Mark–to–Market Reinsurance and Portfolio Selection: Implications for Information Quality*

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Abstract

This paper investigates the optimal mark–to–market reinsurance and asset investment strategies for insurers with complete or partial information on expected return of the risky asset. The insurer with partial information is assumed to have prior belief on the expected return and to update posterior beliefs by exploiting its price information. We show that the strategies of the insurer with partial information depend on prior belief, and that variation in posterior beliefs gives rise to the counter–cyclical investment demand. By comparing the two insurers' strategies, we show that information quality comes from the relative importance between the financial market price of risk and the reinsurance price of risk.

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1. Introduction

Insurance companies and pension funds have paid much attention to asset–liability management, during the recent global financial crisis in particular. For example, the OECD Insurance and Private Pension Committee has advised insurers to not only meet liability obligations to policyholders but also to mitigate several risk exposures. This new regulatory trend gives rise to risk–based capital requirements along with quantitative and qualitative provisions (OECD, 2015). A central aspect of the capital requirements is to determine proper *discount rates* (or expected returns) used to adjust assets and liabilities on a market basis, namely a *mark–to–market valuation*.

Motivated by the recent regulatory trend, this paper studies optimal mark–to–market reinsurance and asset allocation strategies for risk-averse insurers. The fact that discount rates vary over time (Cochrane, 2011) demonstrates that investment opportunities also change over business cycles.[1] The capital requirements subject to variation in expected returns can cause portfolio manages to become increasingly conservative. For example, Solvency II in European countries established a guideline that capital must cover unexpected losses over a one–year horizon with a probability 99.5%. Even with strong savings accumulations, insurers, who are traditionally recognized as long–term institutional investors, also desire to increase their reserve funds against future potential losses. Indeed, investment strategy can be used together with reinsurance strategy as an important hedging tool in reducing insurance claims risk.

This paper considers two different types of models on the expected return from risky investment:

- CI model: a model for an insurer with *complete information*.
- PI model: a model for an insurer with *partial information*.

Commonly, we assume that the expected return follows a two–state continuous–time Markov chain, but that return volatility does not change over time.[2] In addition, both insurers have the utility preference of constant absolute risk aversion (CARA) and aim to maximize their utility functions by controlling the proportional reinsurance rates and risky investment amount. We also assume that insurance/reinsurance opportunities do not vary over time because insurers have difficulty frequently changing premiums that describe time–varying insurance/reinsurance opportunities.

The central difference between the two models arises from whether insurers can perfectly observe the expected return (or investment opportunity set) in terms of *information quality*. For example,

[1] Ang and Bekaert (2002) show that stock returns are much volatile during economic recessions.
[2] Merton (1980) shows that expected return is harder to estimate than is return volatility.

the insurer under the PI model does not exactly observe time variation in the expected returns but updates posterior beliefs on future investment opportunities by exploiting available information on the risky asset prices. In contrast, the insurer under the CI model can exactly know the return variation, because she has full confidence on time–varying investment opportunities.

This paper is a rare research finding an intersection between studies on information quality in asset pricing theory and those on the insurer's optimal behavior.[3] Conventional studies have commonly assumed that insurers have perfect knowledge on financial and insurance/reinsurance markets,[4] which incurs no estimation risk over the indefinite future. Accordingly, the insurer with complete information can only make myopic decisions without a hedging motive.

This paper has three main contributions to the literature. First, we derive an explicit representation of *certainty equivalent wealth* (CEW) for the insurer with partial information. The CEW, which captures the wealth level of changes in future investment and insurance/reinsurance opportunities, can be decomposed into two components: reinsurance costs, and demands on precautionary saving. This representation can help better to understand the mark–to–market strategies in a logical way. For instance, the decomposition provides us to conclude that

- an insurer with high risk aversion cares much about reinsurance costs due to a strong hedging motive for changes in future investment and insurance/reinsurance opportunities, and
- a low correlation between financial assets and insurance claims reduces a risk–sharing effect.

Both cases leads to much strict risk management for insurers because the cases cause a drop in the CEW as a result of low information quality, which has a negative effect on her wealth maximization.

Second, the mark–to–market strategies under the PI model accord well with the recent regulatory trend. For example, the mark–to–market investment strategy can represent a linear combination of pro–cyclical and counter–cyclical ones. The counter–cyclical investment strategy can help to present potential pro–cyclical overreaction, especially during economic recessions (OECD, 2015). Moreover, the mark–to–market reinsurance strategy depends counter–cyclically on prior belief. This counter–cyclical behavior is a simple consequence of the fact that insurance claims are negatively correlated with risky asset returns.

[3] A large body of the literature has provided evidence for stochastically changing investment opportunities driven by the first and second moments of risky assets or stochastic real interest rates (Korn and Kraft, 2002; Campbell et al., 2004; Chacko and Viceira, 2005). In this regard, information–quality implications have been extensively studied in asset pricing theory (David, 1997; Ai, 2010) and on optimal portfolio choice (Honda, 2003; Liu, 2011).

[4] See Cao and Wan (2009), Gu et al. (2010), and Liang et al. (2011). In detail, Jang and Kim (2015) provide an extensive literature review on the insurer's optimal behavior.

Third, information quality comes from the relative importance between the *financial market price of risk* and the *reinsurance price of risk*. To see this robustly, we compare the CEWs under the two models to obtain utility gain called the *certainty equivalent wealth gain by information acquisition* (I–CEWG). As a result, the CEW under the CI model is constant, and is even larger than the CEW under the PI model due to information confidence. This finding demonstrates that the I–CEWG used to measure information quality results mainly from the stochastic characteristics of the CEW under the PI model. We show that these characteristics such as whether to increase or decrease and whether to be convex or concave result from the relative importance of the two risk–adjusted premiums.

The rest of the paper is organized as follows. Section 2 provides our basic setting and formulates the insurer's problems. Sections 3 and 4 introduce the CI model and the PI model and show insurer's optimal strategies, respectively. Section 5 provides numerical implication in relation to information quality by comparing the two models. Section 6 concludes.

2. The Basic Set-Up

2.1. Financial Market

The financial market is frictionless and has two investment assets, one risk–free asset (bond) and one risky asset (stock). The risk–free asset price B_t evolves by

$$dB(t) = rB(t)dt,$$

with a constant instantaneous interest rate r. The return of the risky asset evolves by

$$dR(t) = \mu(t)dt + \sigma dW_R(t), \tag{1}$$

where $\mu(t)$ is a stochastic expected–return process, $\sigma > 0$ is a constant parameter, and $W_R(t)$ is a standard Brownian motion. An insurer considered here is a price-taker in the financial market.

2.2. Insurance Claim

The insurer pays insurance claims to policyholders as a return service of receiving some insurance premium. We assume that there are sufficiently many policyholders so that the arrival of insurance claims are quite frequent. Thus, we can take the diffusion model in Iglehart (1969) as the cumulative insurance claims $C(t)$ satisfying

$$dC(t) = \alpha dt - \beta dW_C(t),$$

where $\alpha > 0$ is its expected growth rate, $\beta > 0$ is its constant volatility, and $W_C(t)$ is a standard Brownian motion. We assume the exogenous shocks on the stock returns and the insurance claims have a correlation of ρ, that is,

$$dW_R(t) \cdot dW_C(t) = \rho dt.$$

2.3. Reinsurance

A reinsurer can take all reinsurance demands of the insurer, and the insurer has a proportional reinsurance strategy. Given a time-t reinsurance rate as $\varepsilon(t)$, the reinsurer pays $\varepsilon(t)dC(t)$-dollar to policyholders for the infinitesimal time period of $[t, t + \Delta t)$ and the insurer pays the rest, i.e., $(1 - \varepsilon(t))dC(t)$.

We assume that the insurer's safety loading is $\theta \geq 0$, and the reinsurer's safety loading is $\eta \geq 0$. Then, the insurance premium $p_I(t)$ must be

$$p_I(t) = (1 + \theta)\alpha,$$

and the reinsurance premium $p_O(t)$ must be

$$p_O(t) = (1 + \eta)\alpha\varepsilon(t).$$

We take the assumption of $\eta \geq \theta$; otherwise, the insurer's cheapest strategy would be to take a perfect reinsurance strategy, i.e., $\varepsilon(t) = 1$.

2.4. The Insurer's Goal

Taking all considerations together, the insurer's surplus process $S(t)$ satisfies

$$\begin{aligned} dS(t) &= p_I(t)dt - (1 - \varepsilon(t))dC(t) - p_O(t)dt \\ &= (\theta - \eta\varepsilon(t))\alpha dt + \beta(1 - \varepsilon(t))dW_C(t), \end{aligned}$$

and thus, the insurer's wealth $X(t)$ evolves as

$$dX(t) = [rX(t) + (\mu(t) - r)\pi(t) + (\theta - \eta\varepsilon(t))\alpha]dt + \beta(1 - \varepsilon(t))dW_C(t) + \sigma\pi(t)dW_R(t), \quad (2)$$

where $\pi(t)$ is the dollar investment amount in the risky asset.

The insurer's goal is to maximize her utility $U(\cdot)$ with respect to terminal wealth $X(T)$ by controlling reinsurance and portfolio strategies, $(\varepsilon(t),\ \pi(t))$. The insurer's utility preference is a CARA type:

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x},$$

where γ is the coefficient of absolute risk aversion. In the long run, the goal is to find

$$\sup_{\varepsilon,\pi} E^{t,x}[U(X(T))],$$

under the condition of Equation (2). Here, $E^{t,x}[\cdot]$ is an expectation operator conditioned on the time-t wealth $X(t) = x$ under a real probability measure $\mathbb{P}$.

