} u\left(W_T^u\right)\right] \geq \mathbb{E}\left[e^{-\beta T} u\left(\widetilde{W}_T^u\right)\right].$$

Since u is a concave utility, we have

$$u\left(W_T^u\right) - u\left(\widetilde{W}_T^u\right) \geq u'\left(W_T^u\right)\left(W_T^u - \widetilde{W}_T^u\right). \tag{.1}$$

We can compute $u'\left(W_T\right)$:

$$\begin{aligned}
u'\left(W_T^u\right) &= u'\left(\max\left(I_u\left(\mathcal{Y}_u\left(W_0^u\right)\xi_T\right), K\right)\right) \\
&= \min\left(\mathcal{Y}_u\left(W_0^u\right)\xi_T, u'\left(K\right)\right) \\
&= \mathcal{Y}_u\left(W_0^u\right)\xi_T - \left(\mathcal{Y}_u\left(W_0^u\right)\xi_T - u'(K)\right)^+.
\end{aligned}$$

Substitute in (.1), we have

$$u\left(W_T^u\right) - u\left(\widetilde{W}_T^u\right) \geq \mathcal{Y}_u\left(W_0^u\right)\xi_T\left(W_T^u - \widetilde{W}_T^u\right) + \left(\mathcal{Y}_u\left(W_0^u\right)\xi_T - u'(K)\right)^+\left(\widetilde{W}_T^u - K\right).$$

The second term is due to that $W_T^u = K$ corresponds to $\mathcal{Y}_u\left(W_0^u\right)\xi_T > u'(K)$. The second term is always greater or equal to zero since $\widetilde{W}_T^u \geq K$. Multiplying $e^{-\beta T}$ and taking expectation under the physical measure of the first term of RHS yields

$$\begin{aligned}
\mathcal{Y}_u\left(W_0^u\right)\mathbb{E}\left[e^{-\beta T}\xi_T\left(W_T^u - \widetilde{W}_T^u\right)\right] &= \mathcal{Y}_u\left(W_0^u\right)\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(W_T^u - \widetilde{W}_T^u\right)\right] \\
&\geq \mathcal{Y}_u\left(W_0^u\right)\left(W_0^u - \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\widetilde{W}_T^u\right]\right) \\
&\geq 0.
\end{aligned}$$

Hence, we obtain the desired inequality. Now, the optimal portfolio weight can be obtained by matching volatility of (5) and $W_t^u = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)}W_T^u\right]$. By Proposition 1, we can easily compute the latter:

$$W_t^u = y_t^{-\frac{1}{\gamma}} e^{-\alpha_u(T-t)} N\left(\delta_1(y_t, T-t)\right) + K e^{-r(T-t)} N\left(-\delta_2(y_t, T-t)\right),$$

where $y_t = \mathcal{Y}_u(W_0^u)\xi_t$. The diffusion part of the above is

$$\operatorname{diff}\left(dW_t^u\right) = \frac{\eta}{\gamma} y_t^{-\frac{1}{\gamma}} e^{-\alpha_u(T-t)} N\left(\delta_1(y_t, T-t)\right).$$

This should be equal to the diffusion part of (5), $\pi_t^u W_t^u \sigma$. Hence, we have

$$\pi_t^u = \frac{\eta}{\gamma\sigma}\left(1 - \varphi_t\right),$$

where $\varphi_t = \frac{K e^{-r(T-t)}}{W_t^u} N\left(-\delta_2\left(y_t, T-t\right)\right) < 1$. □

Proof of Proposition 3. We first compute $G(y)$:

$$G(y) = \mathbb{E}\left[e^{-\beta T}\frac{(y\xi_T)^{1-\frac{1}{\gamma}}}{1-\gamma}(1-1(A)) + e^{-\beta T}\frac{K^{1-\gamma}}{1-\gamma}1(A)\right]$$

Note that the expectation is under the physical measure. The random variable $I_u(y\xi_T)$ can be expressed as

$$I_u(y\xi_T) = y^{-\frac{1}{\gamma}}\exp\left(-\frac{T}{\gamma}(\beta - r - \frac{1}{2}\eta^2) + \frac{\eta}{\gamma}(Z_T - Z_0)\right).$$

