Information and Trading Targets in a Dynamic Market Equilibrium[1]

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ABSTRACT: This paper describes equilibrium interactions between dynamic portfolio rebalancing given a private end-of-day trading target and dynamic trading on long-lived private information. Order-splitting for portfolio rebalancing injects complicated dynamics in the market. The largest driver of portfolio rebalancing order flow is a deterministic component based on the trading target. Learning and sunshine trading effects are also present but smaller. The model has empirical implications about positive co-movement between rebalancing volatility, order-flow autocorrelation, and intraday U-shaped patterns in volume, price volatility, and order-flow autocorrelation.

KEYWORDS: Order-splitting, optimal order execution, trading targets, price discovery, liquidity, portfolio rebalancing, market microstructure

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Trading via dynamic order-splitting algorithms is a pervasive fact in today's financial markets.[2] Informed investors use dynamic order-splitting to increase trading profits by slowing the public revelation of their private information. Order-splitting is not, however, limited to informed investors. Less informed investors — index mutual funds, and comparatively more passive pensions and insurance companies — rely on order-splitting to minimize trading costs for hedging and portfolio rebalancing. As described in O'Hara (2015), portfolio managers transmit *parent orders* — specifying the total amount of a security to be bought or sold over a fixed trading horizon — to brokers who use computer algorithms to break parent orders into sequences of smaller *child orders*.[3] While dynamic informed trading has been modeled extensively (see, e.g., Kyle 1985), order-splitting for portfolio rebalancing is less understood.

Our paper is the first to model a market equilibrium with dynamic trading with both long-lived private information and portfolio rebalancing. We specifically consider a multi-period Kyle (1985) market in which there are two strategic investors with different trading motives who each follow optimal but different dynamic trading strategies. One investor is a standard Kyle strategic informed investor with long-lived private information. The other investor is a strategic portfolio rebalancer who trades over multiple rounds to minimize the cost of hitting a random terminal trading target.

We use our model to investigate the economic motivations for and equilibrium effects of dynamic order-splitting for portfolio rebalancing. Dynamic rebalancing affects the market in three ways: First, there is additional trading noise. However, order-splitting due to a trading constraint leads to autocorrelated rebalancer's orders that are different from the unpredictable informed investor orders and the independently and identically distributed noise trader orders. Second, autocorrelation in the rebalancer orders leads to a type of sunshine trading since predictable orders have no price impact. In particular, market makers in our model try to forecast the remaining

[2] *Pension & Investments* (2007) reported that in a survey of leading institutional investors 72% said they used order execution algorithms. Anecdotal evidence suggests that the use of order execution algorithms has grown further in subsequent years. Optimal execution algorithms are different from computer-based market making, latency arbitrage, and other high frequency trading strategies.

[3] Keim and Madhavan (1995) is the first empirical study of dynamic order-splitting by institutional investors. Recently, van Kervel and Menkveld (2016) estimate an average of 139 child trades per parent order for four large institutions trading on Nasdaq OMX. Korajczyk and Murphy (2016) estimate an average of between 327 and 604 child orders per large parent order depending on whether the parent order is nonstressful (lower three quartiles of large trades) or stressful (top quartile) for Canadian equities. See Johnson (2010) for more on specific dynamic trading algorithms. The SEC (2010) report also discusses the role of trading algorithms in the current market landscape.

future latent trading demand of the rebalancer. Third, there is additional information trading because of endogenous learning by the strategic rebalancer through the trading process. This is because the rebalancer can filter the aggregate order flow better than the market makers by incorporating his knowledge about his own order submissions. In equilibrium, the additional trading noise and the endogenous learning by the rebalancer affect the trading strategy of the informed investor. The resulting changes in aggregate order flow dynamics then affect the equilibrium dynamics of price discovery and liquidity.[4]

Numerical experiments with our model identify a number of testable implications of stochastic dynamic rebalancing:

- Dynamic rebalancing induces intraday U-shaped patterns in expected trading volume, price volatility, and order-flow autocorrelation. In addition, the price impact of the order flow is S-shaped with higher initial price impacts and lower later price impacts relative to those in Kyle (1985). The overall level of aggregate order-flow autocorrelation and the magnitude of the various intraday patterns are all increasing in the volatility of rebalancing target randomness.

- The rebalancer's and insider's orders tend to become negatively correlated over time. Unlike the negative order correlation in Foster and Vishwanathan (1996), the negative correlation in our model arises from the informed investor trading against noise in prices due to the rebalancer's orders. Both investors benefit by providing liquidity to each other symbiotically.

- The rebalancer's orders are driven primarily by a quantitatively large order-splitting component that depends deterministically on the rebalancing target. Components due to speculative trading on endogenous learning and to sunshine trading are much smaller.

Our analysis integrates two literatures on pricing and trading. The first literature is about price discovery. Kyle (1985) described equilibrium pricing and dynamic trading in a market with noise traders and a single investor who has long-lived private information. Subsequent work by Holden and Subrahmanyam (1992), Foster and Viswanathan (1994, 1996), Back (1992), and Back, Cao, and Willard (2000) extends

[4]Uninformed trading noise plays a critical role in markets subject to adverse selection (see Akerlof 1970, Grossman and Stiglitz 1980, Kyle 1985, and Glosten and Milgrom 1985).

the model to allow for multiple informed investors with long-lived information. Our model builds most closely on Foster and Viswanathan (1996), who were the first to model a dynamic equilibrium with multiple investors with different information and to solve the "forecasting the forecasts of others" problem. Given our interest in the fundamental information content and daily dynamics of order flow, the Kyle set-up lets us abstract from the arms race for speed (Hoffmann 2014 and Biais, Foucault, and Moinas 2015), intermediation chains linking multiple market makers (Weller 2013), limit order cancelation and flickering quotes (Hasbrouck and Saar 2007 and Baruch and Glosten 2013), market fragmentation and latency (Menkveld, Yueshen, and Zhu 2014), and other microsecond-level high-frequency trading (HFT) phenomena.

A second literature studies optimal dynamic order execution for uninformed investors with trading targets. This work includes Bertsimas and Lo (1998), Almgren and Chriss (1999, 2000), Gatheral and Scheid (2011), Engel, Ferstenberg, and Russell (2012), Predoiu, Shaikhet, and Shreve (2011), and Boulatov, Bernhardt, and Larionov (2016) as well as Bunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007) on predatory trading in response to predictable uninformed trading. This research takes the price impact function for orders as an exogenously specified model input. In contrast, we model optimal order execution in an equilibrium setting that endogenizes the effect of strategic rebalancing on pricing.[5] In addition, our rebalancer's trading demand is not known ex ante to the market (as in models of predatory trading), but is random and private information. This is arguably the usual situation on normal trading days, as opposed to special days involving futures roles and index reconstitutions.

Models combining both informed trading and optimized uninformed rebalancing have largely been restricted to static settings or to multi-period settings with short-lived information and/or exogenous restrictions on the rebalancer's trading strategies. Admati and Pfleider (1988) study a multi-period market consisting of a series of repeating one-period trading rounds with short-lived information and uninformed discretionary liquidity traders who only trade once but decide when to time their trading. An exception is Seppi (1990) who models an informed investor and a strategic uninformed investor with a trading target in a market in which both can trade dynamically. He solves for separating and partial pooling equilibria with upstairs

[5] In our model, order flow has a price impact due to adverse selection. Alternatively, price impacts can be due to inventory costs and imperfect competition in liquidity provision.

block trading, but only for a restricted set of particular model parameterizations.

Our paper is related to Degryse, de Jong, and van Kervel (DJK 2014). Both papers model dynamic order-splitting by an uninformed investor. Consequently, both models have autocorrelated (predictable) order flows. Order flow autocorrelation is empirically significant but absent in previous Kyle models.[6] However, there are two notable differences between our model and DJK (2014). First, the informed investors in DJK (2014) have short-lived private information; they only have one chance to trade on intraday signals before they become public. In contrast, our informed investor trades on long-lived information over multiple intraday time periods. Consequently, it is harder to distinguish cumulative order imbalances due to rebalancing from imbalances due to information trading. This reduces the gains from sunshine trading for the rebalancer in our model. Second, our rebalancer's orders depend adaptively on the realized path of the aggregate order flow as well as on his rebalancing target, whereas the DJK (2014) rebalancer trades deterministically over time to reach his target. In particular, our rebalancer learns endogenously about the informed investor's information, because he can filter the aggregate order flow better than the market makers. Our analysis is possible because we use the approach of Foster and Vishwanathan (1996) to circumvent the large state-space problem mentioned in DJK (2014).

Our analysis is related to the literature on sunshine trading. One form of sunshine trading exploits dynamic fluctuation in the price impacts of orders as the supply of liquidity is temporarily depleted and then replenished over time (see Predoiu, Shaikhet, and Shreve 2011). Another form of sunshine trading exploits predictable intraday variation in liquidity due to the timing of uninformed trading (see Admati and Pfleiderer 1988). Sunshine trading in our model and DJK (2014) occurs because predictable orders have no incremental information content and thus, absent frictions in the supply of liquidity, no price impact. In addition, symbiotic liquidity provision by informed investors and rebalancers to each other further lowers their trading costs. Looking at optimal rebalancing in a partial equilibrium misses these equilibrium interactions.

[6]For early empirical evidence on order flow autocorrelation in equity markets, see Hasbrouck (1991a,b). More recently, Brogaard, Hendershott, and Riordan (2016) find autocorrelation in orders from non-HFT investors (which is our focus) as well as in HFT orders.

1 Model

We model a multi-period discrete-time market for a risky stock. A trading day is normalized to the interval $[0, 1]$ during which there are $N \in \mathbb{N}$ time points at which trade can occur where $\Delta := \frac{1}{N} > 0$ is the time step. As in Kyle (1985), the stock's final value $\tilde{v}$ becomes publicly known at time $N+1$ after the market closes at the end of the day. The value $\tilde{v}$ is normally distributed with mean zero and variance $\sigma_{\tilde{v}}^2 > 0$. Additionally, there is a money market account that pays a zero interest rate.

Four types of investors trade in our model:

- An informed investor (who we call a hedge fund portfolio manager) knows the final stock value $\tilde{v}$ at the beginning of trading and has zero initial positions in the stock and the money market account. The hedge fund manager is risk-neutral and maximizes the expected value of the fund's final wealth. The hedge fund's order for the stock at time n, $n = 1, ..., N$, is denoted by $\Delta\theta_n^I$ where θ_n^I is its accumulated total stock position at time n with $\theta_0^I := 0$.

- A constrained investor (who we call the rebalancer) needs to rebalance his portfolio by buying or selling stock to reach a terminal trading target constraint $\tilde{a}$ on his ending stock position θ_N^R by the end of the trading day. For example, he might be a portfolio manager for a large index fund or a passive pension plan or an insurance company who needs to rebalance his portfolio or to respond to fund inflows/outflows. The target is private knowledge of the rebalancer. In practice, such investors trade dynamically using optimal order execution algorithms to minimize their trading costs. He starts the day with zero initial positions in the stock ($\theta_0^R := 0$) and his money market account.[7] The target $\tilde{a}$ is jointly normally distributed with the stock value $\tilde{v}$ and has a mean of zero, a variance $\sigma_{\tilde{a}}^2 > 0$, and a correlation $\rho \in [0, 1]$ with $\tilde{v}$. When ρ is 0, the rebalancer is initially uninformed. If $\rho > 0$, we think of the rebalancer as being initially informed about $\tilde{v}$ but subject to random binding non-public risk limits.[8] Importantly, our rebalancer rationally understands the extent to which

[7]Both the hedge fund and the rebalancer finance their stock trading by borrowing/lending. This assumption simplifies the notation for their objective functions but is without loss of generality.

[8]The fact that the terminal value $\tilde{v}$ is measured in dollars while the trading target $\tilde{a}$ is measured in shares is not problematic for $\tilde{v}$ and $\tilde{a}$ being correlated random variables.

he is uninformed.[9] The rebalancer is risk-neutral and maximizes the expected value of his final wealth subject to the terminal stock position constraint. The rebalancer's order for the stock at time n, $n = 1, ..., N$, is denoted by $\Delta\theta_n^R$, and the terminal constraint requires $\Delta\theta_N^R = \tilde{a} - \theta_{N-1}^R$ at time N.

- Noise traders (who we think of as small non-strategic retail investors) submit net stock orders at times n, $n = 1, ..., N$, that are exogenously given by Brownian motion increments Δw_n. These increments are normally distributed with zero-means and variances $\sigma_w^2\Delta$ for a constant $\sigma_w > 0$ and are independent of $\tilde{v}$ and $\tilde{a}$.

- Competitive risk-neutral market makers observe the aggregate net order flow y_n at times n, $n = 1, ..., N$, where

$$y_n := \Delta\theta_n^I + \Delta\theta_n^R + \Delta w_n, \quad y_0 := 0. \tag{1.1}$$

  Given competition and risk-neutrality, market makers clear the market (i.e., trade $-y_n$) at the stock price p_n set to be

$$p_n = \mathbb{E}[\tilde{v}|\sigma(y_1, ..., y_n)], \quad n = 1, 2, ..., N, \quad p_0 := 0, \tag{1.2}$$

  where $\sigma(y_1, ..., y_n)$ is the sigma-algebra generated by the order flow history. In the past, market makers were dealers on the floor of an exchange. Today, market making is performed by high frequency firms running algorithms on servers colocated near an exchange's market crossing engine. These market-making algos process order-flow information in real-time when setting prices.

The presence of the rebalancer with a trading constraint is the main difference between our setting and Kyle (1985) as well as the multi-agent settings in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1994, 1996). In particular, at each time n, the rebalancer has latent demand to trade the remaining $\tilde{a} - \theta_{n-1}^R$ shares over the rest of the day. Previous microstructure theory says very little about

[9] Alternatively, if some investors trade under the mistaken belief that they are informed, but the signals they condition on are just noise, then their orders should have the same functional form as actual informed investor orders (see Kyle and Obizhaeva 2016). In our model, informed investors and rebalancers trade differently because their trading motives are different.

markets with daily latent trading demand. As we shall see, this latent trading demand produces new stylized features in the market such as autocorrelated order flow.

Because all initial positions are zero (i.e., $\theta_0^I = \theta_0^R = 0$), the informed hedge fund chooses orders $\Delta\theta_n^I \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$ at times n, $n = 1, 2, ..., N$, to maximize

$$\mathbb{E}\left[\theta_N^I(\tilde{v} - p_N) + \theta_{N-1}^I \Delta p_N + ... + \theta_1^I \Delta p_2 \Big| \sigma(\tilde{v})\right] = \mathbb{E}\left[\sum_{n=1}^{N} (\tilde{v} - p_n)\Delta\theta_n^I \Big| \sigma(\tilde{v})\right]. \quad (1.3)$$

On the other hand, the rebalancer faces the terminal constraint $\theta_N^R = \tilde{a}$. Therefore, he submits orders $\Delta\theta_n^R \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$ at times n, $n = 1, 2, ..., N-1$, to maximize

$$\mathbb{E}\left[\tilde{a}(\tilde{v} - p_N) + \theta_{N-1}^R \Delta p_N + ... + \theta_1^R \Delta p_2 \Big| \sigma(\tilde{a})\right] = \frac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}^2 - \mathbb{E}\left[\sum_{n=1}^{N} (\tilde{a} - \theta_{n-1}^R)\Delta p_n \Big| \sigma(\tilde{a})\right], \quad (1.4)$$

given the trading constraint $\theta_N^R = \tilde{a}$. The equality in (1.4) follows from $p_N = \sum_{n=1}^N \Delta p_n$, $p_0 = 0$, and $\mathbb{E}[\tilde{v}|\sigma(\tilde{a})] = \frac{\rho\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}$. We prove in Appendix A that the hedge fund's problem (1.3) and the rebalancer's problem (1.4) are both quadratic. The hedge fund's, rebalancer's, and the market makers' information sets are not nested.

Definition 1.1. A *Baysian Nash* equilibrium is a collection of functions $(\theta_n^I, \theta_n^R, p_n)$ such that:

(i) Given the functions (θ_n^R, p_n), the strategy θ_n^I maximizes the hedge fund's objective (1.3).

(ii) Given the functions (θ_n^I, p_n), the strategy θ_n^R maximizes the rebalancer's objective (1.4).

(iii) Given the functions (θ_n^I, θ_n^R), the pricing rule p_n satisfies (1.2).[10]

We construct a linear Bayesian Nash equilibrium with the following structure:

[10] To clarify this definition we recall the Doob-Dynkin lemma: For any random variable B and any $\sigma(B)$-measurable random variable A, there is a deterministic function f such that $A = f(B)$. Therefore, we can write $\theta_n^R = f_n^R(\tilde{a}, y_1, \ldots, y_{n-1})$, $\theta_n^I = f_n^I(\tilde{v}, y_1, \ldots, y_{n-1})$, and $p_n = f_n^p(y_1, \ldots, y_n)$ for three deterministic functions f_n^R, f_n^I, and f_n^p. The functions f_n^R, f_n^I, and f_n^p are fixed whereas the realization of the aggregate order flow variables $y_1, ..., y_n$ vary with the controls θ^I and θ^R.

First, the forms of the hedge fund's and rebalancer's optimal trading strategies are[11]

$$\Delta\theta_n^R = \beta_n^R\Big(\tilde{a} - \theta_{n-1}^R\Big) + \alpha_n^R q_{n-1}, \quad \theta_0^R := 0, \tag{1.5}$$

$$\Delta\theta_n^I = \beta_n^I\Big(\tilde{v} - p_{n-1}\Big), \quad \theta_0^I := 0, \tag{1.6}$$

where $(\beta_n^R, \beta_n^I, \alpha_n^R)_{n=1}^N$ are constants with $\beta_N^R = 1$ and $\alpha_N^R = 0$. The rebalancer and hedge fund are not restricted to use linear strategies, but they optimally choose such strategies in the equilibrium we construct.

Second, the q_n process in (1.5) is a structural consequence of the rebalancing constraint in our equilibrium. It denotes the market makers' expectation $\mathbb{E}[\tilde{a} - \theta_n^R|\sigma(y_1, ..., y_n)]$, given the history of aggregate order flows up through time n, of how much the rebalancer still needs to trade to reach his target. In other words, much like p_n gives the market-maker beliefs about the stock valuation, q_n gives the market-maker beliefs about the rebalancer's latent trading demand $\tilde{a} - \theta_n^R$ for the rest of the day. In our linear equilibrium, q_n has dynamics:

$$\Delta q_n = r_n y_n + s_n q_{n-1}, \quad q_0 := 0, \tag{1.7}$$

for constants $(r_n, s_n)_{n=1}^N$. The presence of q_n in (1.5) means that the rebalancer's orders are not limited to be a deterministic function of his target $\tilde{a}$. Rather, they can also depend on the prior order flow history, which is in contrast to the deterministic rebalancer orders in DJK (2014). It also follows from (1.5) that the market makers' expectation of the rebalancer's order at time n is

$$\mathbb{E}[\Delta\theta_n^R|\sigma(y_1, ..., y_{n-1})] = (\alpha_n^R + \beta_n^R)q_{n-1}. \tag{1.8}$$

Consequently, the aggregate order flow is autocorrelated in this market[12]

$$\begin{aligned} \mathbb{E}[y_n|\sigma(y_1, ..., y_{n-1})] &= \mathbb{E}[\Delta\theta_n^I + \Delta\theta_n^R + \Delta w_n|\sigma(y_1, ..., y_{n-1})] \\ &= (\alpha_n^R + \beta_n^R)\, q_{n-1}. \end{aligned} \tag{1.9}$$

[11] If an additional term $\alpha_n^I q_{n-1}$ is included in the hedge fund's strategy in (1.6), we find that α_n^I is zero in equilibrium. Contact the authors for a proof of this result.

[12] The second equality in (1.9) follows from (i) the independence between $\tilde{v} - p_{n-1}$ and past order flows, (ii) the assumption that the noise trader orders are zero–mean and i.i.d. over time, and (iii) the expression for expected rebalancer orders in (1.8).

Third, the pricing rule in our linear equilibrium has dynamics:

$$\begin{aligned}\Delta p_n &= \lambda_n \big(y_n - \mathbb{E}[y_n|\sigma(y_1, ..., y_{n-1})]\big) \\ &= \lambda_n \big(y_n - (\alpha_n^R + \beta_n^R) q_{n-1}\big),\end{aligned} \tag{1.10}$$

for $n = 1, ..., N$ where $(\lambda_n)_{n=1}^N$ are constants.[13] In a linear equilibrium of this form, the price at time n is not affected by orders at date n that are predictable given past orders. Thus, the $(\alpha_n^R + \beta_n^R)q_{n-1}$ term in (1.10) represents sunshine trading.

Optimal trading for portfolio rebalancing reflects a number of considerations: First, the rebalancer needs to reach his trading target $\tilde{a}$ at time N. Second, he wants to reach this target at the lowest cost possible. Cost minimization occurs through several channels:

- The rebalancer splits up his orders to spread their price impact over time taking into account intraday patterns of the price impact coefficients λ_n.

- The rebalancer takes advantage of sunshine trading. Early orders signal predictable future orders at later dates, which, from (1.10), have no price impact.