3. The CI Model: A Model with Complete Information

3.1. A Completely Informed Insurer

As a benchmark model, we first explore a model of optimal reinsurance and portfolio selection under complete information (henceforth, the CI model). We characterize the CI model as a two–state continuous–time Markov chain in which the expected return of the risky asset stochastically jumps between two states.

Suppose that $Y(t)$ follows the *observable* Markov chain with two states: *High regime* (or regime H), and *Low regime* (or regime L). In the CI Model, we take the assumption that the expected return satisfies

$$dR(t) = \mu_i dt + \sigma dW_R(t),$$

where $\mu_i \equiv \mu(Y(t) = i)$ for $i \in \{H, L\}$ satisfying $\mu_H > \mu_L$. Regime i jumps to regime j $(j \neq i)$ for $i, j \in \{H, L\}$ with intensity λ_i. Hence, for a infinitesimal length of time Δt, μ_i remains unchanged with probability of $1 - \lambda_i \Delta t$ or shifts to μ_j $(j \in \{H, L\},\ j \neq i)$ with probability of $\lambda_i \Delta t$. We assume that such Poisson-type jumps independently occur each other and are independent of the market risk $(W_R(t))$ and the insurance claims risk $(W_C(t))$.

In the CI model, all information concerning market risk, insurance claims risk, and regime risk is assumed to be immediately revealed to the insurer. Specifically, she has full information confidence on the filtration of

$$\{\mathcal{F}(t)\} = \{\mathcal{F}^{W_R,Y}(t)\} \otimes \{\mathcal{F}^{W_C}(t)\},$$

where $\{\mathcal{F}^{W_R,Y}(t)\}$ stands for the filtration generated by information of the return shocks $W_R(u)$ and regime information $Y(u)$ up to time $u \le t$ for $t \in [0,T]$, and $\{\mathcal{F}^{W_C}(t)\}$ is the filtration generated by the information of insurance claims shocks $W_C(t)$.

3.2. The Insurer's Optimal Strategies

The insurer's problem under the CI model is to find

$$V_i(t,x) = \sup_{\varepsilon_i,\pi_i} E^{t,x}[U(X(T)],$$

subject to

$$dX(t) = [rX(t) + (\mu_i - r)\pi_i + (\theta - \eta\varepsilon_i(t))\alpha]dt + \beta(1-\varepsilon_i)dW_C(t) + \sigma\pi_i dW_R(t),$$

by taking the reinsurance and portfolio strategies (ε_i, π_i) in each regime $i \in \{H, L\}$.

The dynamic programming principle yields a system of the two Hamilton–Jacobi–Bellman (HJB) equations for the value functions V_i:

$$\begin{aligned} V_{i,t} + \sup_{\varepsilon_i,\pi_i} \Big[& \{rx + (\mu_i - r)\pi_i + (\theta - \eta\varepsilon_i)\alpha\}V_{i,x} \\ & + \frac{1}{2}\{\beta^2(1-\varepsilon_i)^2 + \sigma^2\pi_i^2 + 2\rho\beta(1-\varepsilon_i)\sigma\pi_i\}V_{i,xx} + \lambda_i(V_j - V_i)\Big] = 0 \end{aligned} \tag{3}$$

with the two terminal conditions

$$V_i(T, X(T)) = U(X(T)).$$

Here, $V_{i,t}$ and $V_{i,x}$ are respectively the first derivatives with respect to time t and wealth x, and $V_{i,xx}$ is the second derivative with respect to wealth. The first–order conditions lead us to the optimal strategies:

$$\varepsilon_i^* = \Big[\frac{\alpha\sigma\eta - \rho\beta(\mu_i - r)}{\beta^2\sigma(1-\rho^2)}\Big]\frac{V_{i,x}}{V_{i,xx}} + 1, \quad \text{and} \quad \pi_i^* = \Big[\frac{\rho\alpha\sigma\eta - \beta(\mu_i - r)}{\beta\sigma^2(1-\rho^2)}\Big]\frac{V_{i,x}}{V_{i,xx}}. \tag{4}$$

Theorem 1. *The value function in regime $i \in \{H, L\}$ is*

$$V_i(t,x) = -\frac{1}{\gamma}e^{-\gamma c(t)(x + f_i(t))},$$

where

$$c(t) = e^{r(T-t)},$$

and $f_i(t)$ *satisfies*

$$f_i'(t) - rf_i(t) + \frac{e^{-r(T-t)}}{\gamma}\lambda_i\left[1 - e^{-\gamma\exp\{r(T-t)(f_j(t)-f_i(t))\}}\right] + h_i(t) = 0, \quad f_i(T) = 0, \tag{5}$$

for $j \in \{H, L\}$ $(j \neq i)$ *and*

$$h_i(t) = \alpha(\theta - \eta) + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left[\frac{(\mu_i - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\mu_i - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right].$$

Proof. See Appendix A. □

Theorem 1 shows that the level of $f_i(t)$ examines the insurer's different behaviors across regimes.[5] Hereafter, we call $f_i(t)$ the *certainty equivalent wealth* (CEW) under the CI model, which captures the wealth level of stochastic regime risks in the future.

4. The PI Model: A Model with Partial Information

4.1. A Partially Informed Insurer

Contrary to the CI model, the counterpart model with partial information (henceforth, the PI model) takes the assumption that the insurer does not know exactly the current regime $Y(t)$ and thus, the corresponding expected rate of risky return, $\mu(t)$. In this case, the information set the insurer accesses can be represented as

$$\{\mathcal{G}(t)\} = \{\mathcal{F}^{W_R}(t)\} \otimes \{\mathcal{F}^{W_C}(t)\},$$

where $\{\mathcal{F}^{W_R}(t)\}$ is the filtration generated by the past history of the realized risky returns.[6] Instead, she has own *prior* belief from the market information $\{\mathcal{F}^{W_R}(t)\}$ in order to infer the current regime of the financial market. We define the time–t prior belief $p(t)$ as the probability of regime H inferred from the past information:

$$\text{probability of} \quad \{Y(t) = H|\,\mathcal{G}(t)\}.$$

[5] We found f_i by utilizing a simple numerical scheme.

[6] Clearly, $\{\mathcal{F}^{W_R}(t)\} \subset \{\mathcal{F}^{W_R,Y}(t)\}$, implying that the information set of the CI model is bigger than that of the PI model.

Accordingly, the resulting filtered expected return $\bar{\mu}(t)$ corresponds to a weighted average of μ_i for $i \in \{H, L\}$:

$$\bar{\mu}(t) \equiv E\left[\mu(t)|\mathcal{G}(t)\right] = p(t) \cdot \mu_H + (1 - p(t)) \cdot \mu_L = \mu_L + (\mu_H - \mu_L)p(t).$$

We now can apply a non–linear filtering theory in Liptser and Shiryaev (2001) so that all the parameters are adapted to the filtration $\{\mathcal{G}(t)\}$. We employ an *innovation* process generated by the insurer's prior belief on the current regime as

$$\widehat{W}_R(t) = \int_0^t \frac{dR(s) - \bar{\mu}(s)ds}{\sigma}.$$ [7]

Then, we can rewrite the filtered return dynamics as

$$dR(t) = \bar{\mu}(t)dt + \sigma d\widehat{W}_R(t). \tag{6}$$

The insurer's *posterior* belief on the current regime follows the relationship of

$$dp(t) = [\lambda_L - (\lambda_H + \lambda_L)p(t)]dt + \frac{\mu_H - \mu_L}{\sigma}p(t)(1 - p(t))d\widehat{W}_R(t). \tag{7}$$ [8]

Note that the insurer's belief $p(t)$ and the risky asset prices $R(t)$ are perfectly correlated under the PI model, implying that the financial market under the PI model is *complete* from the insurer's point of view.[9]

One advantage of incorporating the non-linear filtering theory into the model is that we can consider a positive learning effect between the realized and expected risky returns. Specifically, the covariance between realized risky returns and the revision in expected risky returns must be

$$Cov(dR(t), d\bar{\mu}(t)) = (\mu_H - \mu_L)^2 p(t)(1 - p(t)) \geq 0, \quad \text{for } p(t) \in [0, 1].$$

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[7] By the Girsanov theorem, $\widehat{W}_R(t)$ serves as a *new* Brownian motion under a new measure and the filtration $\{\mathcal{F}^{\widehat{W}_R}(t)\}$ generated by $\widehat{W}_R(t)$ is equivalent to the filtration $\{\mathcal{F}^R(t)\}$. See Theorem 9.1 of Liptser and Shiryaev (2001) for the details.

[8] David (1997) shows that Equation (7) satisfies Lipshitz and growth conditions and thus, a unique solution exists, although it is hard to obtain the explicit distribution for $p(t)$. More properties regarding the posterior belief can be obtained from David (1997), Honda (2003), and Liu (2011).

[9] In contrast, the market under the CI model is *incomplete* because there is no financial vehicle for hedging the regime risk.

The learning effect under the PI model is a concave function with respect to prior belief, whereas it does not exist under the CI model.[10]

4.2. The Insurer's Optimal Strategies

The insurer's problem under the CI model is to find the value function $V(t,x,p)$:

$$V(t,x,p)=\sup_{\varepsilon,\pi} E^{t,x,p}[U(X(T))]$$

subject to

$$dX(t)=[rX(t)+(\bar{\mu}(t)-r)\pi(t)+(\theta-\eta\varepsilon(t))\alpha]dt+\beta(1-\varepsilon(t))dW_C(t)+\sigma\pi(t)d\widehat{W}_R(t),$$

where $E^{t,x,p}[\cdot]$ is the expectation conditioned on $X(t)=x$ and prior belief $p(t)\equiv p$.