The event A is equivalent to $x < -\delta_3(y,T)$, where x is a standard normal random variable, and $\delta_3(y,T)$ is given by $\delta_3(y,T) = \delta_2(y,T) + \eta\sqrt{T}$, since $Z_T - Z_0$ is normally distributed with zero mean and variance of T. If we follow similar steps as Proposition 1, we can obtain (10). (11) is the direct result of Theorem 2. Take the first derivative of (9), then we have

$$\begin{aligned}
G^{'}(y) &= \mathbb{E}\left[e^{-\beta T}u^{'}\left(I_u\left(y\xi_T\right)\right)I_u^{\prime}\left(y\xi_T\right)\xi_T(1-1(A))\right] \\
&= \mathbb{E}\left[e^{-\beta T}y\xi_T^2 I_u^{\prime}\left(y\xi_T\right)(1-1(A)))\right] \\
&= y\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}I_u^{\prime}\left(y\xi_T\right)\xi_T(1-1(A))\right] \\
&= y\mathcal{W}_u^{'}(y).
\end{aligned}$$

From (11), we have

$$\begin{aligned}
J^{'}\left(W_0^u\right) &= G_u^{'}\left(\mathcal{Y}_u\left(W_0^u\right)\right)\mathcal{Y}_u^{'}\left(W_0^u\right) \\
&= \mathcal{Y}_u\left(W_0^u\right)\mathcal{W}_u^{'}\left(\mathcal{Y}_u\left(W_0^u\right)\right)\mathcal{Y}_u^{'}\left(W_0^u\right) \\
&= \mathcal{Y}_u\left(W_0^u\right).
\end{aligned}$$

□

Proof of Proposition 4. We can interchange the integral and expectation:

$$\mathcal{W}_\phi(y) = \int_0^T e^{-rt}\mathbb{E}^{\mathbb{Q}}[I_\phi(y\xi_t)]dt,$$

where

$$I_\phi(y\xi_t) = \left(\frac{y}{k}\right)^{\frac{1}{\theta-1}}\exp\left(\frac{t}{\theta-1}(\beta - r + \frac{1}{2}\eta^2) - \frac{\eta}{\theta-1}(Z_t^{\mathbb{Q}} - Z_0^{\mathbb{Q}})\right).$$

The inner expectation is

$$\mathbb{E}^{\mathbb{Q}}[I_\phi(y\xi_t)] = \left(\frac{y}{k}\right)^{\frac{1}{\theta-1}}\exp\left(\frac{t}{\theta-1}\left(\beta - r + \frac{\theta\eta^2}{2(\theta-1)}\right)\right).$$

Now, we can express $\mathcal{W}_\phi(y)$ as

$$\begin{aligned}
\mathcal{W}_\phi(y) &= \left(\frac{y}{k}\right)^{\frac{1}{\theta-1}}\int_0^T e^{-\alpha_\phi t}dt \\
&= \left(\frac{y}{k}\right)^{\frac{1}{\theta-1}}\frac{1-e^{-\alpha_\phi T}}{\alpha_\phi},
\end{aligned}$$

where $\alpha_\phi = \frac{\theta}{\theta-1}\left(r - \frac{\eta^2}{2(\theta-1)}\right) - \frac{\beta}{\theta-1}$. The first derivative is straightforward. □

Proof of Theorem 5. Consider any random variable $\tilde{Y}$, whose present value is greater than X_0. This implies that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\tilde{Y}_t dt\right] \geq X_0.$$

Then, we want to show that

$$\mathbb{E}\left[\int_0^T e^{-\beta t}\phi\left(Y_t\right) dt\right] \leq \mathbb{E}\left[\int_0^T e^{-\beta t}\phi\left(\tilde{Y}_t\right) dt\right].$$

Since ϕ is a convex disutility, we have

$$\begin{aligned}
\phi\left(Y_t\right) &\leq \phi\left(\tilde{Y}\right) + \phi'\left(Y_t\right)\left(Y_t - \tilde{Y}\right) \\
&\leq \phi\left(\tilde{Y}\right) + \mathcal{Y}_\phi\left(X_0\right)\xi_t\left(Y_t - \tilde{Y}\right).
\end{aligned}$$