- The rebalancer trades on information about the asset value $\tilde{v}$ to reduce his costs and even, sometimes, to earn a trading profit. If $\rho > 0$, the rebalancer starts out with stock valuation information. However, even if the rebalancer is initially uninformed about $\tilde{v}$ (i.e., $\rho = 0$), he still learns information endogenously over time through the trading process (see 1.12 below).

- The rebalancer understands that price pressure from his trades creates incentives for the hedge fund to trade, which can be beneficial for the rebalancer. If early uninformed rebalancer orders raise prices, then, in expectation, the hedge fund should buy less/sell more in the future, thereby putting downward pressure on later prices which, in turn, reduces the expected cost of subsequent rebalancer buying.

Despite the complexity of the multiple drivers of rebalancing trading, we show that the rebalancer's orders take the simple linear form in (1.5). To gain intuition,

[13]The first equality (1.10) follows because conditional expectations are linear projections given the jointly Gaussian structure of the linear equilibrium. The second equality follows from (1.9).

we rearrange the rebalancer's order at time n from (1.5) as follows:

$$\Delta\theta_n^R = (\alpha_n^R + \beta_n^R)q_{n-1} + \beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1}). \tag{1.11}$$

The first component, $(\alpha_n^R + \beta_n^R)q_{n-1}$, as noted in (1.8), is the market makers' expectation of the rebalancer's order at time n. From the sunshine trading result in (1.10), this amount is traded at time n with no price impact. The second component, $\beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1})$, in (1.11) reflects two effects: First, the difference $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ is the additional amount the rebalancer still needs to trade beyond the market makers' expectation of his remaining latent trading demand. Second, it summarizes the private information the rebalancer has about stock price misvaluation in the market:[14]

$$\begin{aligned}\mathbb{E}[\tilde{v} - p_{n-1}|\sigma(\tilde{a}, y_1, ..., y_{n-1})] &= \mathbb{E}[\tilde{v} - p_{n-1}|\sigma(\tilde{a} - \theta_{n-1}^R - q_{n-1}, y_1, ..., y_{n-1})] \\ &= \mathbb{E}[\tilde{v} - p_{n-1}|\sigma(\tilde{a} - \theta_{n-1}^R - q_{n-1})].\end{aligned} \tag{1.12}$$

Thus, $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ is, in general, informative about $\tilde{v}$ beyond the information already reflected in p_{n-1}. In particular, it is informative about market pricing errors $\tilde{v} - p_n$ at times $n \geq 2$, even if $\rho = 0$ (i.e., $\tilde{a}$ and $\tilde{v}$ are ex ante independent), because knowledge about his own past orders lets the rebalancer filter the prior order flow history to learn about $\tilde{v}$ better than the market makers. This dynamic learning is absent from deterministic rebalancing as in DJK (2014). Intuitively, we expect β_n^R to be positive because trading in the direction of $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ moves the rebalancer's holdings towards his target and also because it exploits his private valuation information. In addition, we expect $0 \leq \alpha_n^R + \beta_n^R \leq \beta_n^R$ (i.e., $-\beta_n^R \leq \alpha_n^R \leq 0$) because trading in the direction of q_n also moves the rebalancer's stock holdings in the direction of his target, but without the additional speculative benefit.

Turning to the informed investor, the term $\tilde{v} - p_{n-1}$ in (1.6) plays two roles in the hedge fund's strategy: It is private information about the stock value and also, in equilibrium, information about the remaining latent trading demand $\tilde{a} - \theta_{n-1}^R$ for

[14] The first equality in (1.12) follows from $q_{n-1}, \theta_{n-1}^R \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$. The second follows from independence between $\tilde{v} - p_{n-1}$ and $y_1, ..., y_{n-1}$ and between $\tilde{a} - \theta_{n-1}^R - q_{n-1}$ and $y_1, ..., y_{n-1}$.

the rebalancer:[15]

$$
\begin{aligned}
&\mathbb{E}[\tilde{a} - \theta^R_{n-1} | \sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&\quad = q_{n-1} + \mathbb{E}[\tilde{a} - \theta^R_{n-1} - q_{n-1} | \sigma(\tilde{v} - p_{n-1}, y_1, ..., y_{n-1})] \\
&\quad = q_{n-1} + \mathbb{E}[\tilde{a} - \theta^R_{n-1} - q_{n-1} | \sigma(\tilde{v} - p_{n-1})].
\end{aligned} \tag{1.13}
$$

2 Equilibrium

In this section we give sufficient conditions for a linear Bayesian Nash equilibrium as in (1.5) through (1.7). Our analysis uses the logic of Foster and Viswanathan (1996), which we extend to allow for a trading constraint. Their approach solves the "forecasting the forecasts of others" problem when showing deviations from equilibrium strategies are suboptimal. Appendix A presents the analysis in greater detail.

To begin, consider a set of possible candidate values for the equilibrium constants

$$\lambda_n, \mu_n, r_n, s_n, \beta^R_n, \alpha^R_n, \beta^I_n, \quad n = 1, \dots, N, \tag{2.1}$$

with

$$\beta^R_N = 1, \quad \alpha^R_N = 0. \tag{2.2}$$

The restrictions in (2.2) at date N follow because the rebalancer must achieve his target $\tilde{a}$ after his last round of trade. Given a set of candidate constants (2.1)-(2.2), we define a system of "hat" price and order flow processes

$$\Delta\hat{\theta}^I_n := \beta^I_n(\tilde{v} - \hat{p}_{n-1}) \quad \hat{\theta}^I_0 := 0, \tag{2.3}$$

$$\Delta\hat{\theta}^R_n := \beta^R_n(\tilde{a} - \hat{\theta}^R_{n-1}) + \alpha^R_n \hat{q}_{n-1}, \quad \hat{\theta}^R_0 := 0, \tag{2.4}$$

$$\hat{y}_n := \Delta\hat{\theta}^I_n + \Delta\hat{\theta}^R_n + \Delta w_n, \quad \hat{y}_0 := 0, \tag{2.5}$$

$$\Delta\hat{p}_n := \lambda_n \hat{y}_n + \mu_n \hat{q}_{n-1}, \quad \hat{p}_0 := 0, \tag{2.6}$$

$$\Delta\hat{q}_n := r_n \hat{y}_n + s_n \hat{q}_{n-1}, \quad \hat{q}_0 := 0, \tag{2.7}$$

which denote the processes that agents conjecture that other agents believe describe the equilibrium. In equilibrium, conjectured beliefs must, of course, be correct in that

[15] The first equality in (1.13) follows from $q_{n-1}, p_{n-1} \in \sigma(y_1, ..., y_{n-1})$. The second equality follows from independence between $\tilde{v} - p_{n-1}$ and $y_1, ..., y_{n-1}$ and between $\tilde{a} - \theta^R_{n-1} - q_{n-1}$ and $y_1, ..., y_{n-1}$.

$p_n = \hat{p}_n$ (the price process is the conjectured price process), $\theta_n^R = \hat{\theta}_n^R$ (the rebalancer's orders follow the conjectured strategy), etc. The conjectured processes (2.3)-(2.7) make problems (1.3) and (1.4) analytically tractable in that both the hedge fund's problem and the rebalancer's problem can be described by low dimensional state processes (see 2.22 and 2.29 below).

The conjectured system $(\Delta\hat{\theta}_n^I, \Delta\hat{\theta}_n^R, \hat{y}_n, \Delta\hat{p}_n, \Delta\hat{q}_n)$ is fully specified (autonomous) by the coefficients (2.1). Given the zero-mean and joint normality of $\tilde{v}$, $\tilde{a}$, and w, the conjectured system (2.3)-(2.7) is zero-mean and jointly normal. The variances and covariance for the conjectured dynamics are denoted[16]

$$\Sigma_n^{(1)} := \mathbb{V}\big[\tilde{a} - \hat{\theta}_n^R - \hat{q}_n\big], \tag{2.8}$$

$$\Sigma_n^{(2)} := \mathbb{V}\big[\tilde{v} - \hat{p}_n\big], \tag{2.9}$$

$$\Sigma_n^{(3)} := \mathbb{E}\big[\big(\tilde{a} - \hat{\theta}_n^R - \hat{q}_n\big)(\tilde{v} - \hat{p}_n)\big]. \tag{2.10}$$

These moments are "post-trade" at time n in that they reflect trading up-through and including the time n order flow y_n. In other words, they are inputs for trading decisions and pricing in round $n+1$. The initial variances and covariance at $n = 0$ are exogenously given by

$$\Sigma_0^{(1)} = \sigma_{\tilde{a}}^2, \quad \Sigma_0^{(2)} = \sigma_{\tilde{v}}^2, \quad \Sigma_0^{(3)} = \rho\sigma_{\tilde{a}}\sigma_{\tilde{v}}. \tag{2.11}$$

In equilibrium, the constants (2.1) must satisfy certain consistency restrictions which we explain in two steps:

Step 1: The first set of restrictions on the coefficients $(\lambda_n, \mu_n, s_n, r_n)_{n=1}^N$ is that in equilibrium $\hat{p}_n$ and $\hat{q}_n$ must be consistent with Bayesian updating. In particular, for the conjectured prices $\hat{p}_n$ to be conditional expectations $\mathbb{E}[\tilde{v}|\sigma(\hat{y}_1, \dots, \hat{y}_n)]$ for the conjectured system, the same logic as for the equilibrium prices p_n in (1.10), implies

$$\begin{aligned}\Delta\hat{p}_n &= \lambda_n\big(\hat{y}_n - \mathbb{E}[\hat{y}_n|\sigma(\hat{y}_1, ..., \hat{y}_{n-1})]\big)\\ &= \lambda_n\big(\hat{y}_n - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1}\big),\end{aligned} \tag{2.12}$$

[16] We note that $\Sigma_n^{(2)}$ must be non-increasing over time (as in Kyle 1985) but $\Sigma_n^{(1)}$ might not be.

for $n = 1, ..., N$ where λ_n equals the projection coefficient

$$\frac{\mathrm{Cov}(\tilde{v} - \hat{p}_{n-1}, \hat{y}_n - \mathbb{E}[\hat{y}_n|\sigma(\hat{y}_1, ..., \hat{y}_{n-1})])}{\mathbb{V}(y_n - \mathbb{E}[\hat{y}_n|\sigma(\hat{y}_1, ..., \hat{y}_{n-1})])}. \tag{2.13}$$

This imposes restrictions on the coefficients of the price process in terms of the hedge fund and rebalancer strategy coefficients. A similar logic gives restrictions for the $\hat{q}_n$ process to equal the conditional expectation $\mathbb{E}[\tilde{a} - \hat{\theta}_n^R|\sigma(\hat{y}_1, \ldots, \hat{y}_n)]$. These calculations lead to four restrictions on the state variable and strategy constants in a linear Bayesian Nash equilibrium for $n = 1, ..., N$ (see the proof of Lemma A.1 in Appendix A.1):

$$\lambda_n = \frac{\beta_n^I \Sigma_{n-1}^{(2)} + \beta_n^R \Sigma_{n-1}^{(3)}}{(\beta_n^I)^2 \Sigma_{n-1}^{(2)} + (\beta_n^R)^2 \Sigma_{n-1}^{(1)} + 2\beta_n^I \beta_n^R \Sigma_{n-1}^{(3)} + \sigma_w^2 \Delta}, \tag{2.14}$$

$$r_n = \frac{(1 - \beta_n^R)\big(\beta_n^I \Sigma_{n-1}^{(3)} + \beta_n^R \Sigma_{n-1}^{(1)}\big)}{(\beta_n^I)^2 \Sigma_{n-1}^{(2)} + (\beta_n^R)^2 \Sigma_{n-1}^{(1)} + 2\beta_n^I \beta_n^R \Sigma_{n-1}^{(3)} + \sigma_w^2 \Delta}, \tag{2.15}$$

$$\mu_n = -\lambda_n(\alpha_n^R + \beta_n^R), \tag{2.16}$$

$$s_n = -(1 + r_n)(\alpha_n^R + \beta_n^R). \tag{2.17}$$

The conditional variances and covariance in (2.8)-(2.10) are computed recursively as

$$\Sigma_n^{(1)} = (1 - \beta_n^R)\big((1 - \beta_n^R - r_n \beta_n^R)\Sigma_{n-1}^{(1)} - r_n \beta_n^I \Sigma_{n-1}^{(3)}\big), \tag{2.18}$$

$$\Sigma_n^{(2)} = (1 - \lambda_n \beta_n^I)\Sigma_{n-1}^{(2)} - \lambda_n \beta_n^R \Sigma_{n-1}^{(3)}, \tag{2.19}$$

$$\Sigma_n^{(3)} = (1 - \beta_n^R)\big((1 - \lambda_n \beta_n^I)\Sigma_{n-1}^{(3)} - \lambda_n \beta_n^R \Sigma_{n-1}^{(1)}\big). \tag{2.20}$$

Note the "block" structure here: The values of the updating coefficients λ_n and r_n just depend on the strategy coefficients β_n^R and β_n^I at date n and the incoming variances and covariance from time $n-1$ (along with the exogenous noise trading variance σ_w^2). The post-trade variances and covariance $\Sigma_n^{(1)}$, $\Sigma_n^{(2)}$, and $\Sigma_n^{(3)}$ at time n just depend on the updating coefficients λ_n and r_n at time n, the strategy coefficients at time n, and the prior variances and covariance from time $n-1$. Lastly, μ_n and s_n depend on λ_n and μ_n and the rebalancer's strategy coefficients (β_n^R, α_n^R).

Step 2: The second set of restrictions on the price and order flow coefficients is that $(\lambda_n, \mu_n, s_n, r_n)_{n=1}^N$ must be consistent with optimal trading strategies for the rebalancer and the hedge fund.

Consider first the hedge fund at a generic time n. For a conjectured strategy $\hat{\theta}^I$ to be the hedge fund's equilibrium strategy, deviations from $\hat{\theta}^I$ cannot be profitable. Proving this requires allowing for the effects of possible past suboptimal play. As in Foster and Viswanathan (1996), the hedge fund not only knows the final stock value $\tilde{v}$, but also the extent to which the actual prices, quantity expectations, and rebalancer's positions (i.e., p_n, q_n, and θ_n^R in 1.10, 1.7, and 1.5 given its actual orders $\Delta\theta_1^I, \ldots, \Delta\theta_n^I$) deviate from their conjectured values (i.e., $\hat{p}_n$, $\hat{q}_n$, and $\hat{\theta}_n^R$ from 2.6, 2.7, and 2.4 given the conjectured orders $\Delta\hat{\theta}_1^I, \ldots, \Delta\hat{\theta}_n^I$ in 2.3). Hence, the "un-hatted" processes depend on actual orders whereas the conjectured "hat" processes only depend on conjectured orders. When the rebalancer's strategy is fixed by (1.5), it is characterized by the two sequences of coefficients $\beta_1^R, \ldots, \beta_N^R$ and $\alpha_1^R, \ldots, \alpha_N^R$. However, even though the rebalancer's strategy is fixed, its realizations are subject to the hedge fund's choice of θ^I because the aggregate order flow affects the rebalancer's orders. Similar statements apply to the prices p_n and the latent trading demand expectations q_n.

A natural set of possible state variables to consider for the hedge fund's problem in (1.3) are:

$$\tilde{v} - \hat{p}_n, \quad \hat{q}_n, \quad \hat{\theta}_n^I - \theta_n^I, \quad \hat{\theta}_n^R - \theta_n^R, \quad \hat{q}_n - q_n, \quad \hat{p}_n - p_n. \tag{2.21}$$

The first two quantities in (2.21) describe market pricing errors (given the hedge fund's private valuation information) and the predicted future latent rebalancer trading demand (given market information) in the conjectured equilibrium. The next four quantities describe the hedge fund's private information about its own past order submissions and the extent to which they induced deviations in the rebalancer's holdings, the market expectations about future latent trading demand, and prices in the conjectured equilibrium. However, the state space for the hedge fund can be simplified, because some of these state variables only matter in combination for the hedge fund's optimization problem. Appendix A shows that the following two composite state variables are sufficient for the hedge fund's value function:

$$X_n^{(1)} := \tilde{v} - p_n, \quad X_n^{(2)} := (\hat{\theta}_n^R - \theta_n^R) + (\hat{q}_n - q_n) + \tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\big(\tilde{v} - \hat{p}_n\big), \quad n = 0, ..., N. \tag{2.22}$$

From a technical point of view, this is a substantial reduction from the six state variables in (2.21). Two seems likely to be the minimum number of state variables necessary for the hedge fund's problem. Lemma A.2 in Appendix A ensures that the $X_n^{(1)}$ and $X_n^{(2)}$ processes are observable for the hedge fund. In equilibrium, with $p_n = \hat{p}_n$, $q_n = \hat{q}_n$, and $\theta_n^R = \hat{\theta}_n^R$, it follows from (2.22) that

$$X_n^{(2)} = \tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}} X_n^{(1)}, \quad n = 0, 1..., N. \tag{2.23}$$

Thus, on the equilibrium path, the hedge fund's state space reduces to just $\tilde{v} - p_n$, which is consistent with the form of its equilibrium order in (1.6).

Lemma A.2 in Appendix A shows that the hedge fund's value function for $n = 0, 1, ..., N$ has the quadratic form

$$\begin{aligned} &\max_{\substack{\Delta\theta_k^I \in \sigma(\tilde{v}, y_1, ..., y_{k-1}) \\ n+1 \le k \le N}} \mathbb{E}\Big[\sum_{k=n+1}^{N} (\tilde{v} - p_k)\Delta\theta_k^I \Big| \sigma(\tilde{v}, y_1, ..., y_n)\Big] \\ &\qquad = I_n^{(0)} + I_n^{(1,1)}(X_n^{(1)})^2 + I_n^{(1,2)} X_n^{(1)} X_n^{(2)} + I_n^{(2,2)}(X_n^{(2)})^2, \end{aligned} \tag{2.24}$$

where $I_n^{(0)}, I_n^{(1,1)}, I_n^{(1,2)}$, and $I_n^{(2,2)}$ are constants. Lemma A.2 also shows that the hedge fund's problem (2.24) is quadratic in its orders $\Delta\theta_n^I$. The first-order-condition for (2.24) gives the hedge fund's optimal orders:

$$\Delta\theta_n^I = \gamma_n^{(1)} X_{n-1}^{(1)} + \gamma_n^{(2)} X_{n-1}^{(2)}, \quad n = 1, ..., N, \tag{2.25}$$

where $\gamma_n^{(1)}$ and $\gamma_n^{(2)}$ are functions of the hedge fund value function coefficients, and the parameters of the conjectured price, latent trading demand and rebalancer strategy processes given in (A.10) and (A.11) in Appendix A. The associated second-order condition for the hedge fund's optimal strategy is

$$I_n^{(2,2)} r_n^2 + I_n^{(1,2)} r_n \lambda_n + I_n^{(1,1)} \lambda_n^2 < \lambda_n, \quad n = 1..., N. \tag{2.26}$$

By inserting the hedge fund's candidate strategy (2.25) and (A.10)-(A.11) into the expectation in (2.24), we can determine the hedge fund's value function coefficients recursively. Appendix A computes the expectation in equation (A.8), and the resulting recursions are in equations (A.22)-(A.24).

By equating the coefficients in (2.25) with (1.6) and using the equilibrium condition (2.23), we get the following restriction on the hedge fund's strategy coefficient

$$\beta_n^I = \gamma_n^{(1)} + \gamma_n^{(2)} \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}, \quad n = 1..., N. \tag{2.27}$$

For fixed $\Sigma_n^{(1)}, \Sigma_n^{(2)}$, and $\Sigma_n^{(3)}$, we can use the linear equations (2.18)-(2.20) to express $\Sigma_{n-1}^{(1)}, \Sigma_{n-1}^{(2)}$, and $\Sigma_{n-1}^{(3)}$ in terms of $r_n, \lambda_n, \beta_n^I, \beta_n^R$. Equations (A.10)-(A.11) and (2.14)-(2.15) can then be used to see that (2.27) is a fifth–degree polynomial in (β_n^R, β_n^I) whenever $\Sigma_n^{(i)}$, $i = 1, 2, 3$, and $I_n^{(i,j)}$, $i = 1, 2$ and $i \leq j \leq 2$, are fixed.