The HJB equation is

$$\begin{aligned}
&V_t+\sup_{\varepsilon,\pi}\Big[\{rx+(\bar{\mu}-r)\pi+(\theta-\eta\varepsilon)\alpha\}V_x+\frac{1}{2}\{\beta^2(1-\varepsilon)^2+\sigma^2\pi^2+2\rho\beta(1-\varepsilon)\sigma\pi\}V_{xx}\\
&+\{\lambda_L-(\lambda_H+\lambda_L)p\}V_p+\frac{(\mu_H-\mu_L)^2}{2\sigma^2}p^2(1-p)^2V_{pp}+\{(\mu_H-\mu_L)\pi p(1-p)\\
&+\rho\beta(1-\varepsilon)\frac{\mu_H-\mu_L}{\sigma}p(1-p)\}V_{xp}\Big]=0
\end{aligned}\tag{8}$$

with the terminal condition

$$V(T,X(T),p(T))=U(X(T)).$$

Here, V_t, V_x, and V_p are respectively first–order derivatives with respect to time t, wealth x, and belief p; V_{xx} and V_{pp} are respectively second–order derivatives with respect to wealth x and belief p; and $V_{xp}=\partial^2V/\partial x\partial p$.

Clearly, variation in posterior beliefs related to changes in future investment opportunities can affect the insurer's decision. In particular, V_p captures the marginal effect of mean–reversion updating, V_{pp} captures the marginal effect of stochastic–belief updating on her value function, and V_{xp} reflects the marginal effect of the covariance between her wealth and belief dynamics. The first–order conditions give us the optimal strategies:

$$\varepsilon^*=\Big[\frac{\alpha\sigma\eta-\rho\beta(\bar{\mu}-r)}{\beta^2\sigma(1-\rho^2)}\Big]\frac{V_x}{V_{xx}}+1,\quad\text{and}\quad\pi^*=\Big[\frac{\rho\alpha\sigma\eta-\beta(\bar{\mu}-r)}{\beta\sigma^2(1-\rho^2)}\Big]\frac{V_x}{V_{xx}}-\frac{\mu_H-\mu_L}{\sigma^2}p(1-p)\frac{V_{xp}}{V_{xx}}.\tag{9}$$

[10] The dependence on the prior belief contrasts with Zhang et al. (2012), who consider the case where the learning effect does not change over time. See pp. 203 of Zhang et al. (2012).

Theorem 2. *By following the standard arguments (see, e.g., Zariphopoulou (2001)), the value function is*

$$V(t,x,p) = -\frac{1}{\gamma}e^{-\gamma c(t)(x+f(t,p))}$$

with the terminal conditions

$$c(T) = 1 \quad and \quad f(T,p(T)) = 0.$$

Let $f_t \equiv \partial f/\partial t$, $f_p \equiv \partial f/\partial p$, and $f_{pp} \equiv \partial^2 f/\partial p^2$,[11] *Then,*

$$f_t(t,p) - rf(t,p) + \widetilde{\mu}(t,p)f_p(t,p) + \frac{1}{2}\widetilde{\sigma}(t,p)f_{pp}(t,p) + h(t,p) = 0, \tag{10}$$

where

$$\begin{aligned}
\widetilde{\mu}(t,p) = &\ \{\lambda_L - (\lambda_L + \lambda_H)p\} - \frac{\bar{\mu} - r}{\sigma^2}(\mu_H - \mu_L)p(1-p), \\
\widetilde{\sigma}(t,p) = &\ \left(\frac{\mu_H - \mu_L}{\sigma}\right)^2 p^2(1-p)^2, \\
h(t,p) = &\ \alpha(\theta - \eta) + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left[\frac{(\bar{\mu} - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\bar{\mu} - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right].
\end{aligned}$$

The solution f exists, and the two boundary conditions hold:

$$\begin{aligned}
f_t(t,0) - rf(t,0) + \lambda_L f_p(t,0) + h(t,0) = 0, \\
f_t(t,1) - rf(t,1) - \lambda_H f_p(t,1) + h(t,1) = 0.
\end{aligned}$$

Sketch of the Proof. The proof of the existence of the solution f coincides with that of the solution $V(t,x,p)$ to the HJB equation.[12] We can show that derivatives such as $f_p(t,p)$, $f_{pp}(t,p)$, and $f_t(t,p)$ are bounded and sufficiently differentiable (see Proposition 2 of Honda (2003) in particular). In detail, we provide the derivation of Equation (10) in Appendix B. □

Theorem 2 shows that the level of $f(t,p)$ represents the insurer's belief adjustment of investment and insurance/reinsurance opportunities. This argument follows from the fact that the belief adjustment affects the conjectured value function V only through $f(t,p)$.[13] Hereafter, we call $f(t,p)$ the *certainty equivalent wealth* (CEW) under the PI model, which captures the wealth level of changes in future investment and insurance/reinsurance opportunities.

[11] We assume that f is twice continuously differentiable with respect to $p \in [0,1]$ and one–time continuously differentiable with respect to $t \in [0,T]$.

[12] The PDE has a degenerate form near the boundaries at $p = 0$ or 1. Thus, the potential degenerate PDE enables us to require whether f is sufficiently smooth enough to be differentiable.

[13] We can easily get $f(t,p)$ by exploiting a numerical method.

4.3. The Properties of Information Quality

We can study *information quality* from the CEW $f(t,p)$. The reason is that at the decision time t, the CEW $f_i(t)$ under each regime is constant due to complete information confidence. Given that information quality is simply inferred from the difference between $f_i(t)$ and $f(t,p)$, we first examine the properties of the CEW $f(t,p)$.

Define the conditional *market price of risk* $\vartheta(t)$ under the PI model as

$$\vartheta(t) \equiv \frac{\bar{\mu}(t) - r}{\sigma} = \frac{\mu_L + (\mu_H - \mu_L)p(t) - r}{\sigma}.$$

Note that $\vartheta(t)$ is within the interval of $[(\mu_L - r)/\sigma, (\mu_H - r)/\sigma]$.[14] Thus, there exists an equivalent Martingale measure $\widetilde{\mathbb{P}}$ with respect to the original measure $\mathbb{P}$ given by $d\widetilde{\mathbb{P}}/d\mathbb{P} = Z(T)$[15] for

$$Z(t) = \exp\left\{-\int_0^t \vartheta(s)d\widetilde{W}_R(s) - \frac{1}{2}\int_0^t \vartheta^2(s)ds\right\}, \quad \text{with} \quad \widetilde{W}_R(t) = \widehat{W}_R(t) + \int_0^t \vartheta(s)ds, \tag{11}$$

where $Z(t)$ is a positive $\widetilde{\mathbb{P}}$–Martingale due to $Z(0) = 1$.

Under the Martingale measure $\widetilde{\mathbb{P}}$, Theorem 2 shows that the CEW $f(t,p)$ follows the discounted Feynman–Kac partial differential equation like the the Black–Scholes–Merton equation (Shreve, 2004). Hence, $f(t,p)$ can be options on posterior beliefs regarding time–varying investment opportunities together with constant insurance/reinsurance ones. For example, the fact that the terminal payoff is $f(T,p(T)) = 0$ indicates no information uncertainty at the terminal time T.

Proposition 1. *The CEW $f(t,p)$ has the Feynman–Kac representation:*[16]

$$\begin{aligned} f(t,p) = &\ \widetilde{E}^{t,p}\left[\int_t^T e^{-r(s-t)} h(s,p(s))ds\right] \\ = &\ \frac{\alpha(\theta-\eta)}{r}\left[1 - e^{-r(T-t)}\right] + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\widetilde{E}^{t,p}\left[\int_t^T \frac{1}{2}\vartheta(s)^2 - \rho\cdot\frac{\alpha\eta}{\beta}\vartheta(s) + \frac{\alpha^2\eta^2}{2\beta^2}ds\right], \end{aligned}$$

where $\widetilde{E}^{t,p}[\cdot]$ is an expectation conditioned on the time-t condition $p(t) = p$ under $\widetilde{\mathbb{P}}$.

Proof: The sketch of the proof is in Appendix C. □

[14] Obviously, the bounded market price of risk satisfies Novikov's condition (the Dominated Convergence Theorem). Moreover, the bounded condition makes the wealth dynamics in the PI model to satisfy the Lipschitz and growth conditions, which must be met to justify use of the standard verification theorem (Dybvig et al., 1999).

[15] See Proposition 5.B of Duffie (2001).

[16] See Theorem 7.6 of Karatzas and Shreve (1991).

Proposition 1 implies the CEW f can be decomposed into two components:

- The CEW by reinsurance costs, $f^C(t)$:

$$f^C(t) \equiv \frac{\alpha(\theta-\eta)}{r}\left[1-e^{-r(T-t)}\right] \leq 0,$$

which depends on both time to maturity $(T-t)$ and the difference between the safety loadings, $\theta - \eta \leq 0$. Note that $f^C \leq 0$, implying the insurer perceives it as capital losses.

- The CEW by demands on precautionary saving, $f^S(t,p)$:

$$f^S(t,p) \equiv \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\widetilde{E}^{t,p}\left[\int_t^T \frac{1}{2}\vartheta(s)^2 - \rho\cdot\frac{\alpha\eta}{\beta}\vartheta(s) + \frac{\alpha^2\eta^2}{2\beta^2}ds\right] \geq 0.$$

Since $(\alpha\eta/\beta)^2(\rho^2-1) \leq 0$, the second component f^S driven by demands on precautionary saving against both market risk and insurance claims risk has a non-negative value.