Multiplying $e^{-\beta t}$ and taking integral and expectation under the physical measure of the second term of RHS yields

$$\begin{aligned}
\mathcal{Y}_\phi\left(X_0\right)\mathbb{E}\left[\int_0^T e^{-\beta t}\xi_t\left(Y_t - \tilde{Y}\right) dt\right] &= \mathcal{Y}_\phi\left(X_0\right)\left(\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}Y_t dt\right] - \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\tilde{Y} dt\right]\right) \\
&= \mathcal{Y}_\phi\left(X_0\right)\left(X_0 - \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\tilde{Y} dt\right]\right) \\
&\leq 0.
\end{aligned}$$

Hence, we obtain the desired inequality. Now, the optimal hedging of contributions can be obtained by matching volatility of (12) and $X_t = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}Y_s ds\right]$. By Proposition 4, we can easily compute the latter:

$$X_t = \left(\frac{y_t}{k}\right)^{\frac{1}{\theta-1}}\frac{1-e^{-\alpha_\phi(T-t)}}{\alpha_\phi}, \tag{.2}$$

where $y_t = \mathcal{Y}_\phi(X_0)\xi_t$. The diffusion part of X_t is

$$\text{diff}\left(dX_t\right) = -\frac{\eta}{\theta-1}X_t.$$

This should be equal to the diffusion part of (12), $\pi_t^\phi X_t\sigma$. Hence, we have

$$\pi_t^\phi = -\frac{\eta}{(\theta-1)\sigma}.$$

□

Proof of Proposition 6. We first compute $C(y)$:

$$\begin{aligned}
C(y) &= \mathbb{E}\left[\int_0^T e^{-\beta t}\frac{k}{\theta}\left(\frac{y\xi_t}{k}\right)^{\frac{\theta}{\theta-1}} dt\right] \\
&= \frac{k}{\theta}\left(\frac{y}{k}\right)^{\frac{\theta}{\theta-1}}\int_0^T e^{-\beta t}\mathbb{E}\left[\xi_t^{\frac{\theta}{\theta-1}}\right] dt \\
&= \frac{k}{\theta}\left(\frac{y}{k}\right)^{\frac{\theta}{\theta-1}}\int_0^T e^{-\alpha_\phi t} dt \\
&= \frac{k}{\theta}\left(\frac{y}{k}\right)^{\frac{\theta}{\theta-1}}\frac{1-e^{-\alpha_\phi T}}{\alpha_\phi}.
\end{aligned}$$

(16) is the direct result of Theorem 5. Take the first derivative of (15) is

$$\begin{aligned}
C^{'}(y) &= \mathbb{E}\left[\int_0^T e^{-\beta t}\phi^{'}\left(I_\phi\left(y\xi_t\right)\right)I_\phi^{'}\left(y\xi_t\right)\xi_t dt\right] \\
&= y\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\xi_t I_\phi^{'}\left(y\xi_t\right)dt\right] \\
&= y\mathcal{W}_\phi^{'}(y).
\end{aligned}$$

From (16), we have

$$\begin{aligned}
L^{'}\left(X_0\right) &= C^{'}\left(\mathcal{Y}_\phi\left(X_0\right)\right)\mathcal{Y}_\phi^{'}\left(X_0\right) \\
&= \mathcal{Y}_\phi\left(X_0\right).
\end{aligned}$$

□

Proof of Theorem 7. We can compute the present value of arbitrary contribution policy. Let

$$\begin{aligned}
X_0 &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt}\tilde{Y}_t dt\right] \\
W_0^u &= W_0 + X_0.
\end{aligned}$$

Then, $(\tilde{\pi}, \tilde{Y})$ satisfies the following static budget constraint:

$$W_0^u \geq \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\widetilde{W}_T\right],$$

where $\widetilde{W}$ is a corresponding wealth process to $(\tilde{\pi}, \tilde{Y})$. Hence, $\tilde{\pi}$ is a feasible trading strategy to the first problem with the initial wealth W_0^u, and $\tilde{Y}$ is a feasible contribution policy to the second problem with the present value of contribution X_0. Let π_t^u and π^ϕ be the optimal trading strategy to the first and the optimal hedging strategy to the second problem, respectively. Also, let W_t^u and X_t be the optimal path of asset value to the first problem, and the optimal path of internal resources for hedging contributions to the second problem, respectively. Finally, let Y denote the optimal contribution policy to the second problem. Then, we can construct the following portfolio and contribution policy, and path of the pension plna's asset:

$$\pi_t = \frac{\pi_t^u W_t^u - \pi^\phi X_t}{W_t^u - X_t} \tag{.3}$$

$$Y_t = Y_t \tag{.4}$$

$$W_t = W_t^u - X_t. \tag{.5}$$

We need to prove that these policies are feasible for the original problem. Consider the discounted pension plan's asset:

$$\begin{aligned}
e^{-rt}W_t &= e^{-rt}W_t^u - e^{-rt}X_t \\
&= W_0 + X_0 + \int_0^t e^{-rs}\pi_s^u\sigma W_s^u dZ_s^{\mathbb{Q}} - X_0 + \int_0^t e^{-rs}Y_s ds - \int_0^t e^{-rs}\pi^\phi\sigma X_s dZ_s^{\mathbb{Q}} \\
&= W_0 + \int_0^t e^{-rs}Y_s ds + \int_0^t e^{-rs}\pi_s\sigma W_s dZ_s^{\mathbb{Q}} \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-rT}W_T\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-rs}Y_s ds\right],
\end{aligned}$$

since $W_T = W_T^u - X_T = W_T^u$. Hence, (π, Y) is a admissible portfolio and contribution policy to the original problem. Then, we have

$$\begin{aligned}
\mathbb{E}\left[e^{\beta T} u\left(W_T\right)\right] &\geq \mathbb{E}\left[e^{\beta T} u\left(\tilde{W}_T\right)\right] \\
\mathbb{E}\left[\int_0^T e^{-\beta t} \phi\left(Y_t\right) dt\right] &\leq \mathbb{E}\left[\int_0^T e^{-\beta t} \phi\left(\tilde{Y}_t\right) dt\right].
\end{aligned}$$

Hence, we have a desired inequality. Then, (17) is straightforward. □

Proof of Proposition 8. From (6), we can easily see that $\mathcal{W}_u(y)$ is decreasing, $\lim_{y\to 0} \mathcal{W}_u(y) = \infty$, and $\lim_{y\to\infty} \mathcal{W}_u(y) = Ke^{-rT}$. Also, from (13) we can see that $\mathcal{W}_\phi(y)$ is increasing, $\lim_{y\to 0} \mathcal{W}_\phi(y) = 0$, and $\lim_{y\to\infty} \mathcal{W}_\phi(y) = \infty$. □

Proof of Proposition 9. Suppose that we find y solving (18), i.e. the optimal present value of the contribution, X_0. Then, we can set the optimal portfolio and contribution policy, and the optimal path of the pension plan's asset to the original problem as (.3), (.4), (.5) using solutions to the first and second problems. Then, the optimal portfolio weight is straightforward. Note that by (.2) the optimal path of the internal resources for hedging contributions is

$$X_t = \left(\frac{y\xi_t}{k}\right)^{\frac{1}{\theta-1}} \frac{1 - e^{-\alpha_\phi (T-t)}}{\alpha_\phi} = Y_t \frac{1 - e^{-\alpha_\phi (T-t)}}{\alpha_\phi}.$$

Hence, the optimal contribution rate is

$$\frac{Y_t}{W_t} = \frac{X_t}{W_t} \frac{\alpha_\phi}{1 - e^{-\alpha_\phi (T-t)}}.$$

□

Proof of Proposition 10. The first part of (6) is the present value of the terminal pension plan's asset, $I_u(y\xi_T)$ if $I_u(y\xi_T) > K$, otherwise zero. Hence, $\mathcal{W}_u^{BC}(y)$ can be easily computed from that. The first derivative is straightforward. Now, we can express $\mathcal{W}_u(y)$ as

$$\mathcal{W}_u(y) = y^{-\frac{1}{\gamma}} e^{-\alpha_u T} + Ke^{-rT} N(-\delta_2(y,T)) - y^{-\frac{1}{\gamma}} e^{-\alpha_u T} N(-\delta_1(y,T)) > \mathcal{W}_u^{BC}(y).$$

The last two terms are the present value of $(K - I_u(y\xi_T))^+$, and thus positive. The remaining part can be proved following similar procedures as in Theorem 2 and Proposition 3. □

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