We next turn to the rebalancer's problem. Again, it would be natural to consider six possible state variables for the rebalancer's problem in (1.4) :

$$\tilde{a} - \hat{\theta}_n^R, \quad \hat{q}_n, \quad \hat{\theta}_n^R - \theta_n^R, \quad \hat{\theta}_n^I - \theta_n^I, \quad \hat{q}_n - q_n, \quad \hat{p}_n - p_n. \tag{2.28}$$

The first two quantities in (2.28) describe the rebalancer's latent trading demand (give his private information about his target and past orders) and the market maker prediction of his future latent trading demand (given public order flow information) in a conjectured equilibrium. The next four quantities describe the rebalancer's private information about its own past orders and how they caused the hedge fund's holdings, the market's latent trading demand predication, and prices to deviate from the conjectured equilibrium. However, the rebalancer's state space can also be simplified. Just three composite state variable are sufficient for the rebalancer's value function:

$$Y_n^{(1)} := \tilde{a} - \theta_n^R, \quad Y_n^{(2)} := (\hat{p}_n - p_n) + \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(\tilde{a} - \hat{\theta}_n^R - \hat{q}_n), \quad Y_n^{(3)} := q_n, \quad n = 0, 1, ..., N. \tag{2.29}$$

Lemma A.4 in Appendix A ensures that these processes are observable for the rebalancer. In equilibrium, with $p_n = \hat{p}_n$, $q_n = \hat{q}_n$, and $\theta_n^I = \hat{\theta}_n^I$, it follows from (2.29) that

$$Y_n^{(2)} = \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(Y_n^{(1)} - Y_n^{(3)}), \quad n = 1, ..., N. \tag{2.30}$$

Thus, on the equilibrium path, the state space for the rebalancer at time n reduces to just two state variables, $\tilde{a} - \theta_n^R$ and q_n, which is consistent with (1.5). When the hedge fund's strategy is fixed as in (1.6), Lemma A.4 in Appendix A shows that the rebalancer's value function is quadratic in the rebalancer state variables

$$\begin{aligned} \max_{\substack{\Delta\theta_k^R \in \sigma(\tilde{a}, y_1, ..., y_{k-1}) \\ n+1 \le k \le N-1}} & - \mathbb{E}\Big[\sum_{k=n+1}^{N} (\tilde{a} - \theta_{k-1}^R) \Delta p_k \Big| \sigma(\tilde{a}, y_1, ..., y_n) \Big] \\ &= L_n^{(0)} + \sum_{1 \le i \le j \le 3} L_n^{(i,j)} Y_n^{(i)} Y_n^{(j)}, \end{aligned} \tag{2.31}$$

where $L_n^{(0)}, ..., L_n^{(3,3)}$ are constants. Lemma A.4 also ensures that the rebalancer's problem (2.31) is quadratic in his orders $\Delta\theta_n^R$. The corresponding first-order-condition gives the rebalancer's optimal orders:

$$\Delta\theta_n^R = \delta_n^{(1)} Y_{n-1}^{(1)} + \delta_n^{(2)} Y_{n-1}^{(2)} + \delta_n^{(3)} Y_{n-1}^{(3)}, \quad n = 1, ..., N, \tag{2.32}$$

where $\delta_n^{(1)}$, $\delta_n^{(2)}$, and $\delta_n^{(3)}$ are functions of the rebalancer's value function coefficients, and the parameters of the conjectured price, latent trading demand and hedge fund's strategy processes given in (A.19)–(A.21) in Appendix A. The associated second-order condition for the rebalancer's optimal strategy is

$$L_n^{(1,1)} + L_n^{(3,3)} r_n^2 + L_n^{(1,2)} \lambda_n + L_n^{(2,2)} \lambda_n^2 < L_n^{(1,3)} r_n + L_n^{(2,3)} r_n \lambda_n, \quad n = 1, ..., N. \tag{2.33}$$

Similar to the hedge fund's problem, by inserting the rebalancer's candidate strategy (2.32) and (A.19)-(A.21) into the expectation in (2.31), we can find the rebalancer's value function coefficients recursively (see equations (A.25)-(A.31) in Appendix A.5).

By equating the coefficients in (2.32) with (1.5) and using the equilibrium condition (2.30), we get two restrictions:

$$\beta_n^R = \delta_n^{(1)} + \delta_n^{(2)} \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}, \quad \alpha_n^R = \delta_n^{(3)} - \delta_n^{(2)} \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}, \quad n = 1, ..., N. \tag{2.34}$$

Similarly to (2.27), the first equation in (2.34) is a fifth–degree polynomial in (β_n^R, β_n^I) whenever $\Sigma_n^{(i)}$, $i = 1, 2, 3$, and $L_n^{(i,j)}$, $i = 1, 2, 3$ and $i \le j \le 3$, are fixed. The second equation in (2.34) is a linear equation in α_n^R once all of the other parameters are determined.

Our main theoretical result is the following:

Theorem 2.1. *Consider constants* (2.1) *satisfying* (2.2). *Given the initial variances and covariance in* (2.11), *these constants describe a linear Bayesian Nash equilibrium* *in the form* (1.5)-(1.7)*, and* (1.10) *if, for all n, the following restrictions hold: (i) The pricing and latent trading prediction coefficient relations in* (2.14)-(2.17) *hold where* $\Sigma^{(1)}_{n-1}, \Sigma^{(2)}_{n-1}$*, and* $\Sigma^{(3)}_{n-1}$ *are computed recursively using* (2.18)-(2.20)*. (ii) The equilibrium conditions* (2.27) *and* (2.34) *are satisfied with the second-order-conditions* (2.26) *and* (2.33) *holding where the value function coefficients are computed as follows: First define the constants*

$$I_N^{(1,1)} := ... := I_N^{(2,2)} := L_N^{(1,1)} := ... := L_N^{(3,3)} := 0,$$

and then define the value function coefficients

$$(I_n^{(i,j)})_{1\leq i\leq j\leq 2}, (L_n^{(i,j)})_{1\leq i\leq j\leq 3}, \quad n = 1, ..., N-1, \tag{2.35}$$

via the recursions (A.22)-(A.24) *and* (A.25)-(A.31)*. In this case we have*

$$r_N = 0, \quad \mu_N = -\lambda_N, \quad s_N = -1, \quad \beta^I_N = \big(\frac{1}{2\lambda_N} - \frac{\Sigma^{(3)}_{N-1}}{2\Sigma^{(2)}_{N-1}}\big), \quad \lambda_N > 0. \tag{2.36}$$

Theorem 2.1 establishes sufficient conditions for a linear equilibrium. It extends Proposition 1 in Foster and Viswanathan (1996) to allow for an investor with a trading constraint. As in most discrete-time Kyle models, including Foster and Viswanathan (1996), we do not have analytic expressions for the equilibrium. Actual equilibria must be computed numerically. Appendix A.6 describes our numerical algorithm.

3 Numerical results

Our analysis in this section investigates two quantitative questions: What do dynamic rebalancing trading strategies look like in our market? And what are the equilibrium effects of the rebalancing constraint? To answer these questions, we conduct a variety of numerical experiments. Our baseline specification has $N = 10$ rounds of trading, the variance of the terminal stock value $\tilde{v}$ is normalized to $\sigma^2_{\tilde{v}} = 1$, the total variance of the Brownian motion noise trading order flow over the N periods is fixed at $\sigma^2_w = 4$, the variance of the trading target $\tilde{a}$ is $\sigma^2_{\tilde{a}} = 1$, and the correlation between the trading target $\tilde{a}$ and the terminal stock value $\tilde{v}$ is $\rho = 0$ (i.e., $\tilde{v}$ and $\tilde{a}$ are ex ante independent). Given the prevalence of order-splitting in real-world markets, we investigate dynamic rebalancing in markets in which rebalancing is large: A one standard deviation rebalancer target in our baseline calibration is one half of the standard deviation of the cumulative daily noise trader order imbalance. In our analysis, we vary the rebalancing target variance $\sigma_{\tilde{a}}^2$ and the informativeness of the target ρ.

We assess the impact of strategic rebalancing by comparing our model with two alternative models. For $\rho = 0$, we compare our equilibrium with Kyle (1985). For $\rho > 0$, we compare our model with a variant of the Foster and Viswanathan (1994) model which we call the *modified FV model*. In the modified FV model, one investor has superior information in that she knows the terminal security value $\tilde{v}$, while a less-informed investor receives a noisy signal $\tilde{a}$ which has a correlation $\rho > 0$ with $\tilde{v}$.[17] The variable $\tilde{a}$ in the modified FV model is a noisy signal about the stock's terminal value $\tilde{v}$ with the same distribution as the target $\tilde{a}$ in our rebalancing model. In other words, $\tilde{a}$ in the modified FV model is just payoff information (and not a rebalancing constraint). The only difference between the modified FV model and the original Foster and Viswanathan (1994) model is that the better-informed investor in the modified FV does not know the less-informed investor's information (i.e., in Foster and Viswanathan 1994 the better-informed investor knows both $\tilde{v}$ and $\tilde{a}$). Hence, our strategic rebalancing model and the modified FV model have identical information structures. Comparing the equilibria in our model and the modified FV model identifies the effect of the rebalancing constraint when $\rho > 0$. The modified FV model is described in more detail in Appendix B and in the Internet Appendix.

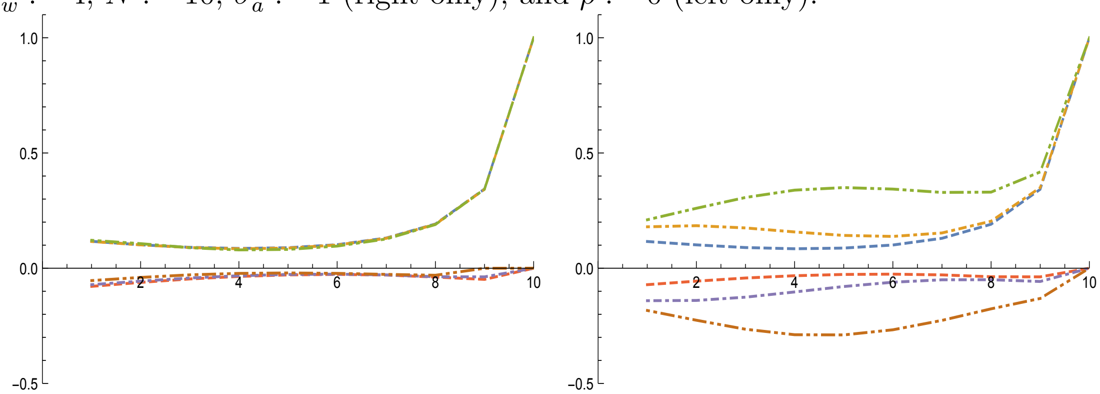
3.1 Dynamic rebalancing

The rebalancer's orders are described by the strategy coefficients β_n^R and α_n^R. Figure 1 shows trajectories for these strategy coefficients. We use the decomposition in (1.11) to interpret them. The fact that β_n^R is positive means that the rebalancer trades in the direction of his private information $\tilde{a} - \theta_{n-1}^R - q_{n-1}$. One reason for this is mechanical: The larger $\tilde{a}$ is relative to θ_{n-1}^R (given q_{n-1}), the more the rebalancer must trade to achieve his target compared to the market makers' expectation of his remaining trading demand. The second reason is informational: The smaller θ_{n-1}^R is relative to q_{n-1} (given $\tilde{a}$), the less the rebalancer has actually bought relative to the market

[17]The modified FV model reduces to the Kyle (1985) model when $\rho = 0$ since then the less-informed investor has no private information and, thus, in equilibrium does not trade.

makers' expectation, which, in turn, implies that, given the prior observed aggregate order flows, the hedge fund has bought more, in expectation, than the market makers' realize, which is informative about the pricing error $\tilde{v} - p_n$. Given this information, the rebalancer buys more/sells less stock at times $n < N$. Next, consider the sunshine trading component $(\alpha_n^R + \beta_n^R)q_{n-1}$ of his order. The sum $\alpha_n^R + \beta_n^R$ is positive but small for most of the day, which means that the rebalancer only trades a relatively small fraction of his expected latent trading gap q_{n-1} over time until close to time N.

Figure 1: Plots of the rebalancer's strategy coefficients $(\alpha_n^R)_{n=1}^N$ (below the x-axis) and $(\beta_n^R)_{n=1}^N$ (above the x-axis) for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).



A: $\sigma_{\tilde{a}}^2 = 0.48$ (−−), $\sigma_{\tilde{a}}^2 = 1$ (− · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · ·−).

B: $\rho = 0$ (−−), $\rho = 0.25$ (− · −), $\rho = 0.47$ (− · ·−).

Since market maker expectation q_n of the remaining latent rebalancing demand in (1.11) is an endogenous process, we further decompose the rebalancer's orders as linear functions of the underlying exogenous random variables — the rebalancing target $\tilde{a}$, the terminal stock payoff $\tilde{v}$, and noise trader orders Δw_j— in the market:

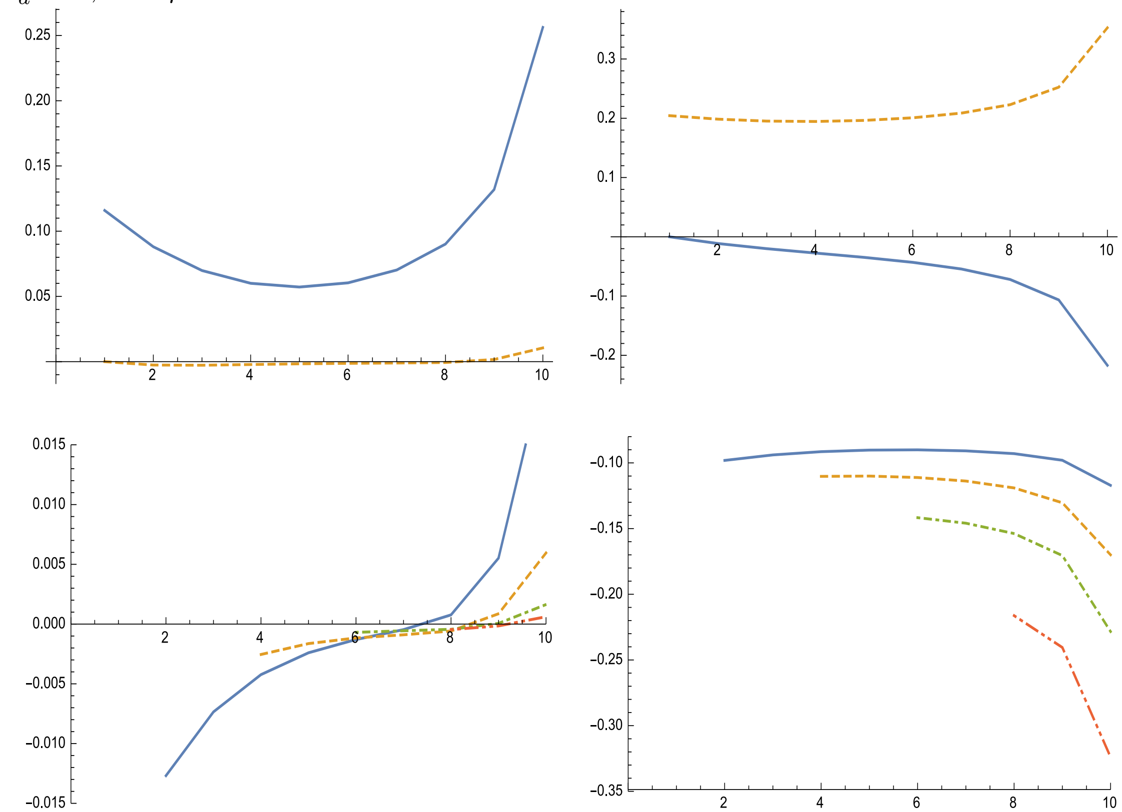
$$\Delta\theta_n^R = A_n^R \tilde{a} + B_n^R \tilde{v} + \sum_{j=1,\dots n-1} c_{n,j}^R \Delta w_j. \tag{3.1}$$

This decomposition follows from the joint linearity of prices, orders, and the q_n processes. The dependence on $\tilde{v}$ and the noise trader orders Δw_j comes through the q_n process and its dependence on lagged aggregate orders. The dependence on the target $\tilde{a}$ is both direct and indirect through the lagged θ_{n-1}^R and q_{n-1} terms in (1.11). This linear decomposition implies that part of the rebalancer's orders depends deterministically on the target $\tilde{a}$, but that — because of endogenous learning and sunshine trading — a portion is random, after controlling for $\tilde{a}$, due to the impact of $\tilde{v}$ and the noise trader orders on the aggregate orders and, thus, on the q_n process.

Figure 2A shows the linear decomposition coefficients for the rebalancer orders over time for our baseline parametrization. One factor affecting these intertemporal patterns is the terminal rebalancing constraint ($\theta_N^R = \tilde{a}$), which, by construction, requires $\sum_{n=1,...N} A_n^R = 1$, $\sum_{n=1,...N} B_n^R = 0$, and $\sum_{n=j+1,...N} c_{n,j}^R = 0$ for $j = 1, \ldots, N-1$. Quantitatively, the target $\tilde{a}$ is clearly the dominant driver of the rebalancer's orders. Perhaps surprisingly, the rebalancer decomposition coefficient on $\tilde{v}$ is initially negative at time 2. The intuition can be explained using (1.11). Holding everything else fixed, larger values of $\tilde{v}$ (which are not directly observed by the rebalancer) imply larger informed trader orders at time 1, which increases the aggregate order flow y_1 (which is observable but which the rebalancer cannot distinguish from increased buying by the noise traders). This increases q_1 (i.e., increased buying at time 1 makes market makers predict more buying by the rebalancer over the rest of the day). This makes the $\tilde{a} - \theta_1^R - q_1$ term smaller in (1.11). In words, when $\tilde{v}$ is bigger, the rebalancer knows that higher future rebalancer buying predicted by market makers in this case is, on average, inflated and, thus, that prices will tend to fall when subsequent aggregate order flows are less than expected. This leads the rebalancer to buy less at time 2, as indicated by the positive coefficient β_2^R (see Figure 1) and the minus sign in front of q_{n-1} in the second (informational) term in (1.11). A complication here is that higher values of $\tilde{v}$ have a positive impact on the sunshine trading term $(\alpha_2^R + \beta_2^R)q_1$ given that $\alpha_2^R + \beta_2^R > 0$. However, the net effect of $\tilde{v}$ via q_1 on the rebalancer's order can be negative, as shown by this example, given that $\alpha_2^R < 0$ (see Figure 1 again). Later in the day, the sign of the coefficient on $\tilde{v}$ switches since at some point the rebalancer must start unwinding speculative positions given the terminal rebalancing constraint.

As discussed in Section 1 the rebalancer's trading strategy coefficients α_n^R and β_n^R reflect the combined effects of a variety of economic considerations. Our goal in the rest of this section is to disentangle these various economic considerations and to assess their relative quantitative importance for trading by the rebalancer. In this discussion, we distinguish between the deterministic component of the rebalancer's orders that depends on the target $\tilde{a}$ and an adaptive component that depends on fluctuations in the realized aggregate order flow history over the trading day.

Figure 2: Plots of coefficients in the linear decompositions for the rebalancer orders in (3.1) and the informed trader orders in (3.4). The top figures show the coefficients for $\tilde{a}$ (—) and $\tilde{v}$ (- -), and the lower figures show the coefficients for Δw_1 (—), Δw_3 (- -), Δw_5 $(- \cdot -)$, and Δw_7 $(- \cdot \cdot -)$. The parameters are $N := 10$, $\sigma_w^2 := 4$, $\sigma_{\tilde{v}}^2 := 1$, $\sigma_{\tilde{a}}^2 = 1$, and $\rho := 0$.



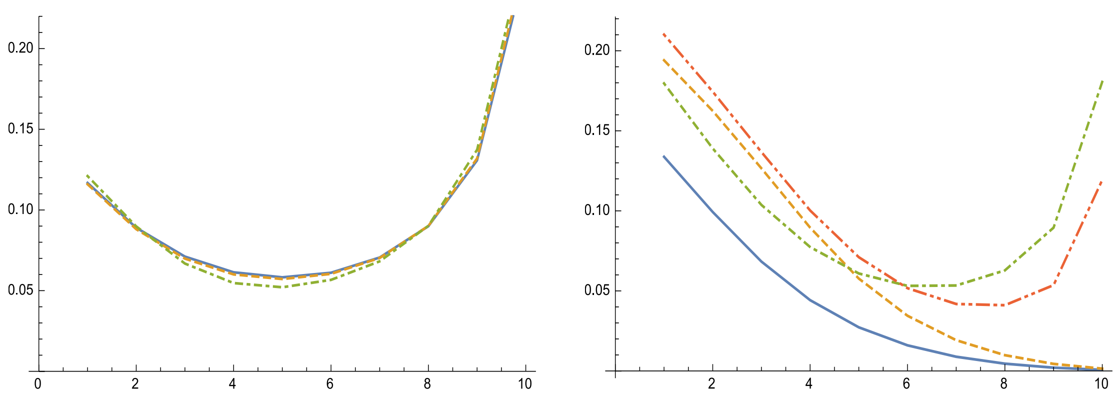
A: Coefficients in $\Delta\theta_n^R$'s decomposition. B: Coefficients in $\Delta\theta_n^I$'s decomposition.

Figure 3 shows the ratio of the rebalancer's ex ante expected orders relative to his target $\tilde{a} \neq 0$ over the day

$$\frac{\mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{a})]}{\tilde{a}} = A_n^R + B_n^R \frac{\mathbb{E}[\tilde{v}|\sigma(\tilde{a})]}{\tilde{a}} = A_n^R + B_n^R \rho \frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}. \tag{3.2}$$

In (3.2) the expectations are taken over the terminal stock price $\tilde{v}$ and the noise trader orders Δw. This ratio does not depend on the realized target $\tilde{a}$. The second term in (3.2) is zero if $\rho = 0$. Figure 3A shows that — ignoring adaptive responses in the rebalancer's orders to realtime fluctuations in the aggregate order flows over the day — the trading target $\tilde{a}$ induces a U–shaped pattern in rebalancing trading volume over the day when $\rho = 0$. DJK (2014) obtain a similar result in their model with short–lived information for the informed investor and deterministic trading for the rebalancer. Thus, our results show that the U-shaped pattern of rebalancing trading does not depend on short-lived information.[18] Figure 3B shows that the U-shape skews toward more trading earlier in the day when $\rho > 0$ because of the incentive to trade on the valuation information in $\tilde{a}$ before it is impounded in prices later in the day. Figure 3B also shows the intraday patterns of expected trades for the less-informed investor in the modified FV model for two ρ values (the solid blue and dashed yellow lines). Comparing the two models, the intraday pattern for our rebalancer is very different from the less-informed investor in the modified FV model. Because of the rebalancing constraint, the rebalancer orders are larger, and they have a big upturn at the end of the day.