Note that the more information quality an insurer has, the more $f(t,p)$ increases. This argument occurs because an increase in $f(t,p)$ has a positive effect on the wealth (and thus the value function) (Theorem 2). Given this central fact, the CEW decomposition offers some implications about information quality.

First, high risk aversion γ leads to much strict risk management as a result of low information quality; this conclusion is consistent with our intuition. This result arises because an insurer, who has a stronger precautionary motive, has less information confidence about future investment and insurance/reinsurance opportunities until the terminal time T. Therefore, she cares much about the capital losses, i.e., f^C. For example, suppose that γ goes to infinity:

$$\lim_{\gamma\to\infty} f(t,p) = f^C(t) + \lim_{\gamma\to\infty} f^S(t,p) = f^C(t) \leq 0.$$

The fact that $f^C \leq 0$ implies that low information quality has a negative effect on the wealth.

Second, larger correlation ρ helps risk management much easier as a result of high information quality. This argument follows from the fact that ρ is regarded as a measure of how the claims risk is easily shared by the financial market. For instance, suppose that risky returns are perfectly correlated with insurance claims:

$$\lim_{|\rho|\to 1} f(t,p) = \lim_{|\rho|\to 1} f^S(t,p) = \infty.$$

As a result, the insurer has the strongest desire to exploit the *risk–sharing effect*. This risk–sharing effect is surely attributed to high information quality, which has a positive effect on the wealth.

Third, information quality comes from the relative importance between the market price of risk $\vartheta(s)$ for $s \geq t$ and reinsurance price of risk $\alpha\eta/\beta$. To see this specifically, we rewrite the CEW by precautionary saving f^S as

$$f^S(t,p) = \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)} \widetilde{E}^{t,p}\left[\int_t^T \frac{1}{2}\left\{\vartheta(s) - \rho\cdot\frac{\alpha\eta}{\beta}\right\}^2 + \frac{1}{2}\left(\frac{\alpha\eta}{\beta}\right)^2 (1-\rho^2)ds\right] \geq 0.$$

At the decision time t, all information is available to the insurer except for the relative importance of the two risk–adjusted premiums until the terminal time T. Here, the correlation ρ functions as a adjusting vehicle of the two markets like an exchange rate used to adjust two different currencies. Next, we further elaborate their relative importance on information quality.

Proposition 2. *The first–order marginal CEW $f_p(t,p) > 0$ if, for all $t \in [0,T]$, $\vartheta(t) > \rho\alpha\eta/\beta$ holds. Moreover, the second–order marginal CEW $f_{pp}(t,p)$ can have both positive and negative values.*

Proof: Applying the Malliavin calculus (Honda, 2003), the boundedness and differentiablity of $f(t,p)$ (see the Sketch of the Proof in Theorem 2) yield

$$\begin{aligned} f_p(t,p) = &\ \widetilde{E}^{t,p}\Big[\int_t^T e^{-r(s-t)}\frac{\partial h(s,p(s))}{\partial p(s)}\frac{\partial p(s)}{\partial p}ds\Big] \\ = &\ \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\frac{\mu_H-\mu_L}{\sigma}\widetilde{E}^{t,p}\left[\int_t^T\left\{\vartheta(s)-\rho\cdot\frac{\alpha\eta}{\beta}\right\}I(s)ds\right] > 0, \end{aligned}$$

where $I(s) \equiv \partial p(s)/\partial p$ with $I(0) = 1$. We present a stochastic differential equation of $I(t)$ in Appendix D. Moreover, f can be either convex or concave with respect to p, depending on the value of f_{pp}. Applying the Malliavin calculus again, we get

$$\begin{aligned} f_{pp}(t,p) = &\frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left(\frac{\mu_H-\mu_L}{\sigma}\right)^2\widetilde{E}^{t,p}\left[\int_t^T I(s)^2 ds\right] \\ &+ \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\frac{\mu_H-\mu_L}{\sigma}\widetilde{E}^{t,p}\left[\int_t^T\left\{\vartheta(s)-\rho\cdot\frac{\alpha\eta}{\beta}\right\}J(s)ds\right], \end{aligned}$$

where $J(t) \equiv \partial I(t)/\partial p$ with $J(0) = 0$. Since $J(0) = 0$, $J(t)$ can be positive or negative. □

The first statement of Proposition 2 implies that a sufficiently high expected risky return in Low regime μ_L guarantees the increasing property of the CEW f with the initial belief p:

$$\vartheta_L > \rho \cdot \frac{\alpha\eta}{\beta} \iff \mu_L > r + \rho \cdot \frac{\alpha\eta}{\beta}\sigma.$$

This result can be justifiable when the worst investment opportunity ϑ_L always outperforms the (adjusted) reinsurance opportunity $\rho\alpha\eta/\beta$ until the terminal time T. The second statement of Proposition 2 also shows that the CEW f can be a convex or concave function of prior belief, depending on the financial and (re)insurance opportunity sets. In sum, the relative importance of the two risk–adjusted premiums is a main source of information quality.

5. Implications

5.1. Parameters

We use parameters in Jang and Kim (2015), who solve the ruin minimization problem for an insurer under the two-state observable Markov chain. Jang and Kim (2015) estimate market parameters and insurance claim parameters by utilizing KOSPI data and Korean property and casualty insurance market data. In fact, Jang and Kim (2015) estimate the parameters in the two different models, and we choose the parameters of Model 1 which restricts only the drift of insurance claims process as a constant value across the two regimes. We set the other parameters by utilizing the relationship of

$$\text{Parameter}_{ave} = \frac{\lambda_L}{\lambda_L + \lambda_H}\text{Parameter}_H + \frac{\lambda_H}{\lambda_L + \lambda_H}\text{Parameter}_L.$$

Note that $\frac{\lambda_L}{\lambda_L+\lambda_H}$ is the expected duration of staying in regime H, and $\frac{\lambda_H}{\lambda_L+\lambda_H}$ is that in regime L.

Specifically, the benchmark parameters are $\mu_H = 0.1188$, $\mu_L = -0.2592$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$. The sign of ρ differs from that presented in Jang and Kim (2015). This difference results from the negative relation of the claims process $dC(t)$ to claims shock $dW_C(t)$.[17] We set the baseline coefficient of absolute risk aversion as $\gamma = 20$.

[17] See footnote 14 in Jang and Kim (2015).

5.2. Optimal Reinsurance Strategy

In this section, we compare the optimal reinsurance rates ε^* (the PI model) with ε_i^* for $i \in \{H, L\}$ (the CI model). Specifically, we rewrite ε^* and ε_i^* in terms of the two risk–adjusted premiums and then study the corresponding implications.

[Insert Figure 1 here.]

The mark–to–market reinsurance policy ε^* (Figure 1) depends on prior belief p. This argument can be clearly justified by the existence of the market price of risk $\vartheta(0) \equiv \frac{\mu_L+(\mu_H-\mu_L)p-r}{\sigma}$:

$$\varepsilon^* = 1 + \frac{e^{-rT}}{\gamma\beta(1-\rho^2)}\left[\rho \cdot \vartheta(0) - \frac{\alpha\eta}{\beta}\right].$$

In addition, ε^* is a linearly decreasing function of prior belief (Figure 1) with slope

$$\frac{\partial \varepsilon^*}{\partial p} = \frac{e^{-rT}}{\gamma\beta(1-\rho^2)} \frac{\mu_H - \mu_L}{\sigma} \cdot \rho < 0$$

as a result of $\rho < 0$.

This negative slope suggests a *counter–cyclical reinsurance mechanism.* For example, the insurer's pessimistic view $p \to 0$ that is close to regime L induces her to further raise the reinsurance rates to maximize her terminal wealth. Indeed, the counter–cyclical mechanism can help to prevent considerable losses to meet liability obligations, especially during the recent financial crisis (CGFS, 2011).[18]

As a matter of fact, the use of the low correlation $\rho = -0.0222$ makes the slope of ε^* almost flat. It seems to be independent of prior belief (Figure 1). However, this small effect on the mark–to–market policy ε^* may be confined into the Korean markets, in which policy differs from those of other countries (OECD, 2015). In the United States, for example, the insurance sector is highly correlated with the banking sector (CGFS, 2011). This high correlation can result in substantial variation in the counter–cyclical reinsurance mechanism.

In contrast, ε_i^* for $i \in \{H, L\}$ is independent of prior belief:

$$\varepsilon_i^* = 1 + \frac{e^{-rT}}{\gamma\beta(1-\rho^2)}\left[\rho \cdot \vartheta_i - \frac{\alpha\eta}{\beta}\right],$$

[18] CGFS is the abbreviation of "the Committee on the Global Financial System".

where ϑ_H represents the best investment opportunity, but ϑ_L represents the worst investment opportunity. In short, ε_i^* only provides the quantitative reinsurance ceiling and floor; ε_L^* is the upper limit of ε^*, and ε_H^* is the lower limit such that $\varepsilon^* \in [\varepsilon_H^*, \varepsilon_L^*]$ because $\rho < 0$.

Importantly, the relative importance of the two risk–adjusted premiums significantly affects the counter–cyclical reinsurance policy ε^*. Here, the correlation ρ is used to adjust the market price of risk to the reinsurance price of risk like the exchange rate. For example, the condition $\rho \cdot \vartheta(0) = \alpha\eta/\beta$ corresponds to $\varepsilon^* = 1$ so that she wants to perfectly hedge the claims risk with the resulting prior:

$$p_{\varepsilon^*} \equiv \frac{\sigma}{\rho(\mu_H - \mu_L)} \left[\frac{\alpha\eta}{\beta} - \rho \cdot \vartheta_L\right].$$

The insurer' perfect–hedge desire to be $\varepsilon^* = 1$ occurs when she compares the (adjusted) worst investment opportunity $\rho \cdot \vartheta_L$ with her target reinsurance opportunity $\alpha\eta/\beta$. We can exclude the possibility that $\varepsilon^* \geq 1$ because $\rho \cdot \vartheta(0) < \alpha\eta/\beta$ for all $p \in [0, 1]$. What this means is that given the risk–sharing extent ρ, she prefers the current reinsurance opportunity for risk management to the current investment opportunity.