Figure 3: Plots of the ratio $\mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{a})]/\tilde{a}$ of expected rebalancer orders relative to his target conditional on a target $\tilde{a} \neq 0$ for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).



A: $\sigma_{\tilde{a}}^2 = 0.48$ (——), $\sigma_{\tilde{a}}^2 = 1$ (−−), $\sigma_{\tilde{a}}^2 = 3.7$ (− · −).

B: $\rho = 0.25$ (—)[FV], $\rho = 0.47$ (−−)[FV], $\rho = 0.25$ (− · −), $\rho = 0.47$ (− · · −).

The deterministic component in the rebalancer's orders given the target $\tilde{a}$ includes predictable sunshine trading. Figure 4A shows the ratio of the rebalancer's expected

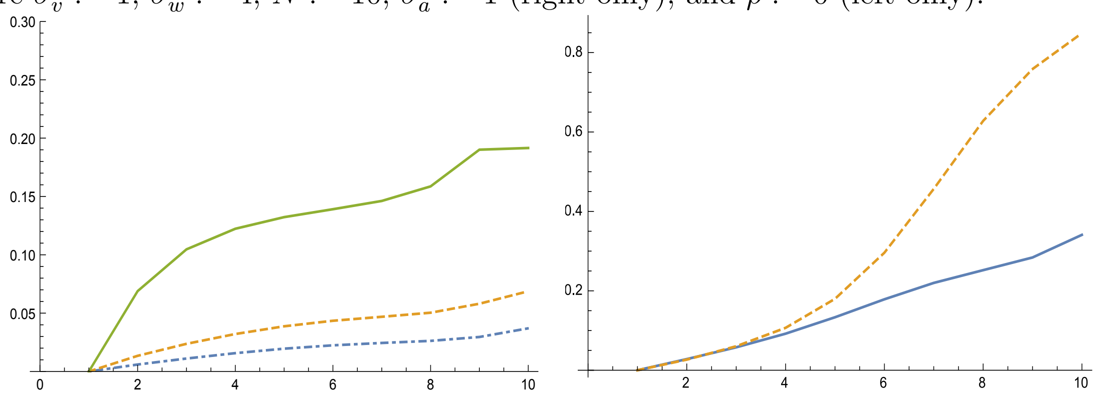
[18] The literature on optimal order execution includes many models that also produce U-shaped optimal strategies, see, e.g., Predoiu, Shaikhet, and Shreve (2011) and the references therein. However, sunshine trading in that literature stems from exogenously specified liquidity resilience and replenishment dynamics. In contrast, liquidity is endogenously determined in our equilibrium model.

sunshine trading orders relative to his expected total orders over time given $\tilde{a} \neq 0$

$$\frac{\mathbb{E}[\mathbb{E}[\Delta\theta_n^R|\sigma(y_1,\dots,y_{n-1})]|\sigma(\tilde{a})]}{\mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{a})]} = \frac{(\alpha_n^R+\beta_n^R)\mathbb{E}[q_{n-1}|\sigma(\tilde{a})]}{\mathbb{E}[\Delta\theta_n^R|\sigma(\tilde{a})]}. \tag{3.3}$$

A sunshine component is present but is not large (less than 5% for $\sigma_{\tilde{a}}^2 = 1$) in Figure 4A. Thus, quantitatively, only a small part of the rebalancer's orders benefit from the zero-price-impact of sunshine trading orders when $\rho = 0$.[19] Thus, in our baseline case, most of the deterministic part of the reblancer's orders appears to be due to intertemporal smoothing. However, sunshine trading is more important when $\rho > 0$, as shown in Figure 4B.

Figure 4: Plots of the ratio of the expected sunshine trading portion of the rebalancer's orders relative to his expected orders from (3.3) given a target $\tilde{a} \neq 0$. The parameters are $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).



A: $\sigma_{\tilde{a}}^2 = 0.48$ $(- \cdot -)$, $\sigma_{\tilde{a}}^2 = 1$ $(--)$, $\sigma_{\tilde{a}}^2 = 3.7$ (—).

B: $\rho = 0.25$ (—), $\rho = 0.47$ $(--)$.

The rebalancer orders also have a component that responds adaptively to realtime fluctuations in the aggregate order flow over the trading day. This randomness in the rebalancer's orders, after controlling for the target $\tilde{a}$, occurs because of the q_n term in (1.11), which leads, in turn, to the two terms involving the terminal payoff $\tilde{v}$ and the history of lagged noise orders Δw_j in (3.1). This randomness is due to speculative trading by the rebalancer (given his endogenous learning through trading over time) and as an adaptive sunshine trading response to fluctuations in the market maker's expectations q_n. Figures 5A-B show that the standard deviation of rebalancer orders,

[19] In the modified FV model, with no trading constraint, there is no sunshine trading at all.

after conditioning on the target $\tilde{a}$, also has a U-shaped intraday pattern. Figure 5C shows 10 simulated paths of the rebalancer's order flows over time. In this example, the realized stock value $\tilde{v}$ is 1, and the realized trading target $\tilde{a}$ is 0, but the noise trader order paths are random. Along these paths, the rebalancer buys/sells more than his trading target $\tilde{a}$ at early times ($n > 1$) and then unwinds his position later to achieve his trading target. This is not manipulation. Rather, the rebalancer's orders reflect a combination of informed trading motives (about $\tilde{v}$) and the trading constraint $\tilde{a}$. The rebalancer does not trade at time 1 because, given $\tilde{a} = 0$, he does not need to rebalance, and because, initially, he does not have any stock valuation information given $\rho = 0$. However, at time 2 the rebalancer trades based on whether — given the stock valuation information he gleans from filtering the order flow y_1 better than the market makers – he infers the stock is over– or under–valued. Eventually, however, he must unwind these earlier positions in order to achieve his realized trading target $\theta_N^R = \tilde{a} = 0$ at the end of the day.[20] The dispersion in the paths is consistent with the trajectory of the rebalancer order flow standard deviation. Paths for non-zero targets $\tilde{a}$ involve shifting the means of these paths from zero to the appropriate deterministic order component path given $\tilde{a}$.[21] This is illustrated in Figure 5D.

One further factor that reduces the rebalancer's trading costs is the fact that his orders tend to become negatively correlated with the hedge fund's orders over time. Figure 6A shows that, if $\rho = 0$, then the correlation between the hedge fund's orders and the rebalancer's orders is negative at times $n > 1$. This negative correlation is mutually beneficial for both the rebalancer and the hedge fund. By trading in opposite directions (in expectation), they symbiotically provide liquidity to each other with a reduced price impact. In contrast, Figure 6B shows that the correlation of the better-informed and less-informed investors orders is always non-negative in the modified FV model.[22]

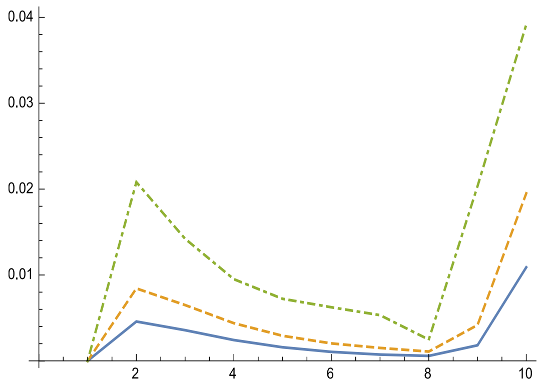
The correlation between the rebalancer's and the informed hedge fund's orders

[20]This is another example of a situation in which different traders acquire information at different times and/or have to unwind positions in advance of definitive public announcements. See also Foucault, Hombert, and Rosu (2016).

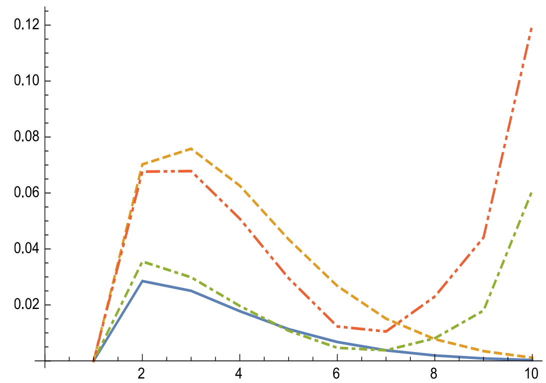
[21]When the realized target $\tilde{a}$ is large, the rebalancer's orders will all tend to be in the same direction over time (e.g., a large positive target $\tilde{a}$ will be associated with a series of buy orders). Randomness in his orders due to the q_n process (connected with sunshine trading and the rebalancer's endogenous learning about the stock's value) just causes the rebalancer to speed up or slow down his trading relative to his expected orders given his target.

[22]This is because the better and less-informed trader orders both load positively on the common information reflected in the less-informed signal in the modified FV model.

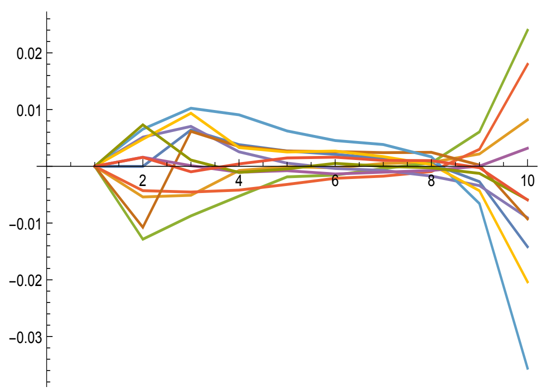
Figure 5: Plots A and B show the conditional standard deviation of the rebalancer's orders $\Delta\theta_n^R$ given a target $\tilde{a}$. Plots C and D show 10 sample paths of $\Delta\theta_n^R$ for two different target realizations. The parameters are $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, and $N := 10$, with $\sigma_{\tilde{a}}^2 := 1$ (B, C, and D only), and $\rho := 0$ (A, C, and D only).



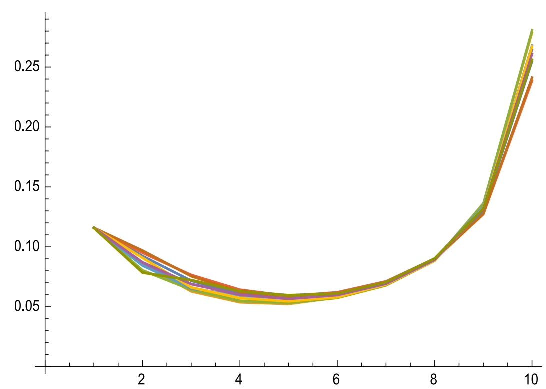
A: $\sigma_{\tilde{a}}^2 = 0.48$ (——), $\sigma_{\tilde{a}}^2 = 1$ (– –), $\sigma_{\tilde{a}}^2 = 3.7$ (– · –).



B: $\rho = 0.25$ (—)[FV], $\rho = 0.47$ (– –)[FV], $\rho = 0.25$ (– · –), $\rho = 0.47$ (– · · –).



C: 10 sample paths of $\Delta\theta_n^R$ for the realizations $\tilde{a} = 0$ and $\tilde{v} = 1$.



D: 10 sample paths of $\Delta\theta_n^R$ for the realizations $\tilde{a} = 1$ and $\tilde{v} = 1$.

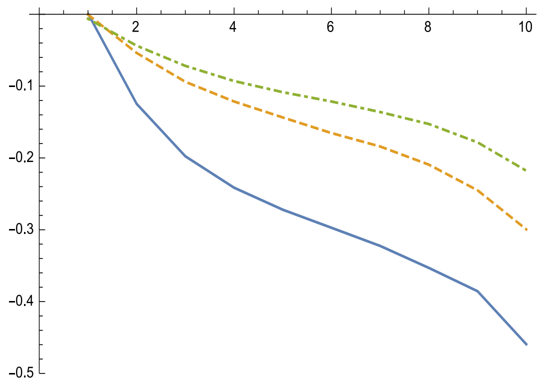
can be understood using the linear order decompositions. In particular, Figure 2B shows the linear decomposition coefficients for the informed hedge fund[23]

$$\Delta\theta_n^I = A_n^I \tilde{a} + B_n^I \tilde{v} + \sum_{j=1,\dots n-1} c_{n,j}^I \Delta w_j. \tag{3.4}$$

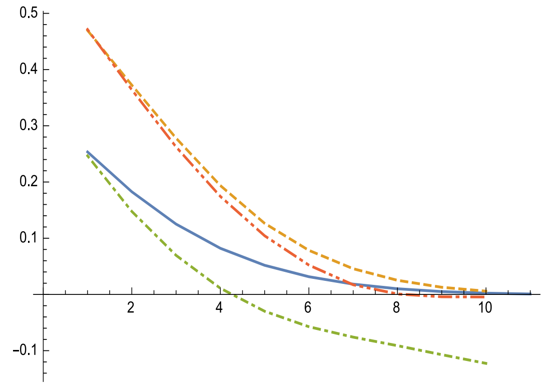
As expected, the hedge fund orders load positively on $\tilde{v}$ and negatively on $\tilde{a}$ and the

[23] We discuss the hedge fund's trading strategy in more detail later in this section.

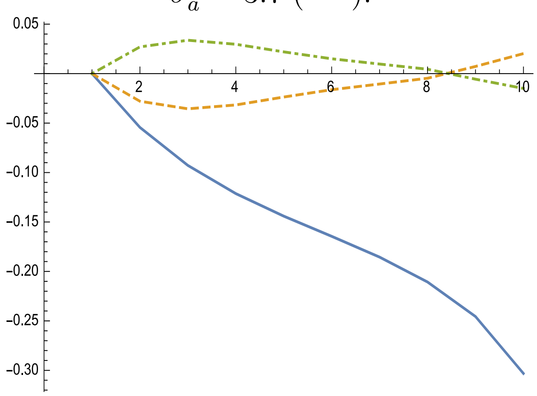
Figure 6: Plots of $\mathrm{corr}(\Delta\theta_n^I, \Delta\theta_n^R)$ for $n = 1, 2, ..., 10$ (unconditional) and correlation decomposition components. The parameters are $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (B and C only), and $\rho := 0$ (A and C only).



A: $\sigma_{\tilde{a}}^2 = 0.48$ $(- \cdot -)$, $\sigma_{\tilde{a}}^2 = 1$ $(--)$, $\sigma_{\tilde{a}}^2 = 3.7$ (——).



B: $\rho = 0.25$ (——) [FV], $\rho = 0.47$ $(--)$[FV], $\rho = 0.25$ $(- \cdot -)$, $\rho = 0.47$ $(- \cdot \cdot -)$.



C: Correlation components from $\tilde{a}$ (—), $\tilde{v}$ $(--)$, noise orders $(- \cdot -)$.

noise trader orders (since buying by the rebalancer and noise traders both inflate the stock price). Using the rebalancer and hedge fund linear order decompositions, we can decompose their order correlations into components due to the two investors' loadings on $\tilde{v}$, $\tilde{a}$, and the noise trader orders. Figure 6C shows that a large negative correlation component due to the target $\tilde{a}$ accounts for a large part of the negative correlation between the rebalancer's and the hedge fund's orders in Figure 6A in our baseline calibration. The rebalancer trades in the direction of his target $\tilde{a}$ while the hedge fund trades opposite this noise. This mechanism for negative order correlation is different from the Bayesian signal correlation mechanism in Foster and Vishwanathan (1996) when investors have different but equally informative noisy signals.

Next, we consider a somewhat different approach to shed light on the economic considerations that drive the rebalancer's orders. This approach takes the equilibrium pricing rule p_n and informed investor trading strategy θ_n^I as given, and then computes a number of constrained trading strategies for the rebalancer that ignore various combinations of the different economic considerations that determine the rebalancer's optimal equilibrium strategy. Specifically, we consider constrained trading strategies in which the rebalancer

1. Trades just once at time 1 to reach his full target $\tilde{a}$.

2. Trades just once to reach his full target $\tilde{a}$ but optimizes his choice of the time in which he trades so as to minimize his expected trading cost.

3. Trades determinstically to reach his target $\tilde{a}$ by splitting his orders equally over time (i.e., trading the same amount $\tilde{a}/N$ at each date n).

4. Trades deterministically to reach his target $\tilde{a}$ but optimizes his orders to minimize his expected trading cost given the time pattern of the equilibrium price impact coefficients λ_n. However, in doing so, he ignores the impact of the sunshine trading adjustment $-(\alpha_n^R + \beta_n^R)q_{n-1}$ in prices in (1.10).

5. Trades deterministically to reach his target $\tilde{a}$ but optimizes his orders to minimize his expected trading cost taking into account both the time pattern of the equilibrium price impact coefficients and the sunshine trading adjustment.

These constrained strategies are off-equilibrium deviations that incorporate different combinations of the considerations affecting the rebalancer's equilibrium orders. Any adaptive trading in response to the realized order flow paths is excluded. Conveniently, each of the constrained strategies is linear in the target $\tilde{a}$. The orders for constrained strategies $j = 1, \ldots, 5$ at time n are denoted by x_n^j.

Table 1 measures the distance between the different constrained strategies and the equilibrium rebalancer strategy using mean squared errors (MSEs). We average the squared errors $(\Delta\theta_n^R - x_n^j)^2$ at each date n given a target level $\tilde{a}$ across different simulated values of $\tilde{v}$ and noise trader orders and then sum them across all N dates. Not surprisingly, the MSEs are large for the single-date constrained strategies, but they are quite small for the multiperiod constrained strategies. This indicates that the rebalancer's orders in equilibrium are primarily driven by the trading target realization $\tilde{a}$ and less so by dynamic sunshine trading and endogenous learning about the asset's payoff. Table 2 in the Internet Appendix (Appendix C) reinforces this point by showing that the expected profit for the rebalancer for constrained strategy 5 is very close to the equilibrium strategy.

Table 1: MSE for constrained strategies (listed in the paper) relative to the equilibrium rebalancer strategy. The parameters are $N := 10$, $\sigma_w^2 := 4$, $\sigma_{\tilde{v}}^2 := 1$, $\rho := 0$ (top part only), and $\sigma_{\tilde{a}}^2 := 1$ (lower part only).

Strategy	$\sigma_{\tilde{a}}^2 := 0.48$	$\sigma_{\tilde{a}}^2 := 1$	$\sigma_{\tilde{a}}^2 := 3.7$
1	$0.0002 + 0.8972\,\tilde{a}^2$	$0.0005 + 0.9009\,\tilde{a}^2$	$0.0028 + 0.8949\,\tilde{a}^2$
2	$0.0002 + 0.6283\,\tilde{a}^2$	$0.0005 + 0.6203\,\tilde{a}^2$	$0.0028 + 0.6110\,\tilde{a}^2$
3	$0.0002 + 0.0308\,\tilde{a}^2$	$0.0005 + 0.0327\,\tilde{a}^2$	$0.0028 + 0.0369\,\tilde{a}^2$
4	$0.0002 + 0.0001\,\tilde{a}^2$	$0.0005 + 0.0005\,\tilde{a}^2$	$0.0028 + 0.0059\,\tilde{a}^2$
5	0.0002	0.0005	0.0028
	$\rho := 0.24$	$\rho := 0.47$	$\rho := 0.86$
1	$0.0068 + 0.7631\,\tilde{a}^2$	$0.0298 + 0.7114\,\tilde{a}^2$	$0.2642 + 0.7024\,\tilde{a}^2$
2	$0.0068 + 0.9148\,\tilde{a}^2$	$0.0298 + 0.8581\,\tilde{a}^2$	$0.2642 + 0.9178\,\tilde{a}^2$
3	$0.0068 + 0.0223\,\tilde{a}^2$	$0.0298 + 0.0316\,\tilde{a}^2$	$0.2642 + 0.0539\,\tilde{a}^2$
4	$0.0068 + 0.0009\,\tilde{a}^2$	$0.0298 + 0.0019\,\tilde{a}^2$	$0.2642 + 0.0028\,\tilde{a}^2$
5	0.0068	0.0298	0.2642

We summarize our analysis as follows: The rebalancer's orders appear to be driven primarily by a large deterministic component depending on the realized trading target $\tilde{a}$. Dynamic effects due to learning and sunshine trading due to the aggregate order via q_n are present but quantitatively small. These conclusions follow from the rebalancer's large order decomposition loadings on $\tilde{a}$ (in Figure 2A), the small rebalancer order standard deviations after conditioning on $\tilde{a}$ (in Figure 5), and the MSE result that the rebalancer's equilibrium orders are close to the optimized deterministic constrained dynamic strategy 5 (in Table 1). If noise due to the trading target $\tilde{a}$ is the largest component of the rebalancer's orders, then it should also be an important driver of the equilibrium effects of rebalancing on prices, the informed trader's orders, and other market variables.

3.2 Equilibrium effects

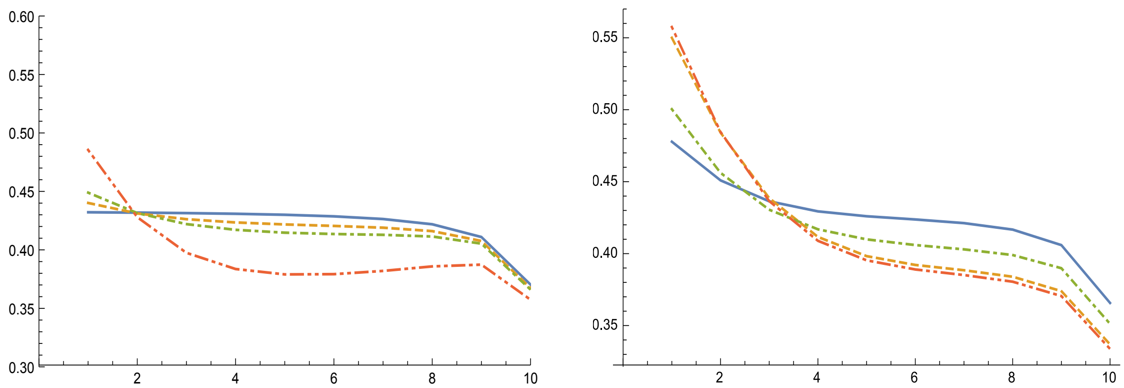
Price dynamics: We first discuss how dynamic rebalancing affects the equilibrium price dynamics in our market. Figure 7 show trajectories for the price-impact-of-order-flow parameter λ_n over time. For comparison, the solid blue line in Figure 7A is the corresponding price impact in Kyle (1985). In the first round of trading at time $n = 1$, rebalancing noise by itself (consistent with 2.14) lowers the value of λ_1 relative to Kyle (1985). However, in equilibrium, the hedge fund's trading strategy also changes. The net effect in this example is that λ_1 increases relative to Kyle (1985).[24] At later times $n > 2$, the price impacts are lower than in Kyle. The result is an S-shaped twist in λ_n over time. Figure 7B shows similar but somewhat smaller differences relative to the modified FV model (solid blue and dashed yellow lines) when $\rho > 0$. The price impact trajectory in our model also differs from DJK (2014) in which price impacts have an inverted U-shape (see their Figure 1).

Figure 7A shows that the S-shaped twist in λ_n increases when there is more trading target volatility $\sigma^2_{\tilde{a}}$. When $\sigma^2_{\tilde{a}}$ is high enough, the price impact of order flow can even be non-monotone over time (see the dashed line corresponding to $\sigma^2_{\tilde{a}} = 3.7$, which is comparable to the total daily noise trader order variance $\sigma^2_w = 4$). Figure 7B shows the target/terminal price correlation ρ has an aysmetric impact on λ_n over time. At early times, λ_n is increasing in the correlation ρ, but at later times, λ_n is decreasing in ρ. This is because increasing ρ changes some rebalancing trades from noise into informative order flow.

Figure 8 shows the trajectory of the variance $\Sigma^{(2)}_n$ of the market pricing errors $\tilde{v} - p_n$ over time, which measures the quality of price discovery. When $\rho = 0$, more information is revealed at early times compared to the Kyle model (due to more aggressive informed trading by the hedge fund, see below), but pricing accuracy is reduced later in the day. When $\rho > 0$ (so that $\tilde{a}$ is informative), the trading target constrains the aggressiveness of the rebalancer's orders relative to the purely informational unconstrained orders of the less informed investor in the modified FV model. This constraint, depending on the parameterization, can cause the rebalancer's orders to be larger or smaller than in the modified FV model. For example, in the case of $\rho = 0.25$ (when the information content of $\tilde{a}$ is relatively low and the less informed investor would not trade much in the modified FV model), the rebalancing

[24] We see in (2.14) that λ_n is non-monotone in the aggressiveness of informed trading. Thus, there may also be parameterizations for which our model has an inverted U-shape for λ_n.

Figure 7: Plots of price impacts $(\lambda_n)_{n=1}^N$ for the parameters $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).



A: Kyle (——), $\sigma_{\tilde{a}}^2 = 0.48$ (– –), $\sigma_{\tilde{a}}^2 = 1$ (– · –), $\sigma_{\tilde{a}}^2 = 3.7$ (– · · –).

B: $\rho = 0.25$ (——)[FV], $\rho = 0.47$ (– –)[FV], $\rho = 0.25$ (– · –), $\rho = 0.47$ (– · · –).

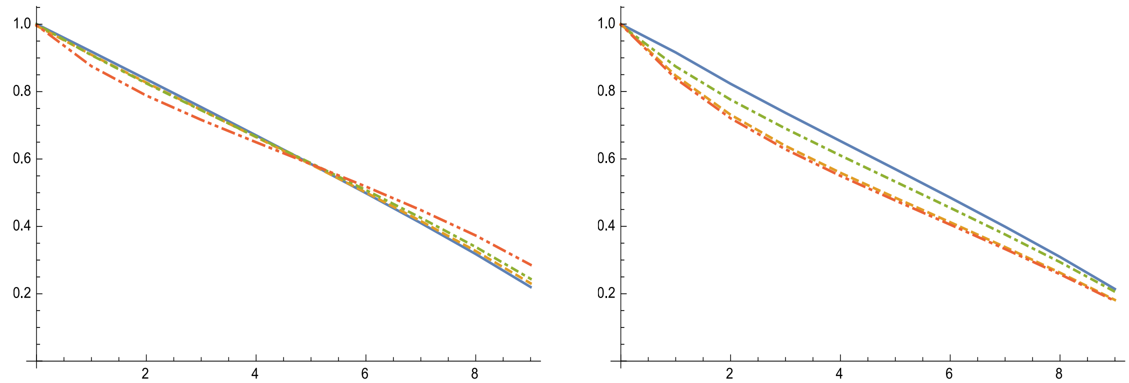
target volatility $\sigma_{\tilde{a}}^2 = 1$ appears to enlarge the rebalancer's orders. This, in turn, increases the aggressiveness of the hedge fund's orders (through the "rat race" effect) and, thereby, increases price accuracy relative to the modified FV model. However, in the case of $\rho = 0.47$, the increased information content of $\tilde{a}$ causes the less informed investor in the modified FV model to trade more aggressively and the difference in pricing accuracy between the two models shrinks.

A novel feature of markets with dynamic rebalancing is that expected orders do not have price impacts (see 1.10). A key variable here is the market makers' order flow expectation q_n of the reblancer's latent remaining trading demand $\tilde{a} - \theta_{n-1}^R$. Figure 9 shows the market makers' uncertainty $\Sigma_n^{(1)} = \mathbb{V}[\tilde{a} - \theta_n^R - q_n]$ about the rebalancer's remaining latent trading demand. Although a priori $\Sigma_n^{(1)}$ does not need to be monotone over time,[25] Figure 9 shows that uncertainty about the remaining latent trading demand is monotonically decreasing for modest values of ρ.

Informed investor: Figure 10 shows the hedge fund's strategy coefficients β_n^I, which determine how aggressively she trades on her private information $\tilde{v} - p_{n-1}$ over time. As in Kyle (1985), the intensity of informed trading in our model increases as time approaches the terminal time N. This is consistent with the fact that the incentive to

[25]This is because θ_n^R is a different random variable at different times n. However, the conditional variances of $\tilde{a}$ are, by definition, non-increasing.

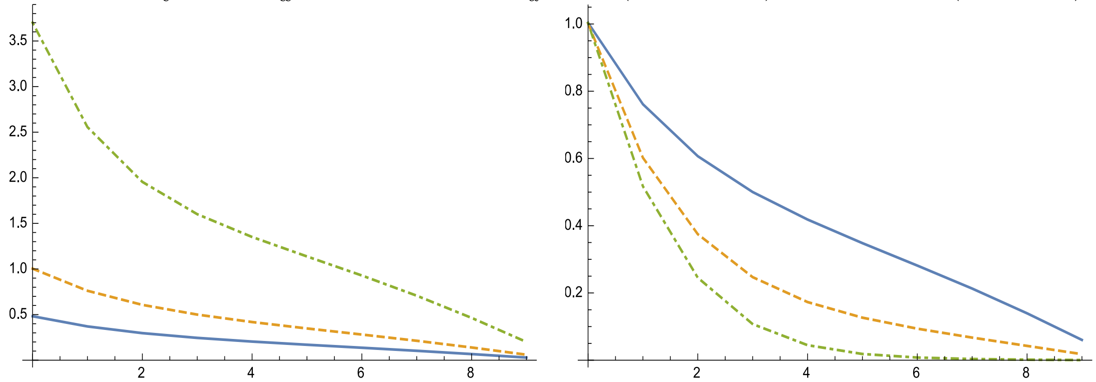
Figure 8: Plots of the pricing error variances $(\Sigma_n^{(2)})_{n=0}^{N-1}$ for the parameters $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).



A: Kyle (——), $\sigma_{\tilde{a}}^2 = 0.48$ (−−), $\sigma_{\tilde{a}}^2 = 1$ (− · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · −).

B: $\rho = 0.25$ (——)[FV], $\rho = 0.47$ (−−)[FV], $\rho = 0.25$ (− · −), $\rho = 0.47$ (− · · −).

Figure 9: Plots of the latent remaining trading demand variances $(\Sigma_n^{(1)})_{n=0}^{N-1}$ for the parameters $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).

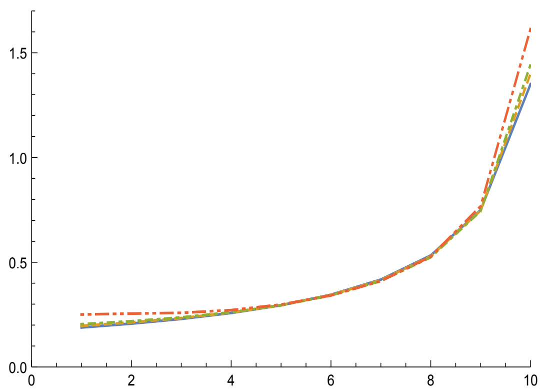


A: $\sigma_{\tilde{a}}^2 = 0.48$ (——), $\sigma_{\tilde{a}}^2 = 1$ (−−), $\sigma_{\tilde{a}}^2 = 3.7$ (− · −).

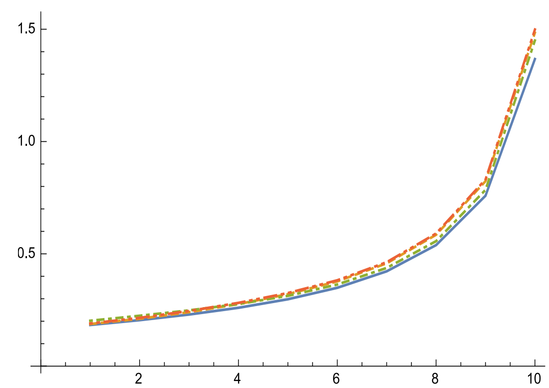
B: $\rho = 0$ (——), $\rho = 0.25$ (−−), $\rho = 0.47$ (− · −).

delay trading on information becomes weaker later in the day as the remaining time available for trading becomes shorter. We also see that as the variance of the trading target $\sigma_{\tilde{a}}^2$ increases, the informed investor trades more aggressively at early dates, less so in the middle, and then slightly more aggressively again towards the end. The informed trader's increased initial aggressiveness reflects the fact that there is more noise (due to the rebalancer's trading target $\tilde{a}$) in which to hide the hedge fund's orders. In addition, if $\rho > 0$, hedge fund trading aggressiveness increases somewhat due to a race–to–trade competition effect. The apparent size of the changes in β_1^I — which are on the order of 10 percent — are visually understated in Figure 10 because of the vertical scaling (due to the size of β_{10}^I).

Figure 10: Plots of hedge fund strategy coefficients $(\beta_n^I)_{n=1}^N$ for the parameters $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only), and $\rho := 0$ (left only).



A: Kyle (——), $\sigma_{\tilde{a}}^2 = 0.48$ (−−), $\sigma_{\tilde{a}}^2 = 1$ (− · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · · −).

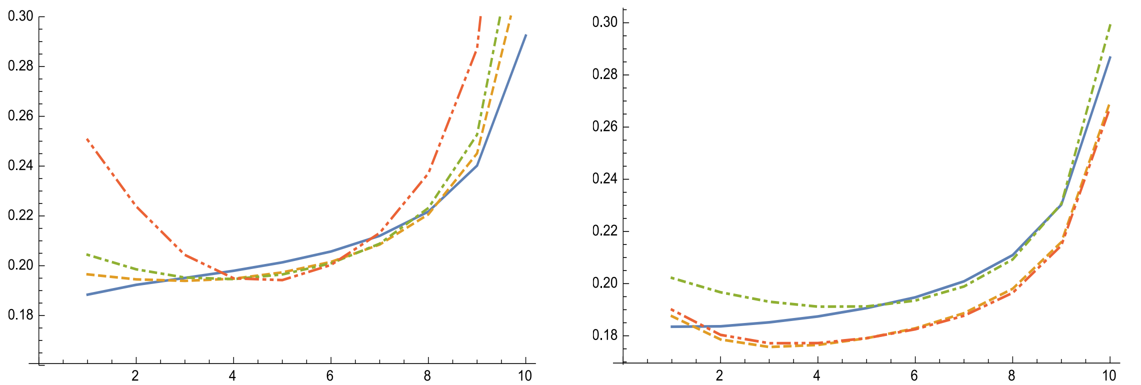


B: $\rho = 0.25$ (——)[FV], $\rho = 0.47$ (−−)[FV], $\rho = 0.25$ (− · −), $\rho = 0.47$ (− · · −).

Using the linear decomposition for the informed hedge fund's orders in (3.4), Figure 11 shows the ratios over the trading day of the hedge fund's expected orders conditional on her information about the terminal stock valuation $\tilde{v}$ relative to $\tilde{v}$ for $\tilde{v} \neq 0$, averaged over $\tilde{a}$ and the noise trader paths w. Unlike in the Kyle model (solid blue line in Figure 11A), our model produces a slight U-shaped trading pattern. Our hedge fund expects ex ante to trade somewhat more initially and again at the end of the day. However, the U-shape is not big. Figure 11B shows that the modified FV model can also produce U-shaped volume for the better-informed trader when ρ is sufficiently large (the dashed yellow line corresponding to $\rho = 0.47$). However, the U-shape pattern is even larger for the rebalancer.

The informed hedge fund and the rebalancer behave differently in our model. This can be seen in the different functional forms of their orders in (1.6) and (1.5) as well as from the quantitatively different properties of their orders in our numerical analysis. For example, the U–shaped patterns in expected trading skew towards more trading earlier in the day for the rebalancer (consistent with the benefit of subsequent sunshine trading) versus later in the day for the informed hedge fund. These differences

Figure 11: Plots of the ratio $\mathbb{E}[\Delta\theta_n^I|\sigma(\tilde{v})]/\tilde{v} = A_n^I \rho\sigma_{\tilde{a}}/\sigma_{\tilde{v}} + B_n^I$ of the expected hedge fund orders (conditional on a final stock value $\tilde{v} \neq 0$) relative to $\tilde{v}$ for $n = 1, 2, ..., 10$. The parameters are $\sigma_{\tilde{v}}^2 := 1$, $\sigma_w^2 := 4$, $N := 10$, $\sigma_{\tilde{a}}^2 := 1$ (right only) and $\rho := 0$ (left only).



A: Kyle (——), $\sigma_{\tilde{a}}^2 = 0.48$ (−−), $\sigma_{\tilde{a}}^2 = 1$ (− · −), $\sigma_{\tilde{a}}^2 = 3.7$ (− · ·−).

B: $\rho = 0.25$ (——)[FV], $\rho = 0.47$ (−−)[FV], $\rho = 0.25$ (− · −), $\rho = 0.47$ (− · ·−).

are empirically testable given data about parent and child orders from a cross-section of heterogenous institutional investors who differ in their information and rebalancing motives. In particular, the predicted relation between parent-child orders for large index funds is very different from the parent-child relation for hedge funds. The numerical comparative static results also suggest how rebalancer and informed investor order submissions should change as parameters of the trading environment change.

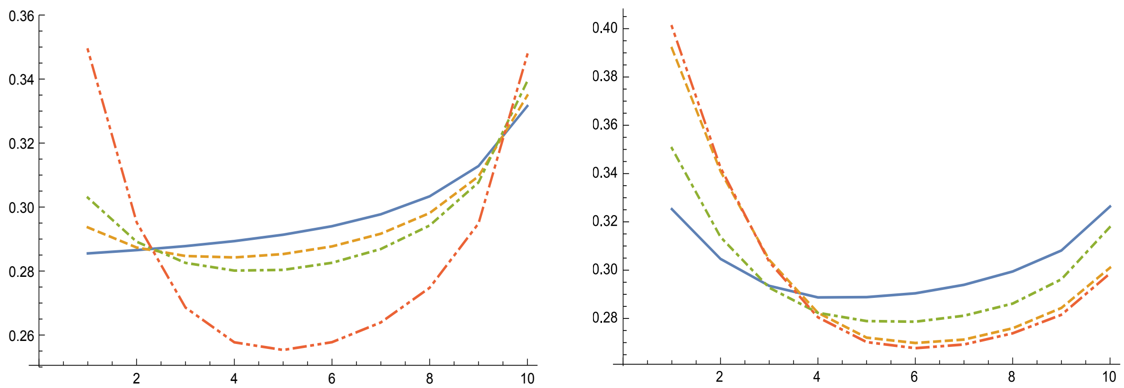
Other intraday patterns: Stock markets have a variety of empirical intraday patterns in prices and order flows.[26] Our model features several intraday patterns in addition to those already discussed above.

Figure 12 shows the unconditional standard deviation for the price changes over time. Price volatility is monotonically increasing in the Kyle model (solid blue line in Figure 12A), whereas our model produces U-shaped dotted lines (for various target variances $\sigma_{\tilde{a}}^2$). In other words, prices in our model are more volatile at the beginning and at the end of the trading day relative to the middle of the trading day. In addition, the U-shape becomes larger when rebalancing volatility is higher. When $\rho > 0$, Figure 12B shows that price volatility is U-shaped in both the modified FV

[26]Intraday patterns are robust properties of volume and price volatility in equity markets that were first documented in Wood, McInish, and Ord (1985) and Jain and Joh (1988).

model (solid blue and dashed yellow lines) and in our model with rebalancing.

Figure 12: Plots of price change standard deviations $\sqrt{\mathbb{E}[(p_n - p_{n-1})^2]}$ for $n = 1, 2, ..., 10$. The parameters are $\sigma^2_{\tilde{v}} := 1$, $\sigma^2_w := 4$, $N := 10$, $\sigma^2_{\tilde{a}} := 1$ (right only), and $\rho := 0$ (left only).



A: Kyle (——), $\sigma^2_{\tilde{a}} = 0.48$ (−−), $\sigma^2_{\tilde{a}} = 1$ (− · −), $\sigma^2_{\tilde{a}} = 3.7$ (− · · −).

B: $\rho = 0.25$ (——)[FV], $\rho = 0.47$ (−−)[FV], $\rho = 0.25$ (− · −), $\rho = 0.47$ (− · · −).

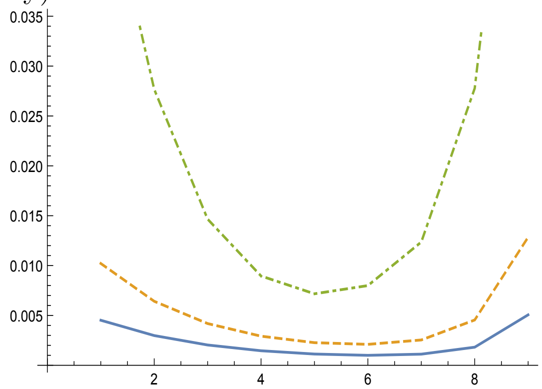
Figure 13 shows the unconditional autocorrelation of the aggregate order flow over time for different values of $\sigma^2_{\tilde{a}}$ and ρ. Although the absolute level of autocorrelation is low, there is a clear U-shaped pattern of higher order flow autocorrelation at the beginning and the end of the day (when, from Figure 3, the rebalancer trades more) and with lower autocorrelation during the middle of the day (when the rebalancer trades less). Somewhat surprisingly, order-flow autocorrelation can be negative in the middle of the day when the target-information correlation ρ is high.[27]

4 Robustness and empirical implications

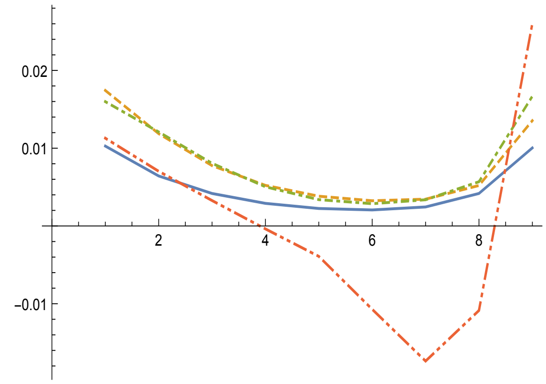
Our model has empirical implications for order flow and prices:

[27]There are many interactions as play here, but part of the intuition for a negative autocorrelation is as follows: The sign of the autocorrelation depends on the autocovariance $\text{cov}(y_n, y_{n-1})$, which simplifies to $(\alpha^R_n + \beta^R_n)\,\text{cov}(q_{n-1}, y_{n-1})$. If ρ is high, the rebalancer's target $\tilde{a}$ is highly informative, making the rebalancer want to take large positions. However, if $\sigma^2_{\tilde{a}}$ is small, the rebalancing constraint limits his terminal position size. In this case, the rebalancer takes large positions $\theta^R_n > \tilde{a}$ at early dates but then later must partially unwind these positions. Part of this unwinding is predictable when y_{n-1} is large later in the day, which leads to the negative autocovariance.