In summary, both ε^* and ε_i^* are myopic decisions. The myopic decisions result from information confidence. This is a simple consequence of our assumption that investment opportunities only change over time. However, insurance companies that are eager to monitor a financial market can adopt the mark–to–market reinsurance strategy that moves on a counter–cyclical basis.

5.3. Optimal Portfolio Strategy

In this section, we compare the optimal portfolio strategy π^* (the PI model) with π_i^* for $i \in \{H, L\}$ (the CI model). Specifically, we restate π^* and π_i^* in terms of the two risk–adjusted premiums and then study the corresponding implications.

The mark–to–market investment strategy π^* depends on prior belief p. We can further decompose π^* into a *myopic demand* π_1^* and an *intertemporal hedging demand* π_2^*:

$$\pi^* = \underbrace{\frac{e^{-rT}}{\gamma\sigma(1-\rho^2)} \left[\vartheta(0) - \rho \cdot \frac{\alpha\eta}{\beta}\right]}_{\pi_1^*} + \underbrace{\frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \cdot -f_p(0, p)}_{\pi_2^*},$$

where

$$f_p(0, p) = \frac{e^{-rT}}{\gamma(1-\rho^2)} \frac{\mu_H - \mu_L}{\sigma} \cdot \widetilde{E}^{0,p} \left[\int_0^T \left\{\vartheta(s) - \rho \cdot \frac{\alpha\eta}{\beta}\right\} I(s) ds\right].$$

The non–myopic decision π_2^* is attributed to her marginal CEW f_p that results from the relative importance of the two risk–adjusted premiums until the terminal time T (Proposition 2).

In contrast, π_i^* for $i \in \{H, L\}$ only provides the quantitative investment ceiling and floor:

$$\pi_i^* = \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\left[\vartheta_i - \rho \cdot \frac{\alpha\eta}{\beta}\right].$$

That is, π_H^* can be the upper bound of π^* at $p = 1$, and π_L^* is the lower bound at $p = 0$ without the intertemporal hedging demand π_2^* such that $\pi^* \in [\pi_L^*, \pi_H^*]$.

This simple difference in investment strategy accords well with the conventional results that future uncertainty in relation to estimation risk generates an intertemporal hedging demand (Campbell et al., 2004; Chacko and Viceira, 2005). Our analysis also supports the argument of Ang and Bekaert (2002), who show that when an economic agent is uncertain about macroeconomic conditions, the regime effect weakens by the hedging demand. Specifically, we attempt to analyze the mark–to–market investment policy π^* with two cases: one in which short sales are allowed (Figure 2), and one in which short sales are not allowed (Figure 3).

[Insert Figure 2 here.]

The short–sale case originates from the condition $\mu_L < r$ (Section 5.1). The insurer can increase her risk–free profit in regime L, so she has no incentive to invest in risky assets. Consequently, the availability of the worst investment opportunity makes short sales optimal, although they are not legally allowed for insurance companies in most countries (OECD, 2015). Thus, we will conduct a study of whether to limit short sales affects the mark–to–market asset allocation π^*.

For example, a simple restriction $\mu_L = r$ can yield no short–sale case. This restriction satisfies $\mu_L > r + \rho \cdot \frac{\alpha\eta}{\beta}\sigma$ due to $\rho < 0$ and also coincides with

$$\vartheta(t) > \rho \cdot \frac{\alpha\eta}{\beta}$$

for all $t \in [0, T]$ (Proposition 2). Likewise, the restriction also affects the mark–to–market reinsurance policy ε^*. However, it only changes the slope variation (Section 5.2), *not* the general pattern.

The myopic demand $\pi_1^* \equiv \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\left[\vartheta(0) - \rho \cdot \frac{\alpha\eta}{\beta}\right]$ is a linearly increasing function of prior belief p with slope:

$$\frac{\partial \pi_1^*}{\partial p} = \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\frac{\mu_H - \mu_L}{\sigma} > 0.$$

Thus, the myopic demand π_1^* corresponds to a *pro–cyclical investment mechanism*: the insurer's optimistic view $p \to 1$ close to regime H increases the amount invested.

The relative importance of the two risk–adjusted premiums significantly affects the pro–cyclical mechanism. Here, the correlation ρ is used to adjust the reinsurance price of risk to the market price of risk. For example, the condition $\vartheta(0) = \rho\alpha\eta/\beta$ corresponds to $\pi_1^* = 0$ so that she has no pro–cyclical demand. The resulting break–even prior to hold $\vartheta(0) = \rho\alpha\eta/\beta$ delivers

$$p_{\pi_1^*} \equiv \frac{\sigma}{\mu_H - \mu_L}\left[\rho \cdot \frac{\alpha\eta}{\beta} - \vartheta_L\right].$$

Her desire to eliminate the pro–cyclical demand arises from the comparison between the reinsurance opportunity $\rho\alpha\eta/\beta$ and the worst investment opportunity ϑ_L. Moreover, the condition that $\vartheta(0) < \rho\alpha\eta/\beta$ results in $\pi_1^* < 0$ implies that short sales occur when the current investment opportunity is not preferable to the reinsurance opportunity for risk management. Thus, a good market signal $\vartheta(0)$ relative to $\rho\alpha\eta/\beta$ gives incentive to increase the pro–cyclical portfolio amount, i.e., $\vartheta(0) > \rho\alpha\eta/\beta$.

[Insert Figure 3 here.]

In the short–sale case (Panel A, Figure 2), π_1^* is negative at $p < 0.70$ but positive at $p > 0.70$. This conclusion indicates that $p_{\pi_1^*} = 0.70$. In the no short–sale case (Panel A, Figure 3), however, π_1^* is always positive as a result of $\vartheta(0) > \rho\alpha\eta/\beta$ for all $p \in [0, 1]$, so no break–even prior $p_{\pi_1}^*$ exists.

The hedging demand $\pi_2^* \equiv \frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \cdot -f_p$ is a non–linear function of prior belief. This demand functions as a *counter–cyclical investment mechanism* to offset the potential overreaction to the pro–cyclical mechanism:

$$\pi_2^* = \frac{e^{-rT}}{\gamma\sigma(1-\rho^2)}\left(\frac{\mu_H - \mu_L}{\sigma}\right)^2 p(1-p) \cdot \widetilde{E}^{0,p}\left[\int_0^T \left\{\rho \cdot \frac{\alpha\eta}{\beta} - \vartheta(s)\right\} I(s)ds\right],$$

where

$$f_p(0,p) = \frac{e^{-rT}}{\gamma(1-\rho^2)}\frac{\mu_H - \mu_L}{\sigma} \cdot \widetilde{E}^{0,p}\left[\int_0^T \left\{\vartheta(s) - \rho \cdot \frac{\alpha\eta}{\beta}\right\} I(s)ds\right].$$

In short, $\rho\alpha\eta/\beta - \vartheta(t)$ for all $t \in [0, T]$ in π_2^* represents the counter–cyclical mechanism, whereas $\vartheta(0) - \rho\alpha\eta/\beta$ in π_1^* represents the pro–cyclical mechanism.

Next, we relate the counter–cyclical mechanism to information quality. In particular, we must study (i) the first–order marginal CEW f_p that determines either the increasing or decreasing property for the CEW f, and (ii) the second–order marginal CEW f_{pp} that determines whether f is convex or concave.

As a consequence, we find that information quality characterized by the CEW f must be convex with respect to prior belief. This convex property can be justified by our numerical results that f_{pp} is strictly positive for all $p \in [0, 1]$, regardless of short–sale allowance. Further, the CEW decomposition $f(t, p) = f^C(t) + f^S(t, p)$ shows that the CEW by demands on precautionary saving f^S is also convex, because the CEW by reinsurance costs f^C is independent of prior belief (Proposition 1). Let us go into further details.

In the short–sale case (Panel B, Figure 2), π_2^* is positive at $p < 0.61$ but negative at $p > 0.61$. Given $\pi_2^* \equiv \frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \cdot -f_p$ and $\frac{\mu_H - \mu_L}{\sigma^2} p(1-p) \geq 0$, the first–order marginal CEW f_p must be negative at $p < 0.61$ but positive at $p > 0.61$. For example, the fact $f_p < 0$ at $p < 0.61$ implies that f is decreasing with respect to prior belief. In other words, the insurer with the initial belief $p < 0.61$ prefers the (adjusted) reinsurance opportunity to the current investment opportunity. To the extent of this relative–importance discrepancy, however, the insurer expects the future investment opportunities to get better until the terminal time T. This mean–reverting property that comes from variation in posterior beliefs in Equation (7) leads to the counter–cyclical mechanism, i.e., $\pi_2^* > 0$ at $p < 0.61$.

In the no short–sale case (Panel B, Figure 3), π_2^* is consistently negative. The central difference in the short–sale case (Figure 2) arises from the consistent predominance of $\vartheta(t)$ over $\rho\alpha\eta/\beta$ for all $t \in [0, T]$. The fact $\vartheta(t) > \rho\alpha\eta/\beta$ for all $t \in [0, T]$ indicates that the marginal CEW f_p is always positive. Likewise, the insurer anticipates the future investment opportunities to get worse until the terminal time T as a result of the mean–reverting property. This is why the counter–cyclical demand always has a negative effect on the total portfolio demand π^*, i.e., $\pi_2^* < 0$ for $p \in [0, 1]$.