Figure 13: Plots of aggregate order flow autocorrelation $\frac{\mathbb{E}[y_n y_{n+1}]}{\sqrt{\mathbb{E}[y_n^2]\mathbb{E}[y_{n+1}^2]}}$ for $n = 1, 2, ..., 9$. The parameters are $N := 10$, $\sigma^2_w := 4$, $\sigma^2_{\tilde{v}} := 1$, $\sigma^2_{\tilde{a}} := 1$ (right only), and $\rho := 0$ (left only).



A: $\sigma^2_{\tilde{a}} = 0.48$ (——), $\sigma^2_{\tilde{a}} = 1$ (−−), $\sigma^2_{\tilde{a}} = 3.7$ (− · −).



B: $\rho = 0$ (——), $\rho = 0.25$ (−−), $\rho = 0.47$ (− · −), $\rho = 0.86$ (− · · −).

1. Predictable order flow (i.e., given public information) should not have a permanent/informational price impact.

2. Autocorrelation of the aggregate order flow should be linked to autocorrelation in the orders of individual investors. An alternative is cross-autocorrelation of order submissions across orders submitted by different investors at different times due to front-running or back-running (see Yang and Zhu 2015).

3. Intraday U-shaped patterns in price volatility and volume are increasing in target volatility $\sigma_{\tilde{a}}$.[28] This prediction is testable by looking at these intraday patterns on stocks and days in which rebalancing trading uncertainty is greater (e.g., days with high mutual fund inflows/outflows). In addition, our model predicts, given the link between rebalancing and aggregate order flow autocorrelation, that time-variation in these intraday patterns and aggregate order flow autocorrelation should be positively correlated. In other words, daily order flow autocorrelation (estimated using intraday data) can be a proxy to track changes in rebalancing volatility.

[28]Order-splitting is certainly not the only cause of U-shaped intraday patterns, since many of the empirically documented intraday patterns predate the widespread use of order-splitting algorithms. However, the magnitude of these U-shaped patters should co-vary with rebalancing volatility.

4. Dynamic trading by informed investors is qualitatively different from order-splitting by rebalancers. In particular, rebalancer order flows are autocorrelated, while informed investor orders are not. This is testable given order data linked to individual institutions (e.g., using IIORC data). We expect lower trading profits for institutions submitting more autocorrelated orders (i.e., likely rebalancers), even after adjusting for transitory price effects.

5. The correlation of orders from profitable (informed) investors and large indexers (and other less-informed or passive investors) should turn negative over the trading day. This is testable given investor-level order data (e.g., from the IIORC).

These predications are mainly about effects of time-variation in the volatility (i.e., second moment) of private portfolio rebalancing trading demand. Thus, they differ from effects of predictable (i.e., first moment) trading demand investigated in Bessembinder, Carrion, Tuttle, and Venkatarman (2016).

The qualitative properties of our analysis are likely to be robust to relaxing some of our model's assumptions. We discuss three here. First, our model assumes a hard rebalancing constraint. Alternatively, the rebalancing constraint could be soft with a quadratic cost for deviations from the target, or investors could have a random private value for the asset that is decreasing in their terminal holdings. In either case, the rebalancer should still engage in order-splitting to reduce their trading costs. These alternative rebalancing motives should result in some amount of price elasticity in the total amount traded by rebalancers. This should increase the importance of the part of rebalancer orders that responds to changing intraday market conditions (e.g., the dependence on the past order flow history via the q_n process in our model).

Second, informed investors and rebalancers only use market orders in our model. In practice, however, order-splitting algorithms also use limit orders (see O'Hara 2015). In addition, limit order flows are also autocorrelated (see Biais, Hillion, and Spatt 1995). While the mathematics of the dynamic programming problems and finding the rational expectations equilibrium fixed point would be even more complicated, we still expect rebalancing to result in order flow autocorrlation and for predictable components of market and limit order flows to have no persistent price impacts.

Third, our market makers are competitive, risk-neutral, and have no order processing costs. As a result, prices are martingales in our model. We do not expect market-making frictions and transitory price effects to eliminate the informational causes of order-splitting. However, it would be interesting to investigate empirically how market frictions and informational causes of order-splitting interact.

5 Conclusion

This paper has explored the equilibrium interactions between dynamic order-splitting for portfolio rebalancing and price discovery, order flow dynamics and market liquidity. Our paper is the first to investigate these issues with both long-lived information and dynamic rebalancing given a terminal trading target. We find that while the rebalancer takes advantage of sunshine trading and endogenous learning, the quantitative impact appears small relative to a deterministic component of order-splitting. However, strategic rebalancing does not just inject additional trading noise in the market; rather strategic rebalancing affects the structure of the market equilibrium. Order flow becomes autocorrelated and liquidity and price discovery dynamics change because of sunshine trading. In addition, orders from the rebalancer and informed trader tend to become negatively correlated over time. Because the hedge fund's and rebalancer's orders partially cancel each other, they can supply liquidity to each other symbioticallty with a reduced price impact.

We have identified a number of testable empirical predictions from our model. There are many interesting possible extensions for future theory. One possible extension is to model trading in continuous-time. Another extension is to relax the assumption that all investors are risk-neutral. For example, it would be natural to consider exponential utilities with different coefficients of absolute risk aversion. Finally, our model could be extended to include multiple hedge funds and rebalancers.

References

[1] Admati, A., and P. Pfleiderer, 1988, A Theory of Intraday Patterns: Volume and Price Variability, *Review of Financial Studies*, 1, 3–40.

[2] Akerlof, G., 1970, The Market for 'Lemons': Quality Uncertainty and the Market Mechanism, *Quarterly Journal of Economics*, 84, 488–500.

[3] Almgren, R., and N. Chriss, 1999, Value Under Liquidation, *Risk*, 12, 61–63.

[4] Almgren, R., and N. Chriss, 2000, Optimal Execution of Portfolio Transactions, *Journal of Risk*, 5–39.

[5] Back, K. (1992): Insider Trading in Continuous Time, *Review of Financial Studies*, 5, 387–409.

[6] Back, K., H. Cao, and G. Willard, 2000, Imperfect Competition among Informed Traders, *Journal of Finance*, 55, 2117–2155.

[7] Baruch, S., and L. Glosten, 2013, Flickering Quotes, working paper, Columbia University and the University of Utah.

[8] Bertsimas, D., and A. Lo, 1998, Optimal Control of Execution Costs, *Journal of Financial Markets*, 1, 1–50.

[9] Bessembinder, H., A. Carrion, L. Tuttle, and K. Venkataraman, 2016, Liquidity, Resiliency and Market Quality around Predictable Trades: Theory and Evidence, *Journal of Financial Economics*, 121, 142-166.

[10] Biais, B., T. Foucault, and S. Moinas, 2015, Equilibrium Fast Trading, *Journal of Financial Economics*, 116, 292-313.

[11] Biais, B., P. Hillion, and C. Spatt, 1995, An Empirical Analysis of the Limit Order Book and the Order Flow in the Paris Bourse, *Journal of Finance*, 50, 1655-1689.

[12] Boulatov, A., D. Bernhardt, and I. Larionov, 2016, Predatory and Defensive Trading in a Dynamic Model of Optimal Execution by Multiple Traders, working paper.

[13] Brogaard J., T. Hendershott, and R. Riordan, 2016, Price Discovery without Trading: Evidence from Limit Orders, working paper, University of Washington.

[14] Brunnermeier, M.K., and L.H. Pedersen, 2005, Predatory Trading, *Journal of Finance*, 60, 1825–1863.

[15] Carlin, B., M. Lobo, and S. Viswanathan, 2007, Episodic Liquidity Crises: Cooperative and Predatory Trading, *Journal of Finance*, 62, 2235-2274.

[16] Degryse, H., F. de Jong, and V. van Kervel, 2014, Does Order Splitting Signal Uninformed Order Flow?, working paper.

[17] Engle, R., R. Ferstenberg, and J. Russell, 2012, Measuring and Modeling Execution Cost and Risk, *Journal of Portfolio Management*, 38, 14–28.

[18] Foster, F., and S. Viswanathan, 1994, Strategic Trading with Asymmetrically Informed Traders and Long-Lived Information, *Journal of Financial and Quantitative Analysis*, 29, 499–518.

[19] Foster, F., and S. Viswanathan, 1996, Strategic Trading when Agents Forecast the Forecasts of Others, *Journal of Finance*, 51, 1437–1478.

[20] Foucault, T., J. Hobert, and I. Rosu, 2016, News Trading and Speed, *Journal of Finance*, 71, 335-382.

[21] Gatheral, J., and A. Schied, 2011, Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework, *International Journal of Theoretical and Applied Finance*, 14, 353–368.

[22] Grossman, S.J., and J.E. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, *American Economic Review*, 70, 393–408.

[23] Glosten, L.R., and P.R. Milgrom, 1985, Bid, Ask, and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders, *Journal of Financial Economics*, 14, 71–100.

[24] Hasbrouck, J., 1991a, Measuring the Information Content of Stock Trades, *Journal of Finance*, 46, 179–207.

[25] Hasbrouck, J., 1991b, The Summary Informativeness of Stock Trades: An Econometric Analysis, *Review of Financial Studies*, 4, 571–595.

[26] Hasbrouck, J., and G. Saar, 2007, Technology and Liquidity Provision: The Blurring of Traditional Definitions, working paper, Stern School of Business.

[27] Hoffmann, P., 2014, A Dynamic Limit Order Market with Fast and Slow Traders, *Journal of Financial Economics*, 113, 156-169

[28] Holden, C.W., and A. Subrahmanyam, 1992, Long-Lived Private Information and Imperfect Competition, *Journal of Finance*, 47, 247–270.

[29] Jain, P., and G. Joh, 1988, The Dependence between Hourly Prices and Trading Volume, *Journal of Financial and Quantitative Analysis*, 23, 269-283.

[30] Johnson, B., 2010, Algorithmic Trading and DMA, 4Myeloma Press, London.

[31] Keim, D., and A. Madhavan, 1995, Anatomy of the Trading Process: Empirical Evidence on the Behavior of Institutional Traders, *Journal of Financial Economics*, 37, 371–398.

[32] van Kervel, V, and A. Menkveld, 2016, High-Frequency Trading around Large Institutional Orders, working paper.

[33] Korajczyk, R., and D. Murphy, 2016, High-Frequency Market Making to Large Institutional Trades, working paper.

[34] Kyle, A., 1985, Continuous auctions and Insider trading, *Econometrica*, 53, 1315–1336.

[35] Kyle, A., and A. Obizhaeva, 2016, Market Microstructure Invariance: A Dynamic Equilibrium Model, working paper.

[36] Menkveld, A., B. Yueshen, and H. Zhu, 2014, Shades of Darkness: A Pecking Order of Trading Venues, working paper.

[37] O'Hara, M., 2015, High Frequency Market Microstructure, *Journal of Financial Economics*, 116, 257-270.

[38] *Pensions & Investments*, 2007, Algorithmic Trading, Already the Preferred Execution choice for Fund..., online, dated: February 12, 2007 12:01 am

[39] Predoiu, S., G. Shaikhet, and S. Shreve, 2011, Optimal Execution of a General One-Sided Limit-Order Book, *SIAM J. Financial Math*, 2, 183–212.

[40] Seppi, D.J., 1990, Equilibrium Block Trading and Asymmetric Information, *Journal of Finance*, 45, 73–94.

[41] U.S. Securities and Exchange Commission, 2010, Concept Release on Equity Market Structure #34–61358.

[42] Weller, B., 2013, Intermediation Chains and Specialization by Speed:Evidence from Commodity Futures Markets, working paper.

[43] Wood, R., T. Mclnish, and J. Ord, 1985, An Investigation of Transaction Data for NYSE Stocks, *Journal of Finance*, 40, 723-741.

[44] Yang, L., and H. Zhu, 2015, Back-Running: Seeking and Hiding Fundamental Information in Order Flows, working paper, MIT.

A Proofs

A.1 Kalman filtering

Lemma A.1. *Consider the conjectured system* (2.3)-(2.7) *corresponding to arbitrary coefficients* $(\beta_n^I, \beta_n^R, \alpha_n^R)_{n=1}^N$. *Whenever* (2.14)-(2.17) *hold, we have*

$$\hat{p}_n = \mathbb{E}[\tilde{v}|\sigma(\hat{y}_1, ..., \hat{y}_n)], \tag{A.1}$$

$$\hat{q}_n = \mathbb{E}[\tilde{a} - \hat{\theta}_n^R|\sigma(\hat{y}_1, ..., \hat{y}_n)], \tag{A.2}$$

where $\hat{p}$ *is defined by* (2.6) *and* $\hat{q}$ *is defined by* (2.7). *Furthermore, the recursions for the variances and covariance* (2.18)-(2.20) *hold.*

Proof. For $n = 1, ..., N$, we have the moment definitions in (2.8)-(2.10) where the starting values are given in (2.11). We then define the process $\hat{z}_n^M$ as

$$\begin{aligned} \hat{z}_n^M :=& \hat{y}_n - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1} \\ =& \beta_n^I(\tilde{v} - \hat{p}_{n-1}) + \beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) + \Delta w_n. \end{aligned} \tag{A.3}$$

These Gaussian variables $\hat{z}_1^M, \hat{z}_2^M, ..., \hat{z}_N^M$ are mutually independent and satisfy $\sigma(\hat{z}_1^M, ..., \hat{z}_n^M) = \sigma(\hat{y}_1, ...\hat{y}_n)$. The projection theorem for Gaussian random variables gives

$$\begin{aligned} \Delta\hat{p}_n =& \mathbb{E}[\tilde{v}|\sigma(\hat{z}_1^M, ..., \hat{z}_n^M)] - \mathbb{E}[\tilde{v}|\sigma(\hat{z}_1^M, ..., \hat{z}_{n-1}^M)] \\ =& \frac{\mathbb{E}[\tilde{v}\,\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M, \\ \Delta\hat{q}_n =& \mathbb{E}[\tilde{a} - \hat{\theta}_n^R|\sigma(\hat{z}_1^M, ...\hat{z}_n^M)] - \mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R|\sigma(\hat{z}_1^M, ..., \hat{z}_{n-1}^M)] \\ =& \mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R|\sigma(\hat{z}_1^M, ...\hat{z}_n^M)] - \mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R|\sigma(\hat{z}_1^M, ..., \hat{z}_{n-1}^M)] - \mathbb{E}[\Delta\hat{\theta}_n^R|\sigma(\hat{z}_1^M, ..., \hat{z}_n^M)] \\ =& \frac{\mathbb{E}[(\tilde{a} - \hat{\theta}_{n-1}^R)\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M - \mathbb{E}\big[\beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) + (\alpha_n^R + \beta_n^R)\hat{q}_{n-1}\big|\sigma(\hat{z}_1^M, ..., \hat{z}_n^M)\big] \\ =& \frac{\mathbb{E}[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M - \beta_n^R\mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}|\sigma(\hat{z}_n^M)] - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1} \\ =& (1 - \beta_n^R)\frac{\mathbb{E}[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1})\hat{z}_n^M]}{\mathbb{V}[\hat{z}_n^M]}\hat{z}_n^M - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1}. \end{aligned}$$

To proceed, we first need to compute

$$
\begin{aligned}
\mathbb{V}[\hat{z}^M_n] =& \mathbb{E}\Big[\Big(\beta^I_n(\tilde{v}-\hat{p}_{n-1}) + \beta^R_n(\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1}) + \Delta w_n\Big)^2\Big] \\
=& (\beta^I_n)^2\Sigma^{(2)}_{n-1} + (\beta^R_n)^2\Sigma^{(1)}_{n-1} + 2\beta^I_n\beta^R_n\Sigma^{(3)}_{n-1} + \sigma^2_w\Delta, \\
\mathbb{E}[\tilde{v}\hat{z}^M_n] =& \mathbb{E}[(\tilde{v}-\hat{p}_{n-1})\hat{z}^M_n] \\
=& \mathbb{E}\Big[(\tilde{v}-\hat{p}_{n-1})\Big(\beta^I_n(\tilde{v}-\hat{p}_{n-1}) + \beta^R_n(\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1}) + \Delta w_n\Big)\Big] \\
=& \beta^I_n\Sigma^{(2)}_{n-1} + \beta^R_n\Sigma^{(3)}_{n-1}, \\
\mathbb{E}[(\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1})\hat{z}^M_n] =& \mathbb{E}\Big[(\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1})\Big(\beta^I_n(\tilde{v}-\hat{p}_{n-1}) + \beta^R_n(\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1}) + \Delta w_n\Big)\Big] \\
=& \beta^I_n\Sigma^{(3)}_{n-1} + \beta^R_n\Sigma^{(1)}_{n-1}.
\end{aligned}
$$

Combining these expressions and by matching coefficients with (2.6) and (2.7), we find the lemma's statement equivalent to the restrictions (2.14)-(2.17). Based on these expressions, the recursion for $\Sigma^{(1)}_n$, $n = 1, ..., N$, in (2.18) is

$$
\begin{aligned}
\Sigma^{(1)}_n :&= \mathbb{V}[\tilde{a}-\hat{\theta}^R_n-\hat{q}_n] \\
&= \mathbb{V}[\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1}-\Delta\hat{\theta}^R_n-\Delta\hat{q}_n] \\
&= \mathbb{V}[\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1}-\Delta\hat{\theta}^R_n-r_n\hat{y}_n-s_n\hat{q}_{n-1}] \\
&= \mathbb{V}\Big[\tilde{a}-\hat{\theta}^R_{n-1}-(1+s_n)\hat{q}_{n-1}-(1+r_n)(\beta^R_n(\tilde{a}-\hat{\theta}^R_{n-1})+\alpha^R_n\hat{q}_{n-1}) \\
&\qquad - r_n\big(\beta^I_n(\tilde{v}-\hat{p}_{n-1})\big) - r_n\Delta w_n\Big], \\
&= \mathbb{V}\Big[\big(1-(1+r_n)\beta^R_n\big)(\tilde{a}-\hat{\theta}^R_{n-1}) - \big(1+s_n+(1+r_n)\alpha^R_n\big)\hat{q}_{n-1} \\
&\qquad - r_n\beta^I_n(\tilde{v}-\hat{p}_{n-1}) - r_n\Delta w_n\Big] \\
&= \mathbb{V}\Big[\big(1-(1+r_n)\beta^R_n\big)(\tilde{a}-\hat{\theta}^R_{n-1}-\hat{q}_{n-1}) - r_n\beta^I_n(\tilde{v}-\hat{p}_{n-1}) - r_n\Delta w_n\Big] \\
&= \big(1-(1+r_n)\beta^R_n\big)^2\Sigma^{(1)}_{n-1} + (r_n\beta^I_n)^2\Sigma^{(2)}_{n-1} + r_n^2\sigma^2_w\Delta - 2\big(1-(1+r_n)\beta^R_n\big)r_n\beta^I_n\Sigma^{(3)}_{n-1} \\
&= (1-\beta^R_n)\big((1-\beta^R_n-r_n\beta^R_n)\Sigma^{(1)}_{n-1} - r_n\beta^I_n\Sigma^{(3)}_{n-1}\big),
\end{aligned}
$$

where the last equality uses (2.15). The recursions for $\Sigma^{(2)}_n$ and $\Sigma^{(3)}_n$, $n = 1, ..., N$, in (2.19) and (2.20) are found similarly.

$\diamondsuit$

A.2 Informed investor's optimization problem

We start with the following lemma which contains most of the calculations we will need later. We recall the hedge fund's state processes $(X_n^{(1)}, X_n^{(2)})$ are defined by (2.22).