The total portfolio demand $\pi^* = \pi_1^* + \pi_2^*$ seems to deviate *little* from the pro–cyclical demand π_1^* (Panels C, Figure 2 and Figure 3). This little deviation may result from the low correlation $\rho = -0.022$ in the Korean markets. We believe that other countries in which the risk–sharing effect is higher than this will exhibit substantial counter–cyclical behavior (CGFS, 2011).

Actually, many insurance companies suffered substantial losses on their portfolios during the recent financial crisis (OECD, 2015). These losses stimulated the development of the counter–cyclical investment mechanism. Our counter–cyclical implications can help to protect against serious losses and can also improve the financial stability of these companies.

5.4. The Certainty Equivalent Wealth Gain by Information Acquisition

Comparing the CI model with the PI model in terms of the CEW can provide economic implication concerning the effect of information quality on the insurer's optimal strategies. To see this clearly, we define utility gain from the two models.

Definition 1. *The certainty equivalent wealth gain by information acquisition (I-CEWG), $\Delta_i(\cdot)$, in each regime $i \in \{H, L\}$ is defined as*

$$V_i(t, x - \Delta_i(p)) = V(t, x, p). \tag{12}$$

We also define the regime–weighted average $\Delta_{ave}(p)$ (Jang et al., 2007):

$$\Delta_{ave}(p) = \frac{\lambda_L}{\lambda_L + \lambda_H}\Delta_H(p) + \frac{\lambda_H}{\lambda_L + \lambda_H}\Delta_L(p).$$

We plot the three kinds of the I-CEWGs with $\gamma = 20$ and $x = 0.5$: the short–sale case $\mu_L < r$ (Panel A, Figure 4), and no short–sale case with the restriction $\mu_L = r$ (Panel B, Figure 4). The main findings are as follows.

[Insert Figure 4 here.]

First, information quality certainly exists as a result of the positive values of $\Delta_i(p)$ for $i \in \{H, L, ave\}$ (Figure 4). This positivity is easily inferred from the exponents in the value functions (Equation (12)):

$$f_i(0) - \Delta_i(p) = f(0, p).$$

The strictly positive $\Delta_i(p) = f_i(0) - f(0, p)$ implies that $f_i(0)$ independent of prior belief p is strictly larger than $f(0, p)$ because of higher information quality. Thus, the insurer with partial information must do conservative risk management.

Second, information quality comes from the relative importance of the risk–adjusted premiums. As of our example (Section 5.3), the CEW f is a convex function of prior belief, regardless of whether to allow short sales. In contrast, $f_i(0)$ is constant even with stochastic regime risk as a consequence of information confidence. Thus, the positive $\Delta_i(p) = f_i(0) - f(0, p)$ naturally corresponds to a concave function of prior belief, which is determined by the relative importance (Proposition 2).

In the short–sale case (Panel A), we find that $\Delta_L(p)$ is larger than $\Delta_H(p)$, implying $f_L(0) > f_H(0)$. This finding follows from the fact that the insurer can get more benefit from worst investment opportunity than from the best investment opportunity because she can confidently short–sale to avail the worst investment opportunity. This interpretation is also associated with $|\pi_L^*| > |\pi_H^*|$ (Panel C, Figure 2). In the no short–sale case (Panel B), however, $\Delta_H(p)$ is larger than $\Delta_L(p)$, leading to $f_H(0) > f_L(0)$. When short sales are not allowed, she only avails the best investment opportunity, which is related to $|\pi_H^*| > |\pi_L^*|$ (Panel C, Figure 3).

6. Conclusion

This paper investigates the optimal mark–to–market reinsurance and asset investment strategies for insurers with complete or partial information on expected return. The insurer with partial information is assumed to have prior belief on the expected return and to update posterior beliefs by exploiting its price information. We show that the strategies of the insurer with partial information can be highly dependent on prior belief, and that variation in posterior beliefs gives rise to her counter–cyclical investment demand. By comparing the two insurers' strategies, we show that information quality comes from the relative importance of the two risk–adjusted premiums.

Appendix A. The Proof of Theorem 1

Fist of all, we have the two terminal conditions $c(T) = 1$ and $f_i(T) = 0$.

Plugging Equation (4) into Equation (3), we get the HJB equations for each regime $i \in \{H, L\}$:

$$\begin{aligned} V_{i,t} + \{rx + \alpha(\theta - \eta)\}V_{i,x} - \frac{1}{1-\rho^2}\left[\frac{(\mu_i - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\mu_i - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right]\frac{V_{i,x}^2}{V_{i,xx}} \\ + \lambda_i\{V_j - V_i\} = 0. \end{aligned} \tag{A.1}$$

The first–order and second–order derivatives are

$$\begin{aligned} V_{i,t} &= -\tfrac{1}{\gamma}[-\gamma c_t(x + f_i) - \gamma c f_{i,t}]e^{-\gamma c(x+f_i)}, \\ V_{i,x} &= c e^{-\gamma c(t)(x+f_i)}, \\ V_{i,xx} &= -\gamma c^2 e^{-\gamma c(t)(x+f_i)}, \end{aligned}$$

where $c_t \equiv \dfrac{\partial c}{\partial t}$ and $f_{i,t} \equiv \dfrac{\partial f_i}{\partial t}$. Rearranging Equation (A.1) gives

$$\begin{aligned} &\Big[c_t + rc\Big]x + c\Big[f_{i,t} + \frac{c_t}{c}f_i + \frac{\lambda_i}{\gamma c}\Big\{1 - e^{-\gamma c(f_j - f_i)}\Big\} \\ &+ \alpha(\theta - \eta) + \frac{1}{\gamma c(1-\rho^2)}\Big\{\frac{(\mu_i - r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\mu_i - r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\Big\}\Big] = 0. \end{aligned}$$

To obtain the solution independent of x, the terms in the two square brackets must be zero. Then Equation (5) in Theorem 1 is straightforward.

The reader can easily provide a verification theorem for V if he/she can just follow the arguments in Jang et al. (2007).

Appendix B. The Proof of Theorem 2

Plugging Equation (9) into Equation (8) corresponds to the HJB equation:

$$\begin{aligned} V_t + (rx + \alpha\theta - \alpha\eta)V_x + \frac{1}{1-\rho^2}\Big[\frac{\alpha\eta\rho(\bar{\mu}-r)}{\beta\sigma} - \frac{(\bar{\mu}-r)^2}{2\sigma^2} - \frac{\alpha^2\eta^2}{2\beta^2}\Big]\frac{V_x^2}{V_{xx}} \\ -\frac{\bar{\mu}-r}{\sigma^2}(\mu_H-\mu_L)p(1-p)\frac{V_xV_{xp}}{V_{xx}} - \frac{(\mu_H-\mu_L)_\mu^2}{2\sigma^2}p^2(1-p)^2\frac{V_{xp}^2}{V_{xx}} \\ +\{\lambda_L - (\lambda_H+\lambda_L)p\}V_p + \frac{(\mu_H-\mu_L)^2}{2\sigma^2}p^2(1-p)^2V_{pp} = 0. \end{aligned} \tag{B.1}$$

Recall that the conjecture value function $V(t,p,x) = -\dfrac{1}{\gamma}e^{-\gamma c(t)(x+f(t,p))}$ with the terminal conditions $c(T) = 1$ and $f(T, p(T)) = 0$. The first–order and second–order derivatives are

$$\begin{aligned} &V_t = -\gamma[c_t(x+f) + cf_t]V, \qquad V_x = -\gamma cV, \qquad V_{xx} = \gamma^2c^2V, \\ &V_p = -\gamma cf_pV, \qquad V_{pp} = -\gamma cf_{pp}V + \gamma^2c^2f_p^2V, \qquad V_{xp} = \gamma^2c^2f_pV. \end{aligned}$$

Substituting the derivatives into Equation (B.1) yields

$$\Big[c_t + rc\Big]x + c\Big[f_t + \frac{c_t}{c}f + \widetilde{\mu}_p(t,p)f_p + \frac{1}{2}\widetilde{\sigma}_p(t,p)f_{pp} + h\Big] = 0,$$

where

$$\begin{aligned} \widetilde{\mu}_p(t,p) &= \{\lambda_L - (\lambda_L+\lambda_H)p\} - \frac{\bar{\mu}-r}{\sigma^2}(\mu_H-\mu_L)p(1-p), \\ \widetilde{\sigma}_p(t,p) &= \left(\frac{\mu_H-\mu_L}{\sigma}\right)^2 p^2(1-p)^2, \\ h(t,p) &= \alpha(\theta-\eta) + \frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left[\frac{(\bar{\mu}-r)^2}{2\sigma^2} - \frac{\alpha\eta\rho(\bar{\mu}-r)}{\beta\sigma} + \frac{\alpha^2\eta^2}{2\beta^2}\right]. \end{aligned}$$

The first square bracket with $c(T) = 1$ leads to $c(t) = e^{r(T-t)}$, and the second square bracket with $f(T, p(T)) = 0$ delivers the following linear second–order PDE:

$$f_t(t,p) - rf(t,p) + \widetilde{\mu}_p(t,p)f_p(t,p) + \frac{1}{2}\widetilde{\sigma}_p(t,p)f_{pp}(t,p) + h(t,p) = 0.$$

Now, we can provide the verification theorem for the PI model. In fact, the standard arguments can be applied (Dybvig et al., 1999), and it is almost a copy of Proposition 1 in Honda (2003).

Appendix C. Proof of Proposition 1

By the Girsanov theorem in Equation (11), we can rewrite Equations (6) and (7) as

$$
\begin{aligned}
dR(t) &= r dt + \sigma d\widetilde{W}_R(t), \\
dp(t) &= \widetilde{\mu}_p(t, p(t)) dt + \widetilde{\sigma}_p(t, p(t)) d\widetilde{W}_R(t),
\end{aligned}
\tag{C.1}
$$

where the drift and diffusion terms of the belief process under $\widetilde{\mathbb{P}}$ are given by

$$
\begin{aligned}
\widetilde{\mu}_p(t, p(t)) &= \lambda_L - (\lambda_H + \lambda_L) p(t) - \left(\frac{\mu_H - \mu_L}{\sigma} \right) p(t)(1 - p(t)) \vartheta(t), \\
\widetilde{\sigma}_p(t, p(t)) &= \left(\frac{\mu_H - \mu_L}{\sigma} \right) p(t)(1 - p(t)).
\end{aligned}
$$

David (1997) shows that the original belief process under $\mathbb{P}$ satisfies the Lipschitz and growth conditions; thus, there exists a unique solution. Indeed, the bounded market price of risk $\vartheta(t)$ also suffices that the adjusted belief process under $\widetilde{\mathbb{P}}$ also has a unique solution.

Next, we briefly sketch the proof of Proposition 1. Define a process $Q(s)$ as

$$
Q(s) = e^{-r(s-t)} f(s, p(s)) + \int_t^s e^{-r(u-t)} h(u, p(u)) du. \tag{C.2}
$$

Applying the Itô formula to $Q(s)$ yields

$$
\begin{aligned}
dQ(s) &= -r e^{-r(s-t)} f + e^{-r(s-t)} \left(f_t + \widetilde{\mu}_p f_p + \frac{1}{2} \widetilde{\sigma}_p^2 f_{pp} + h \right) ds + e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s) \\
&= e^{-r(s-t)} \left(-r f + f_t + \widetilde{\mu}_p f_p + \frac{1}{2} \widetilde{\sigma}_p^2 f_{pp} + h \right) ds + e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s).
\end{aligned}
$$

We easily find that the first parenthesis in the drift term is equal to the PDE in Equation (10) and then rewrite the remaining term as

$$
dQ(s) = e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s).
$$

The Martingale property is easily obtained by

$$
\widetilde{E}^{t,p} \left[Q(T) - Q(t) \right] = \widetilde{E}^{t,p} \left[\int_t^T e^{-r(s-t)} \widetilde{\sigma}_p f_p d\widetilde{W}_R(s) \right] = 0,
$$

provided that f_p is bounded on the domain $p(t) \in [0, 1]$. Here, Proposition 2 of Honda (2003) shows that the marginal function f_p is bounded by the Dominated convergence theorem.

It is straightforward to show from Equation (C.2):

$$\begin{aligned}
\widetilde{E}^{t,p}[Q(T)] &= \widetilde{E}^{t,p}\left[e^{-r(T-t)}f(T,p(T))\right] + \widetilde{E}^{t,p}\left[\int_t^T e^{-r(s-t)}h(s,p(s))ds\right], \\
\widetilde{E}^{t,p}[Q(t)] &= f(t,p),
\end{aligned}$$

with the terminal condition:

$$f(T,p(T)) = 0.$$

This completes the proof:

$$f(t,p) = \widetilde{E}^{t,p}\left[\int_t^T e^{-r(s-t)}h(s,p(s))ds\right], \quad t \in [0,T].$$

Appendix D. Proof of Proposition 2

We start with the belief process under $\widetilde{\mathbb{P}}$ in Equation (C.1). First, we define a process $I(t)$ as

$$I(t) = \frac{\partial}{\partial p}p(t).$$

Then, we can derive the SDE of $I(t)$ by using the the Malliavin calculus:

$$\begin{aligned}
dI(t) &= \frac{\partial\widetilde{\mu}_p(t,p(t))}{\partial p(t)}\frac{\partial p(t)}{\partial p}dt + \frac{\partial\widetilde{\sigma}_p(t,p(t))}{\partial p(t)}\frac{\partial p(t)}{\partial p}d\widetilde{W}_R(t), \\
&= \frac{\partial\widetilde{\mu}_p(t,p(t))}{\partial p(t)}I(t)dt + \frac{\partial\widetilde{\sigma}_p(t,p(t))}{\partial p(t)}I(t)d\widetilde{W}_R(t) \quad \text{with} \quad I(0) = 1,
\end{aligned}$$

where $\frac{\partial\widetilde{\mu}_p(t,p(t))}{\partial p(t)}$ and $\frac{\partial\widetilde{\sigma}_p(t,p(t))}{\partial p(t)}$ are given by

$$\begin{aligned}
\frac{\partial\widetilde{\mu}_p(t,p(t))}{\partial p(t)} &= 3\left(\frac{\mu_H-\mu_L}{\sigma}\right)^2 p(t)^2 + 2\left\{\frac{(\mu_L-r)(\mu_H-\mu_L)}{\sigma^2} - \left(\frac{\mu_H-\mu_L}{\sigma}\right)^2\right\}p(t) \\
&\qquad - \left\{\lambda_L + \lambda_H + \frac{(\mu_L-r)(\mu_H-\mu_L)}{\sigma^2}\right\}, \\
\frac{\partial\widetilde{\sigma}_p(t,p(t))}{\partial p(t)} &= (1-2p(t))\left(\frac{\mu_H-\mu_L}{\sigma}\right).
\end{aligned}$$

Here, $\frac{\partial\widetilde{\mu}_p(t,p(t))}{\partial p(t)}$ and $\frac{\partial\widetilde{\sigma}_p(t,p(t))}{\partial p(t)}$ also satisfy the Lipschitz and growth conditions over the bounded interval $p(t) \in [0,1]$. The fact $I(0) = 1$ demonstrates the strictly positive $I(t)$.

Second, we define a process $J(t)$ as

$$J(t) = \frac{\partial}{\partial p} I(t).$$

The SDE of $J(t)$ evolves as

$$\begin{aligned} dJ(t) = & \left(\frac{\partial \widetilde{\mu}_p}{\partial p(t)} \frac{\partial I(t)}{\partial p} + \frac{\partial^2 \widetilde{\mu}_p}{\partial p(t)^2} \left(\frac{\partial p(t)}{\partial p} \right)^2 \right) dt + \left(\frac{\partial \widetilde{\sigma}_p}{\partial p(t)} \frac{\partial I(t)}{\partial p} + \frac{\partial^2 \widetilde{\sigma}_p}{\partial p(t)^2} \left(\frac{\partial p(t)}{\partial p} \right)^2 \right) d\widetilde{W}_R(t) \\ = & \left(\frac{\partial \widetilde{\mu}_p}{\partial p(t)} J(t) + \frac{\partial^2 \widetilde{\mu}_p}{\partial p(t)^2} I(t)^2 \right) dt + \left(\frac{\partial \widetilde{\sigma}_p}{\partial p(t)} J(t) + \frac{\partial^2 \widetilde{\sigma}_p}{\partial p(t)^2} I(t)^2 \right) d\widetilde{W}_R(t) \quad \text{with} \quad J(0) = 0, \end{aligned}$$

where $\frac{\partial^2 \widetilde{\mu}(t,p(t))}{\partial p(t)^2}$ and $\frac{\partial^2 \widetilde{\sigma}(t,p(t))}{\partial p(t)^2}$ are given by

$$\begin{aligned} \frac{\partial^2 \widetilde{\mu}(t,p(t))}{\partial p(t)^2} &= 6 \left(\frac{\mu_H - \mu_L}{\sigma} \right)^2 p(t) + 2 \left\{ \frac{(\mu_L - r)(\mu_H - \mu_L)}{\sigma^2} - \left(\frac{\mu_H - \mu_L}{\sigma} \right)^2 \right\}, \\ \frac{\partial^2 \widetilde{\sigma}(t,p(t))}{\partial p(t)^2} &= -2 \left(\frac{\mu_H - \mu_L}{\sigma} \right). \end{aligned}$$

Likewise, all the parameters are bounded over the interval $p(t) \in [0,1]$. The fact $J(0) = 0$ implies that $J(t)$ can be negative.