Lemma A.2. *Fix* $\Delta\theta_n^R$ *by* (1.5) *and let the constants* (2.1) *and associated terms* (2.35) *satisfy the pricing coefficient relations* (2.14)-(2.17) *and the variances and covariance recursions* (2.18)-(2.20). *Let* $\Delta\theta_n^I \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$, $n = 1, ..., N$, *be arbitrary for the hedge fund. We define the Gaussian random variables*

$$\hat{z}_n^I := \hat{y}_n - \Delta\hat{\theta}_n^I - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1} - \beta_n^R \tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}(\tilde{v} - \hat{p}_{n-1}), \quad n = 1, ..., N. \tag{A.4}$$

Then $\hat{z}_k^I$ *is independent of* $(\tilde{v}, \hat{y}_1, ..., \hat{y}_{k-1})$ *for* $k \leq N$ *and the following measurability properties are satisfied:*

$$\hat{\theta}_n^R - \theta_n^R \in \sigma(\tilde{v}, y_1, ..., y_n) = \sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_n) = \sigma(\tilde{v}, \hat{z}_1^I, ...\hat{z}_n^I), \quad n = 1, ..., N. \tag{A.5}$$

Furthermore, for $n = 1, ..., N$, *we have the Markovian dynamics*

$$\Delta X_n^{(1)} = -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big) - \lambda_n \hat{z}_n^I, \quad X_0^{(1)} = \tilde{v}, \tag{A.6}$$

$$\Delta X_n^{(2)} = -r_n\Delta\theta_n^I - (1 + r_n)\beta_n^R X_{n-1}^{(2)} - \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\lambda_n \hat{z}_n^I, \qquad X_0^{(2)} = \frac{\rho\sigma_{\tilde{a}}}{\sigma_{\tilde{v}}}\tilde{v}. \tag{A.7}$$

Finally, for any constants $I_n^{(1,1)}, I_n^{(1,2)}$, *and* $I_n^{(2,2)}$, *we have the conditional expectation*

$$\begin{aligned}
&\mathbb{E}\left[(\tilde{v} - p_n)\Delta\theta_n^I + I_n^{(1,1)}\left(X_n^{(1)}\right)^2 + I_n^{(1,2)}X_n^{(1)}X_n^{(2)} + I_n^{(2,2)}\left(X_n^{(2)}\right)^2 \Big|\sigma(\tilde{v}, y_1, ..., y_{n-1})\right] \\
&= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n - \Delta\theta_n^I\lambda_n\beta_n^R X_{n-1}^{(2)} \\
&+ I_n^{(1,1)}\Big(\left(X_{n-1}^{(1)}\right)^2 - 2\lambda_n X_{n-1}^{(1)}\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big) + \lambda_n^2\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)^2 + \lambda_n^2\mathbb{V}[\hat{z}_n^I]\Big) \\
&+ I_n^{(1,2)}\Big(X_{n-1}^{(1)}X_{n-1}^{(2)} - X_{n-1}^{(1)}\Big(r_n\Delta\theta_n^I + (1 + r_n)\beta_n^R X_{n-1}^{(2)}\Big) - X_{n-1}^{(2)}\lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big) \\
&+ \lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)\Big(r_n\Delta\theta_n^I + (1 + r_n)\beta_n^R X_{n-1}^{(2)}\Big) + \lambda_n^2\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\mathbb{V}[\hat{z}_n^I]\Big) \\
&+ I_n^{(2,2)}\Big(\left(X_{n-1}^{(2)}\right)^2 - 2X_{n-1}^{(2)}\Big(r_n\Delta\theta_n^I + (1 + r_n)\beta_n^R X_{n-1}^{(2)}\Big) + \Big(r_n\Delta\theta_n^I + (1 + r_n)\beta_n^R X_{n-1}^{(2)}\Big)^2 \\
&+ \lambda_n^2\left(\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\right)^2\mathbb{V}[\hat{z}_n^I]\Big),
\end{aligned} \tag{A.8}$$

which is quadratic in $\Delta\theta_n^I$, and where the variance $\mathbb{V}[\hat{z}_n^I]$ can be computed to be

$$\mathbb{V}[\hat{z}_n^I] = (\beta_n^R)^2\Big(\Sigma_{n-1}^{(1)} - \frac{\big(\Sigma_{n-1}^{(3)}\big)^2}{\Sigma_{n-1}^{(2)}}\Big) + \sigma_w^2\Delta. \tag{A.9}$$

Proof. The joint normality claim follows by an induction argument. To see the independence claim, we start by noticing

$$\begin{aligned}
&\beta_n^R\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \mathbb{E}\left[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}|\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_{n-1})\right]\Big) + \Delta w_n \\
&= \beta_n^R\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(3)}}(\tilde{v} - \hat{p}_{n-1})\Big) + \alpha_n^R\hat{q}_{n-1} - \alpha_n^R\hat{q}_{n-1} + \Delta w_n \\
&= \hat{y}_n - \Delta\hat{\theta}_n^I - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1} - \beta_n^R\tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(3)}}\big(\tilde{v} - \hat{p}_{n-1}\big),
\end{aligned}$$

which is $\hat{z}_n^I$ (see A.4). To see the independence of the random variables (A.4) we let $k \leq n-1$ be arbitrary. Iterated expectations produce the zero correlation property:

$$\mathbb{E}[\hat{y}_k\hat{z}_n^I] = \mathbb{E}[\mathbb{E}[\hat{y}_k\hat{z}_n^I|\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_k)]] = \mathbb{E}[\hat{y}_k\mathbb{E}[\hat{z}_n^I|\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_k)]] = 0.$$

The independence then follows from the joint normality.

Next, we observe that the last equality in (A.5) follows directly from (A.4). We proceed by induction and observe

$$\begin{aligned}
&\sigma(\tilde{v}, y_1) = \sigma(\tilde{v}, \beta_1^R\tilde{a} + \Delta w_1) = \sigma(\tilde{v}, \hat{y}_1), \\
&\hat{\theta}_1^R - \theta_1^R = 0,
\end{aligned}$$

which follows from $\hat{\theta}_1^I, \theta_1^I \in \sigma(\tilde{v})$. Suppose that (A.5) holds for n. Then,

$$\begin{aligned}
\hat{\theta}_{n+1}^R - \theta_{n+1}^R &= (1 - \beta_{n+1}^R)(\hat{\theta}_n^R - \theta_n^R) + \alpha_{n+1}^R(\hat{q}_n - q_n) \in \sigma(\tilde{v}, y_1, ..., y_n), \\
\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_{n+1}) &= \sigma(\tilde{v}, y_1, ..., y_n, \hat{y}_{n+1}) \\
&= \sigma(\tilde{v}, y_1, ..., y_n, y_{n+1} + \Delta\hat{\theta}_{n+1}^I - \Delta\theta_{n+1}^I + \Delta\hat{\theta}_{n+1}^R - \Delta\theta_{n+1}^R) \\
&= \sigma(\tilde{v}, y_1, ..., y_{n+1}),
\end{aligned}$$

which proves (A.5). The dynamics (A.6) can be seen as follows

$$
\begin{aligned}
\Delta X_n^{(1)} &= -\Delta p_n \\
&= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R(\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1} + \Delta w_n\Big) - \mu_n q_{n-1} \\
&= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R(\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1} + \hat{y}_n - \Delta\hat{\theta}_n^I - \Delta\hat{\theta}_n^R\Big) + \lambda_n(\alpha_n^R + \beta_n^R)q_{n-1} \\
&= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R(\hat{\theta}_{n-1}^R - \theta_{n-1}^R) + \hat{z}_n^I + \beta_n^R(\hat{q}_{n-1} - q_{n-1}) + \beta_n^R\tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}\big(\tilde{v} - \hat{p}_{n-1}\big)\Big) \\
&= -\lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)} + \hat{z}_n^I\Big),
\end{aligned}
$$

The dynamics (A.7) are found similarly using expressions (2.14)-(2.15) and (2.19)-(2.20).

The expression for the variance (A.9) is found as follows:

$$
\begin{aligned}
\mathbb{V}[\hat{z}_n^I] &= \mathbb{V}\left[\beta_n^R\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \mathbb{E}\left[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}|\sigma(\tilde{v}, \hat{y}_1, ..., \hat{y}_{n-1})\right]\Big) + \Delta w_n\right] \\
&= \mathbb{V}\left[\beta_n^R\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}(\tilde{v} - \hat{p}_{n-1})\Big)\right] + \sigma_w^2\Delta \\
&= (\beta_n^R)^2\Big(\Sigma_{n-1}^{(1)} - \tfrac{\left(\Sigma_{n-1}^{(3)}\right)^2}{\Sigma_{n-1}^{(2)}}\Big) + \sigma_w^2\Delta.
\end{aligned}
$$

To compute the conditional expectation (A.8), we compute the four individual terms. The first term in (A.8) equals

$$
\begin{aligned}
&\mathbb{E}[(\tilde{v} - p_n)\Delta\theta_n^I|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= (\tilde{v} - p_{n-1})\Delta\theta_n^I - \Delta\theta_n^I\mathbb{E}[\Delta p_n|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= X_{n-1}^{(1)}\Delta\theta_n^I - \Delta\theta_n^I\lambda_n\mathbb{E}[\Delta\theta_n^I + \beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1})|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n \\
&\quad - \Delta\theta_n^I\lambda_n\beta_n^R\Big(\hat{\theta}_{n-1}^R - \theta_{n-1}^R + \hat{q}_{n-1} - q_{n-1} + \mathbb{E}[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}|\sigma(\tilde{v}, y_1, ..., y_{n-1})]\Big) \\
&= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n - \Delta\theta_n^I\lambda_n\beta_n^R\Big(\hat{\theta}_{n-1}^R - \theta_{n-1}^R + \hat{q}_{n-1} - q_{n-1} + \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}(\tilde{v} - \hat{p}_{n-1})\Big) \\
&= X_{n-1}^{(1)}\Delta\theta_n^I - (\Delta\theta_n^I)^2\lambda_n - \Delta\theta_n^I\lambda_n\beta_n^R X_{n-1}^{(2)}.
\end{aligned}
$$

The second term in (A.8) is

$$
\begin{aligned}
&\mathbb{E}[\big(X_n^{(1)}\big)^2|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \Big(X_{n-1}^{(1)}\Big)^2 + 2X_{n-1}^{(1)}\mathbb{E}[\Delta X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] + \mathbb{E}[\big(\Delta X_n^{(1)}\big)^2|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \Big(X_{n-1}^{(1)}\Big)^2 - 2\lambda_n X_{n-1}^{(1)}\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big) + \lambda_n^2\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)^2 + \lambda_n^2\mathbb{V}[\hat{z}_n^I].
\end{aligned}
$$

The third term in (A.8) is

$$
\begin{aligned}
&\mathbb{E}[X_n^{(1)}X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= X_{n-1}^{(1)}X_{n-1}^{(2)} + X_{n-1}^{(1)}\mathbb{E}[\Delta X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] + X_{n-1}^{(2)}\mathbb{E}[\Delta X_n^{(1)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&+ \mathbb{E}[\Delta X_n^{(1)}\Delta X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= X_{n-1}^{(1)}X_{n-1}^{(2)} - X_{n-1}^{(1)}\Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big) - X_{n-1}^{(2)}\lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big) \\
&+ \lambda_n\Big(\Delta\theta_n^I + \beta_n^R X_{n-1}^{(2)}\Big)\Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big) + \lambda_n^2\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\mathbb{V}[\hat{z}_n^I].
\end{aligned}
$$

Finally, the last term in (A.8) is

$$
\begin{aligned}
&\mathbb{E}[\big(X_n^{(2)}\big)^2|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \Big(X_{n-1}^{(2)}\Big)^2 + 2X_{n-1}^{(2)}\mathbb{E}[\Delta X_n^{(2)}|\sigma(\tilde{v}, y_1, ..., y_{n-1})] + \mathbb{E}[\big(\Delta X_n^{(2)}\big)^2|\sigma(\tilde{v}, y_1, ..., y_{n-1})] \\
&= \Big(X_{n-1}^{(2)}\Big)^2 - 2X_{n-1}^{(2)}\Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big) \\
&+ \Big(r_n\Delta\theta_n^I + (1+r_n)\beta_n^R X_{n-1}^{(2)}\Big)^2 + \lambda_n^2\left(\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\right)^2\mathbb{V}[\hat{z}_n^I].
\end{aligned}
$$

$\diamondsuit$

Theorem A.3. *Fix $\Delta\theta_n^R$ by* (1.5) *and let the constants* (2.1) *and associate terms* (2.35) *satisfy the pricing coefficient relations* (2.14)-(2.17)*, the variances and covariance recursions* (2.18)-(2.20)*, the value function coefficient recursions* (A.22)-(A.24) *and the second-order-condition* (2.26)*. Then the hedge fund's value function has the quadratic form* (2.24) *where $X_n^{(1)}$ and $X_n^{(2)}$ are defined in* (2.22) *and Δp_n is defined by* (1.10)*. Furthermore, the hedge fund's optimal trading strategy is given by* (2.25) *with coefficients*

$$\gamma_n^{(1)} := \frac{-1+I_n^{(1,2)}r_n+2I_n^{(1,1)}\lambda_n}{2(I_n^{(2,2)}r_n^2+\lambda_n(-1+I_n^{(1,2)}r_n+I_n^{(1,1)}\lambda_n))}, \tag{A.10}$$

$$\gamma_n^{(2)} := -\beta_n^R + \frac{-2I_n^{(2,2)}r_n(-1+\beta_n^R)+I_n^{(1,2)}\lambda_n-\beta_n^R\lambda_n(I_n^{(1,2)}+1)}{2(I_n^{(2,2)}r_n^2+\lambda_n(-1+I_n^{(1,2)}r_n+I_n^{(1,1)}\lambda_n))}. \tag{A.11}$$

Proof. We prove the theorem by the backward induction. Suppose that (2.24) holds for $n+1$. The hedge fund's value function in the n'th iteration then becomes

$$\begin{aligned}&\max_{\substack{\Delta\theta_k^I\in\sigma(\tilde{v},y_1,...,y_{k-1})\\ n\le k\le N}} \mathbb{E}\Big[\sum_{k=n}^N(\tilde{v}-p_k)\Delta\theta_k^I\Big|\sigma(\tilde{v},y_1,...,y_{n-1})\Big]\\ &=\max_{\Delta\theta_n^I\in\sigma(\tilde{v},y_1,...,y_{n-1})}\mathbb{E}\Big[(\tilde{v}-p_n)\Delta\theta_n^I+I_n^{(0)}+\sum_{1\le i\le j\le 2}I_n^{(i,j)}X_n^{(i)}X_n^{(j)}\Big|\sigma(\tilde{v},y_1,...,y_{n-1})\Big].\end{aligned} \tag{A.12}$$

Because (2.26) holds, Lemma A.2 shows that the coefficient in front of $(\Delta\theta_n^R)^2$ appearing in (A.12) is strictly negative. Consequently, the first-order condition is sufficient for optimality and the maximizer is (2.25). The value function coefficient recursions (A.22)-(A.24) are obtained by inserting the optimizer (2.25) into (A.12).

$\diamondsuit$

A.3 Rebalancer's optimization problem

In the following analogue of Lemma A.2 we recall that the rebalancer's state variables $(Y_n^{(1)}, Y_n^{(2)}, Y_n^{(3)})$ are defined in (2.29).

Lemma A.4. *We define $\Delta\theta_n^I$ by* (1.6) *and let the constants* (2.1) *and associate terms* (2.35) *satisfy the pricing coefficient relations* (2.14)-(2.17) *and the variances and covariance recursions* (2.18)-(2.20). *Let $\Delta\theta_n^R\in\sigma(\tilde{a},y_1,...,y_{n-1})$, $n=1,...,N$ be arbitrary for the rebalancer. We define the Gaussian random variables*

$$\hat{z}_n^R := \hat{y}_n - \Delta\hat{\theta}_n^R - \beta_n^I\tfrac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}(\tilde{a}-\hat{\theta}_{n-1}^R-\hat{q}_{n-1}), \quad n=1,...,N. \tag{A.13}$$

Then $\hat{z}_k^R$ is independent of $(\tilde{a},\hat{y}_1,....,\hat{y}_{k-1})$ for $k\le N$ and the following measurability *properties are satisfied*

$$\sigma(\tilde{a}, y_1, ..., y_k) = \sigma(\tilde{a}, \hat{y}_1, ..., \hat{y}_k) = \sigma(\tilde{a}, \hat{z}_1^R, ..., \hat{z}_k^R). \tag{A.14}$$

Furthermore, for $n = 1, ..., N$, we have the Markovian dynamics

$$\Delta Y_n^{(2)} = -\lambda_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big) - r_n \tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}} \hat{z}_n^R, \quad Y_0^{(2)} = \frac{\sigma_{\tilde{v}}\rho}{\sigma_{\tilde{a}}}\tilde{a}, \tag{A.15}$$

$$\Delta Y_n^{(3)} = r_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\Big) - (1 + r_n)(\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)} + r_n \hat{z}_n^R, \quad Y_0^{(3)} = 0. \tag{A.16}$$

For constants $L_n^{(1,1)}, L_n^{(1,2)}, L_n^{(1,3)}, L_n^{(2,2)}, L_n^{(2,3)}$, and $L_n^{(3,3)}$ we have the conditional expectation

$$\begin{aligned}
&\mathbb{E}[-(\tilde{a} - \theta_{n-1}^R)\Delta p_n + \sum_{1\le i\le j\le 3} L_n^{(i,j)} Y_n^{(i)} Y_n^{(j)} | \sigma(\tilde{a}, y_1, ..., y_{n-1})] \\
&= -Y_{n-1}^{(1)}\Big(\lambda_n(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}) + \mu_n Y_{n-1}^{(3)}\Big) \\
&+ L_n^{(1,1)}\Big((Y_{n-1}^{(1)} - \Delta\theta_n^R)^2\Big) \\
&+ L_n^{(1,2)}(Y_{n-1}^{(1)} - \Delta\theta_n^R)\big(Y_{n-1}^{(2)} - \lambda_n\big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\big)\big) \\
&+ L_n^{(1,3)}(Y_{n-1}^{(1)} - \Delta\theta_n^R)\big(Y_{n-1}^{(3)} + r_n\big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\big) - (1 + r_n)(\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\big) \\
&+ L_n^{(2,2)}\Big((Y_{n-1}^{(2)})^2 - 2Y_{n-1}^{(2)}\lambda_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big) \\
&\quad + \lambda_n^2\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big)^2 + r_n^2\left(\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\right)^2 \mathbb{V}[\hat{z}_n^R]\Big) \\
&+ L_n^{(2,3)}\Big(Y_{n-1}^{(2)}Y_{n-1}^{(3)} + Y_{n-1}^{(2)}\Big(r_n\big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\big) - (1 + r_n)(\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big) \\
&\quad - Y_{n-1}^{(3)}\lambda_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big) - r_n^2\tfrac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\mathbb{V}[\hat{z}_n^R] \\
&\quad - \lambda_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big)\Big(r_n\big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\big) - (1 + r_n)(\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big)\Big) \\
&+ L_n^{(3,3)}\Big((Y_{n-1}^{(3)})^2 + 2Y_{n-1}^{(3)}\Big(r_n\big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\big) - (1 + r_n)(\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big) \\
&\quad + \Big(r_n\big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\big) - (1 + r_n)(\alpha_n^R + \beta_n^R)Y_{n-1}^{(3)}\Big)^2 + r_n^2 \mathbb{V}[\hat{z}_n^R]\Big),
\end{aligned} \tag{A.17}$$

which is quadratic in $\Delta\theta_n^R$, and where the variance $\mathbb{V}[\hat{z}_n^R]$ is given by

$$\mathbb{V}[\hat{z}_n^R] = (\beta_n^I)^2\Big(\Sigma_{n-1}^{(2)} - \frac{\big(\Sigma_{n-1}^{(3)}\big)^2}{\Sigma_{n-1}^{(1)}}\Big) + \sigma_w^2\Delta. \tag{A.18}$$

Proof. The proof is similar to the proof of Lemma A.2 and is therefore omitted.

$\diamondsuit$

Theorem A.5. *Fix $\Delta\theta_n^I$ by* (1.6) *and let the constants* (2.1) *and associated terms*

(2.35) *satisfy the pricing coefficient relations* (2.14)-(2.17)*, the variances and covariance recursions* (2.18)-(2.20)*, the value function coefficient recursions* (A.25)-(A.31) *and the second-order-condition* (2.33)*. Then for* $n = 0, 1, ..., N-1$ *the rebalancer's value function has the quadratic form* (2.31) *where* $(Y_n^{(1)}, Y_n^{(3)}, Y_n^{(3)})$ *are defined by* (2.29) *and* Δp_n *is defined by* (1.10)*. Furthermore, the rebalancer's optimal trading strategy is given by* (2.32) *with coefficients*

$$\delta_n^{(1)} := \frac{2L_n^{(1,1)} - L_n^{(1,3)} r_n + \lambda_n + L_n^{(1,2)}\lambda_n}{2\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n (L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big)}, \tag{A.19}$$

$$\delta_n^{(2)} := -\beta_n^I + \frac{L_n^{(1,2)} - r_n(L_n^{(2,3)} + L_n^{(1,3)}\beta_n^I) + L_n^{(1,2)}\beta_n^I\lambda_n + 2(L_n^{(1,1)}\beta_n^I + L_n^{(2,2)}\lambda_n)}{2\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n (L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big)}, \tag{A.20}$$

$$\delta_n^{(3)} := \frac{\begin{array}{c}\Big(-2L_n^{(3,3)} r_n - L_n^{(1,3)}(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) + L_n^{(2,3)}\lambda_n \\ +(\alpha_n^R + \beta_n^R)\big(2L_n^{(3,3)} r_n(1 + r_n) + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} - 2L_n^{(2,3)} r_n + 2L_n^{(2,2)}\lambda_n)\big)\Big)\end{array}}{2\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n (L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big)}. \tag{A.21}$$

Proof. The proof is similar to the proof of Theorem A.3 and is therefore omitted.

$\diamondsuit$

A.4 Remaining proof

Proof of Theorem 2.1. Part (iii) of Definition 1.1 holds from Lemma A.1. Parts (i)-(ii) of Definition 1.1 hold from Theorem A.3 and Theorem A.5 as soon as we show that the optimizers (2.25) and (2.32) agree with (2.3) and (2.4). This, however, follows from the equilibrium conditions (2.27) and (2.34).