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Figures

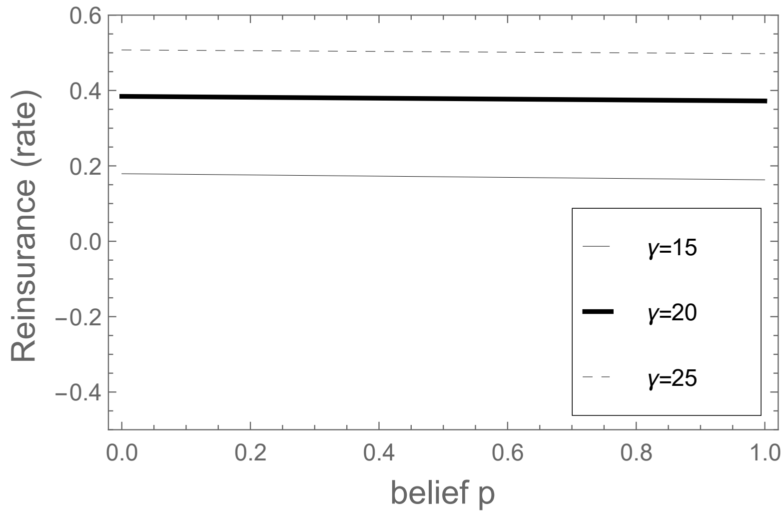
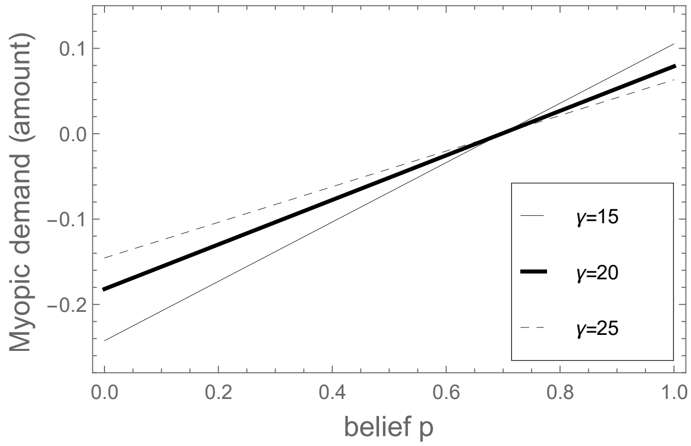
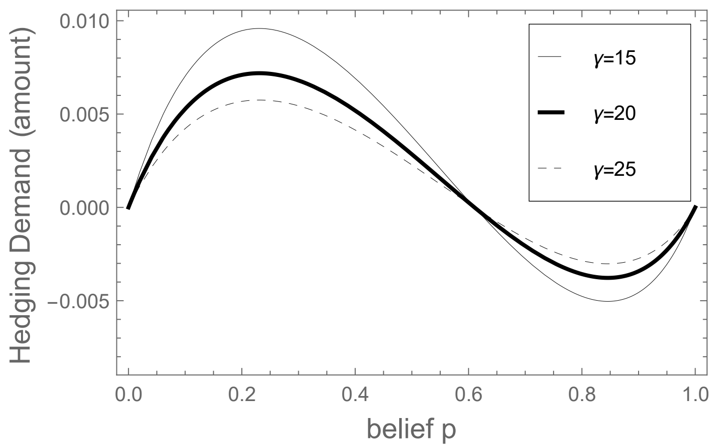


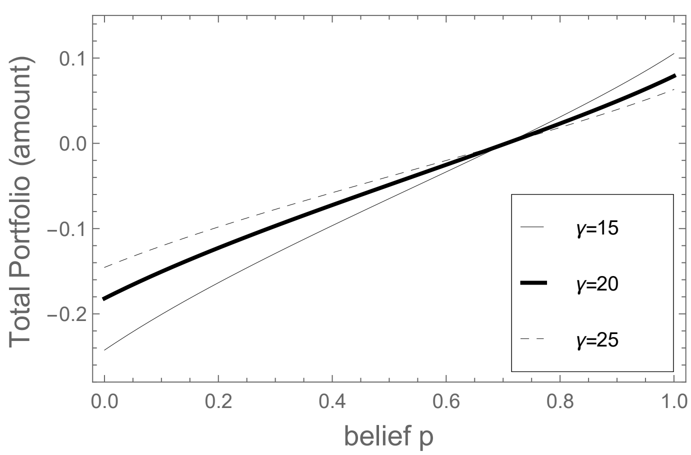
Figure 1: Optimal Reinsurance Rates ε^*. The parameters are $\mu_H = 0.1188$, $\mu_L = -0.2592$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.



(a) Myopic Demand

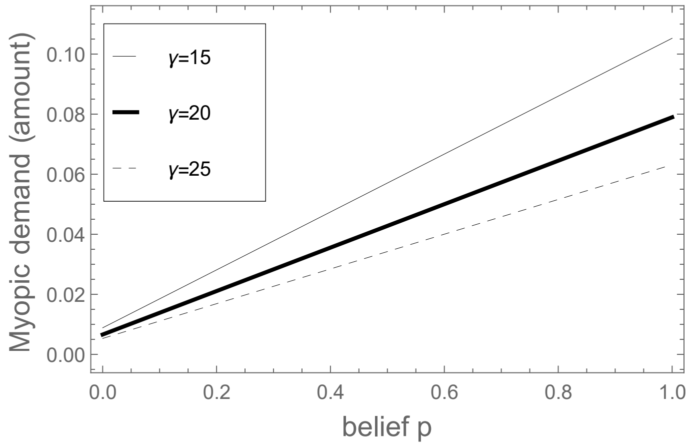


(b) Hedging Demand

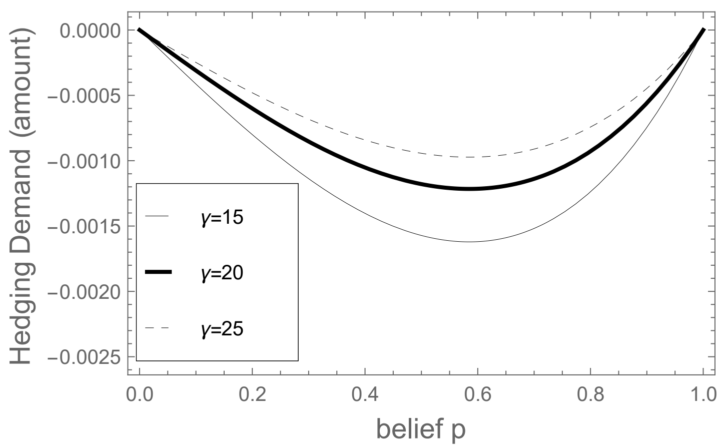


(c) Optimal Portfolio

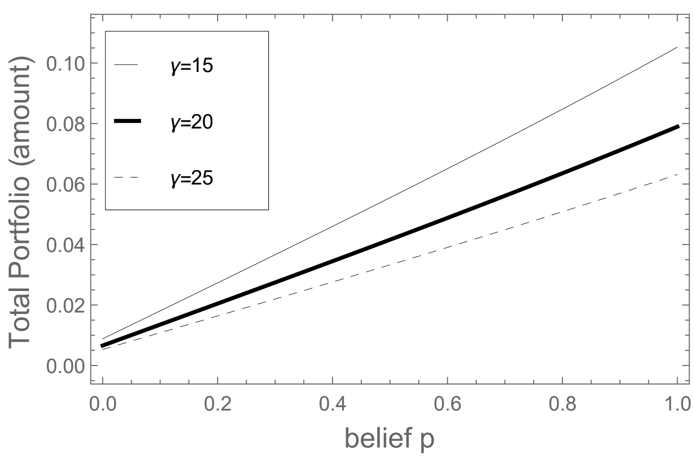
Figure 2: Optimal Portfolio Strategies (short–sale case). The parameters are $\mu_H = 0.1188$, $\mu_L = -0.2592$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.



(a) Myopic Demand

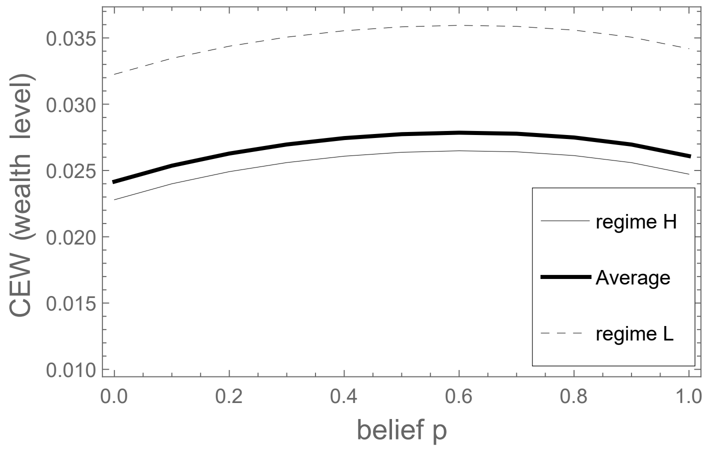


(b) Hedging Demand

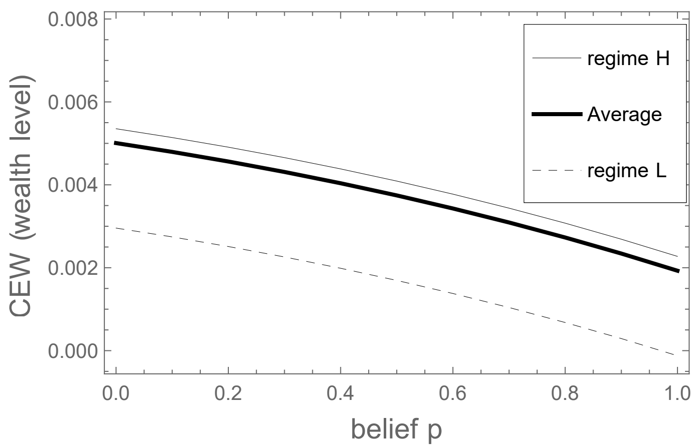


(c) Optimal Portfolio

Figure 3: Optimal Portfolio Strategies (no short–sale case). The parameters are $\mu_H = 0.1188$, $\mu_L = 0.0140$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, and $T = 5$.



(a) Short–Sale Case ($\mu_L < r$)



(b) No Short–Sale Case ($\mu_L = r$)

Figure 4: Certainty Equivalent Wealth (CEW) with $\gamma = 20$. The parameters are $\mu_H = 0.1188$, $\mu_L = 0.0140$, $\lambda_H = 0.275$, $\lambda_L = 1.6304$, $r = 0.0140$, $\sigma = 0.2600$, $\alpha = 1.7136$ $\beta = 0.1239$, $\rho = -0.0222$, $\theta = 0.10$, $\eta = 0.12$, $T = 5$, and $x = 0.5$.