$\diamondsuit$

A.5 Value function coefficients

The recursion for the hedge fund's value function coefficients is given by

$$I_{n-1}^{(1,1)} = \frac{-1 + r_n(2I_n^{(1,2)} - (I_n^{(1,2)})^2 r_n + 4I_n^{(1,1)} I_n^{(2,2)} r_n)}{4(I_n^{(2,2)} r_n^2 + \lambda_n(-1 + I_n^{(1,2)} r_n + I_n^{(1,1)}\lambda_n))}, \tag{A.22}$$

$$I_{n-1}^{(1,2)} = -\frac{(-1 + I_n^{(1,2)} r_n)(I_n^{(1,2)}(-1 + \beta_n^R) + \beta_n^R)\lambda_n + 2I_n^{(2,2)} r_n(-1 + \beta_n^R + r_n\beta_n^R - 2I_n^{(1,1)}(-1 + \beta_n^R)\lambda_n)}{2(I_n^{(2,2)} r_n^2 + \lambda_n(-1 + I_n^{(1,2)} r_n + I_n^{(1,1)}\lambda_n))}, \tag{A.23}$$

$$I_{n-1}^{(2,2)} = \lambda_n \frac{-(I_n^{(1,2)}(-1 + \beta_n^R) + \beta_n^R)^2\lambda_n - 4I_n^{(2,2)}(-1 + \beta_n^R)(-1 + I_n^{(1,1)}\lambda_n + \beta_n^R(1 + r_n - I_n^{(1,1)}\lambda_n))}{4(I_n^{(2,2)} r_n^2 + \lambda_n(-1 + I_n^{(1,2)} r_n + I_n^{(1,1)}\lambda_n))}. \tag{A.24}$$

The recursion for the rebalancer's value function coefficients is given by

$$
\begin{aligned}
L_{n-1}^{(1,1)} &= -\Big((L_n^{(1,3)})^2 r_n^2 - 2(1+L_n^{(1,2)})L_n^{(1,3)} r_n\lambda_n + (1+L_n^{(1,2)})^2\lambda_n^2 \\
&\quad + 4L_n^{(1,1)}(-L_n^{(3,3)}r_n^2 + \lambda_n + L_n^{(2,3)}r_n\lambda_n - L_n^{(2,2)}\lambda_n^2)\Big)\Big/ \qquad \text{(A.25)} \\
&\quad 4\big(L_n^{(1,1)} - L_n^{(1,3)}r_n + L_n^{(3,3)}r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)}r_n + L_n^{(2,2)}\lambda_n)\big), \\
L_{n-1}^{(1,2)} &= -\Big((L_n^{(1,3)}r_n - \lambda_n)(L_n^{(2,3)}r_n + L_n^{(1,3)}r_n\beta_n^I - 2L_n^{(2,2)}\lambda_n) + (L_n^{(1,2)})^2\lambda_n(-1+\beta_n^I\lambda_n) \\
&\quad + L_n^{(1,2)}(r_n(L_n^{(1,3)} - 2L_n^{(3,3)}r_n) + \lambda_n + r_n(L_n^{(2,3)} - 2L_n^{(1,3)}\beta_n^I)\lambda_n + \beta_n^I\lambda_n^2) \\
&\quad + 2L_n^{(1,1)}(-r_n(L_n^{(2,3)} + 2L_n^{(3,3)}r_n\beta_n^I) + (2L_n^{(2,2)} + \beta_n^I + 2L_n^{(2,3)}r_n\beta_n^I)\lambda_n \qquad \text{(A.26)} \\
&\quad - 2L_n^{(2,2)}\beta_n^I\lambda_n^2)\Big)\Big/ 2\big(L_n^{(1,1)} - L_n^{(1,3)}r_n + L_n^{(3,3)}r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)}r_n + L_n^{(2,2)}\lambda_n)\big), \\
L_{n-1}^{(1,3)} &= \Big[(L_n^{(1,3)})^2 r_n(-1+\alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) \\
&\quad + (1+L_n^{(1,2)})\lambda_n\Big(-2L_n^{(3,3)}r_n(-1+\alpha_n^R+\beta_n^R) - L_n^{(2,3)}\lambda_n \\
&\quad + (L_n^{(1,2)} + L_n^{(2,3)})(\alpha_n^R + \beta_n^R)\lambda_n\Big) + 2L_n^{(1,1)}\big(-2L_n^{(3,3)}r_n(-1+\alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) \\
&\quad - L_n^{(2,3)}\lambda_n + (\alpha_n^R+\beta_n^R)\lambda_n(1 + L_n^{(2,3)} + 2L_n^{(2,3)}r_n - 2L_n^{(2,2)}\lambda_n)\big) \qquad \text{(A.27)} \\
&\quad + L_n^{(1,3)}\lambda_n\Big(-1+\alpha_n^R+\beta_n^R + L_n^{(2,3)}r_n(-1+\alpha_n^R+\beta_n^R) \\
&\quad - L_n^{(1,2)}(-1+\alpha_n^R + 2r_n\alpha_n^R + \beta_n^R + 2r_n\beta_n^R) + 2L_n^{(2,2)}\lambda_n - (\alpha_n^R+\beta_n^R)(r_n + 2L_n^{(2,2)}\lambda_n)\Big)\Big]\Big/ \\
&\quad 2(L_n^{(1,1)} - L_n^{(1,3)}r_n + L_n^{(3,3)}r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)}r_n + L_n^{(2,2)}\lambda_n)), \\
L_{n-1}^{(2,2)} &= -\Big[(L_n^{(1,2)})^2(-1+\beta_n^I\lambda_n)^2 - 2L_n^{(1,2)}r_n\big(L_n^{(2,3)} - L_n^{(2,3)}\beta_n^I\lambda_n \\
&\quad + \beta_n^I(-L_n^{(1,3)} + 2L_n^{(3,3)}r_n + L_n^{(1,3)}\beta_n^I\lambda_n)\big) + r_n\Big(\big((L_n^{(2,3)})^2 - 4L_n^{(2,2)}L_n^{(3,3)}\big)r_n \\
&\quad + (L_n^{(1,3)})^2 r_n(\beta_n^I)^2 + L_n^{(1,3)}(4L_n^{(2,2)} + 2L_n^{(2,3)}r_n\beta_n^I - 4L_n^{(2,2)}\beta_n^I\lambda_n)\Big) \qquad \text{(A.28)} \\
&\quad - 4L_n^{(1,1)}\big(L_n^{(2,2)}(-1+\beta_n^I\lambda_n)^2 + r_n\beta_n^I(L_n^{(2,3)} + L_n^{(3,3)}r_n\beta_n^I - L_n^{(2,3)}\beta_n^I\lambda_n)\big)\Big]\Big/ \\
&\quad 4(L_n^{(1,1)} - L_n^{(1,3)}r_n + L_n^{(3,3)}r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)}r_n + L_n^{(2,2)}\lambda_n)),
\end{aligned}
$$

$$\text{(A.29)}$$

$$
\begin{aligned}
L_{n-1}^{(2,3)} = \Big[&\big(L_n^{(1,3)} r_n (L_n^{(2,3)} + L_n^{(1,3)}\beta_n^I) - 2L_n^{(1,1)}(L_n^{(2,3)} + 2L_n^{(3,3)} r_n \beta_n^I)\big)(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) \\
&+ \Big((L_n^{(2,3)})^2 r_n(-1 + \alpha_n^R + \beta_n^R) + 2L_n^{(1,1)} L_n^{(2,3)}\beta_n^I(-1 + \alpha_n^R + 2r_n\alpha_n^R + \beta_n^R + 2r_n\beta_n^R) \\
&+ 4L_n^{(2,2)}(-L_n^{(3,3)} r_n(-1 + \alpha_n^R + \beta_n^R) + L_n^{(1,1)}(\alpha_n^R + \beta_n^R)) \\
&+ L_n^{(1,3)}(L_n^{(2,3)} r_n\beta_n^I(-1 + \alpha_n^R + \beta_n^R) - 2L_n^{(2,2)}(1 + (-1 + r_n)\alpha_n^R + (-1 + r_n)\beta_n^R))\Big)\lambda_n \\
&- 2L_n^{(2,2)}\beta_n^I(L_n^{(1,3)}(-1 + \alpha_n^R + \beta_n^R) + 2L_n^{(1,1)}(\alpha_n^R + \beta_n^R))\lambda_n^2 \\
&+ (L_n^{(1,2)})^2(\alpha_n^R + \beta_n^R)\lambda_n(-1 + \beta_n^I\lambda_n) + L_n^{(1,2)}(L_n^{(2,3)}\lambda_n \\
&- 2L_n^{(3,3)} r_n(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R + \beta_n^I(-1 + \alpha_n^R + \beta_n^R)\lambda_n) \\
&+ L_n^{(2,3)}\lambda_n((-1 + r_n)(\alpha_n^R + \beta_n^R) + \beta_n^I(-1 + \alpha_n^R + \beta_n^R)\lambda_n) \\
&+ L_n^{(1,3)}(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R + \beta_n^I\lambda_n - (1 + 2r_n)\beta_n^I(\alpha_n^R + \beta_n^R)\lambda_n))\Big]\Big/ \\
&2\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big),
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
L_{n-1}^{(3,3)} = -\Big[&(L_n^{(1,3)})^2(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R)^2 + 2L_n^{(1,3)}\lambda_n\big((-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R)\times \\
&\big(L_n^{(2,3)}(-1 + \alpha_n^R + \beta_n^R) - L_n^{(1,2)}(\alpha_n^R + \beta_n^R)\big) - 2L_n^{(2,2)}(-1 + \alpha_n^R + \beta_n^R)(\alpha_n^R + \beta_n^R)\ \lambda_n\big) \\
&- 4L_n^{(1,1)}\Big(L_n^{(3,3)}(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R)^2 + (\alpha_n^R + \beta_n^R)\lambda_n\times \\
&\big(-L_n^{(2,3)}(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) + L_n^{(2,2)}(\alpha_n^R + \beta_n^R)\lambda_n\big)\Big) \\
&+ \lambda_n\Big(\big((L_n^{(2,3)})^2 - 4L_n^{(2,2)}L_n^{(3,3)}\big)(-1 + \alpha_n^R + \beta_n^R)^2\lambda_n + (L_n^{(1,2)})^2(\alpha_n^R + \beta_n^R)^2\lambda_n \\
&- 2L_n^{(1,2)}(-1 + \alpha_n^R + \beta_n^R)(2L_n^{(3,3)}(-1 + \alpha_n^R + r_n\alpha_n^R + \beta_n^R + r_n\beta_n^R) - L_n^{(2,3)}(\alpha_n^R + \beta_n^R)\lambda_n)\Big)\Big]\Big/ \\
&4\big(L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n(L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)}\lambda_n)\big).
\end{aligned}
\tag{A.31}
$$

A.6 Algorithm

This section describes an algorithm for searching numerically for a linear Bayesian Nash equilibrium. The algorithm is similar in logic to the algorithm in Section V in Foster and Viswanathan (1996), except that our algorithm requires three constants as inputs (due to the presence of two strategic agents) whereas Foster and Viswanathan (1996) only requires one constant as an input.

The algorithm starts by taking as inputs three conjectured conditional moments for the final time N round of trading:[29]

$$\Sigma^{(1)}_{N-1} > 0, \quad \Sigma^{(2)}_{N-1} > 0, \quad \Sigma^{(3)}_{N-1} \in \mathbb{R} \quad \text{such that} \quad \big(\Sigma^{(3)}_{N-1}\big)^2 \leq \Sigma^{(1)}_{N-1}\Sigma^{(2)}_{N-1}. \tag{A.32}$$

The algorithm then proceeds through backward induction.

Starting step for trading time N: We need (λ_N, β^I_N) to satisfy (2.14) for $n = N$ and the last two parts of (2.36). Given those two constants (λ_N, β^I_N), we can define

$$\beta^R_N := 1, \quad \alpha^R_N := r_N := 0, \quad \mu_N := -\lambda_N, \quad s_N := -1. \tag{A.33}$$

Because of the rebalancer's terminal constraint, his last round of trading (i.e., at time N) does not involve any optimization, and so we have

$$\mathbb{E}\left[-(\tilde{a} - \theta^R_{N-1})\Delta p_N | \sigma(\tilde{a}, y_1, ..., y_{N-1})\right] = -Y^{(1)}_{N-1}\big(\lambda_N(Y^{(1)}_{N-1} + \beta^I_N Y^{(2)}_{N-1}) - \lambda_N Y^{(3)}_{N-1}\big).$$

This relation implies that the rebalancer's value function coefficients for $n = N - 1$ are given by

$$L^{(1,1)}_{N-1} = -\lambda_N,\ L^{(1,2)}_{N-1} = -\lambda_N \beta^I_N,\ L^{(1,3)}_{N-1} = \lambda_N,\ L^{(2,2)}_{N-1} = L^{(2,3)}_{N-1} = L^{(3,3)}_{N-1} = 0. \tag{A.34}$$

On the other hand, the hedge fund's problem in the last round of trading is similar to her problem in any other round of trading. By inserting the boundary conditions

$$I^{(1,1)}_N = I^{(1,2)}_N = I^{(2,2)}_N = 0$$

into the recursions (A.22)-(A.24), we produce the value function coefficients $I^{(i,j)}_{N-1}$.

Induction step: At each time n the algorithm takes the following terms as inputs:

$$\Sigma^{(1)}_n, \Sigma^{(2)}_n, \Sigma^{(3)}_n, (I^{(i,j)}_n)_{1\leq i\leq j\leq 2}, (L^{(i,j)}_n)_{1\leq i\leq j\leq 3}. \tag{A.35}$$

We first find the constants $(\lambda_n, r_n, \Sigma^{(1)}_{n-1}, \Sigma^{(2)}_{n-1}, \Sigma^{(3)}_{n-1}, \beta^I_n, \beta^R_n)$ by requiring that (2.14)-(2.15), (2.18)-(2.20) with $\Sigma^{(1)}_{n-1} > 0, \Sigma^{(2)}_{n-1} > 0$ and $\big(\Sigma^{(3)}_{n-1}\big)^2 \leq \Sigma^{(1)}_{n-1}\Sigma^{(2)}_{n-1}$, monotonicity

[29] We do not take the post-trade date N moments $(\Sigma^{(1)}_N, \Sigma^{(2)}_N, \Sigma^{(3)}_N)$ as inputs because they are after the last round of trading. In addition, (2.18) and (2.20) together with the terminal condition $\beta^R_N = 1$ imply that $\Sigma^{(1)}_N = \Sigma^{(3)}_N = 0$.

of $\Sigma^{(2)}_{n-1}$, (2.27), the first part of (2.34), as well as the second-order conditions (2.26)-(2.33) hold. These are seven polynomial equations in seven unknown constants. We can then subsequently define (μ_n, s_n) by (2.16)-(2.17) and α^R_n by the second part of (2.34).

Next, the value function coefficients $(I^{(i,j)}_{n-1})_{1\leq i\leq j\leq 2}$ and $(L^{(i,j)}_{n-1})_{1\leq i\leq j\leq 3}$ at time $n-1$ are found by the recursions (A.22)-(A.24) and (A.25)-(A.31).

Termination: The iteration above is continued back to time $n = 0$. If the resulting values at time $n = 0$ satisfy (2.11) the algorithm terminates and the computed coefficients produce a linear Bayesian Nash equilibrium. Otherwise, we adjust the conjectured starting input values in (A.32) and start the algorithm all over.

B Modified Foster and Viswanathan (1994)

Our modification of the Foster and Viswanathan (1994) model has N periods of trade after which the traded security pays off $\tilde{v} \sim N(0, \sigma^2_{\tilde{v}})$ at time $N+1$. Four types of investors trade: First, a strategic risk-neutral investor who knows $\tilde{v}$ at time 0 and who trades dynamically over time using orders $\Delta\theta^I_n$. Second, a strategic risk-neutral less-informed investor who receives an initial signal $\tilde{a} \sim N(0, \sigma^2_{\tilde{a}})$ with $\tilde{a}$ and $\tilde{v}$ being jointly normally distributed random variables with $\mathrm{corr}(\tilde{a}, \tilde{v}) = \rho \in (0,1)$ and who trades dynamically using orders $\Delta\theta^L_n$. The "L" superscript here denotes that this second investor is "less" informed than the first (better-informed) investor with superscript "I". Third, noise traders submit random orders $\Delta w_n \sim N(0, \sigma^2_w \Delta)$ which are independent of $(\tilde{v}, \tilde{a})$. Fourth, competitive risk-neutral market makers see the aggregate order flow at each date

$$y_n := \Delta\theta^I_n + \Delta\theta^L_n + \Delta w_n, \quad y_0 := 0, \tag{B.1}$$

and set prices p_n at which they then clear the market.

In our modified FV model, the better-informed investor does not know $\tilde{a}$, whereas in the original Foster and Viswanathan (1994) the better-informed investor knows both $\tilde{v}$ and $\tilde{a}$. Thus, except for the rebalancing constraint, the modified FV model has the identical information structure as in our model of strategic rebalancing.

A *Bayesian Nash Equilibrium* for the modified FV model consists of: (i) Order strategies that, at each time n, maximize the expected profits of the better-informed and less-informed investors given their their respective information sets $\sigma(\tilde{v}, y_1, \ldots, y_{n-1})$ and $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$, and (ii) A pricing rule that sets prices to be conditional expectations

$$p_n = \mathbb{E}[\tilde{v}|\sigma(y_1, ..., y_n)], \quad n = 1, ..., N. \tag{B.2}$$

Our goal is to find a linear equilibrium in which the price dynamics are given by

$$\Delta p_n = \lambda_n y_n, \quad p_0 := 0. \tag{B.3}$$

The two informed investors' optimal orders take the form:

$$\Delta\theta^I_n = \beta^I_n(\tilde{v} - p_{n-1}), \quad \theta^I_0 := 0, \tag{B.4}$$
$$\Delta\theta^L_n = \beta^L_n(s_{n-1} - p_{n-1}), \quad \theta^L_0 := 0. \tag{B.5}$$

In (B.5) the process s_n denotes the less-informed investor's expectation of the stock payoff $\tilde{v}$ after trade at date n; that is,

$$s_n = \mathbb{E}[\tilde{v}|\sigma(\tilde{a}, y_1, ..., y_n)], \quad s_0 := \rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\,\tilde{a}. \tag{B.6}$$

The dynamics of s_n are given by

$$\begin{aligned}\Delta s_n &= \phi_n\Big(y_n - \mathbb{E}(y_n|\sigma(\tilde{a}, y_1, \ldots, y_{n-1})\Big)\\ &= \phi_n\Big(y_n - (\beta^L_n + \beta^I_n)(s_{n-1} - p_{n-1})\Big)\\ &= \phi_n\Big(\Delta w_n + \beta^I_n(\tilde{v} - s_{n-1})\Big).\end{aligned} \tag{B.7}$$

In particular, the less-informed investor learns about $\tilde{v}$ by updating based on the observed order flow. Because the better-informed investor knows $\tilde{v}$ initially, she does not update her expectations about $\tilde{v}$ over time. The Internet Appendix presents sufficient conditions for a linear Bayesian Nash equilibrium to exist in the modified FV model.

Finally, we remark that unlike in the rebalancing model considered in the main part of the paper, there are no predictable components of the order flow process (predictable for the market makers). Consequently, no q_n process is present and the the aggregate order flow process becomes a martingale with respect to the flow of public information.

C Internet Appendix

C.1 Sufficient conditions for the modified FV model

Our derivation of sufficient conditions for a linear Bayesian Nash equilibrium for the modified FV model follows the same logic as in our dynamic rebalancing model. Given a set $(\lambda_n, \phi_n, \beta_n^I, \beta_n^L)_{n=1}^N$ of model parameters, we define the following set of "hat"-processes:

$$\Delta\hat{\theta}_n^I := \beta_n^I(\tilde{v} - \hat{p}_{n-1}), \quad \hat{\theta}_0^I := 0, \tag{C.1}$$

$$\Delta\hat{\theta}_n^L := \beta_n^L(\hat{s}_{n-1} - \hat{p}_{n-1}), \quad \hat{\theta}_0^L := 0, \tag{C.2}$$

$$\hat{y}_n := \Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^L + \Delta w_n, \quad \hat{y}_0 := 0, \tag{C.3}$$

$$\Delta\hat{p}_n := \lambda_n \hat{y}_n, \quad \hat{p}_0 := 0, \tag{C.4}$$

$$\Delta\hat{s}_n := \phi_n\Big(\hat{y}_n - (\beta_n^L + \beta_n^I)(\hat{s}_{n-1} - \hat{p}_{n-1})\Big), \quad \hat{s}_0 := \rho\frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}. \tag{C.5}$$

These processes must satisfy a variety of restrictions to be a linear Bayesian equilibrium. We derive these restrictions in two steps.

Step 1: The conjectured price and less-informed investor expectation processes must satisfy:

$$\hat{p}_n = \mathbb{E}[\tilde{v}|\sigma(\hat{y}_1, ..., \hat{y}_n)], \tag{C.6}$$

$$\hat{s}_n = \mathbb{E}[\tilde{v}|\sigma(\tilde{a}, \hat{y}_1, ..., \hat{y}_n)]. \tag{C.7}$$

We define the conditional moments for $n = 1, \dots N$:

$$\Sigma_n^{(1)} := \mathbb{V}\big[\tilde{v} - \hat{p}_n\big], \tag{C.8}$$

$$\Sigma_n^{(2)} := \mathbb{V}\big[\hat{s}_n - \hat{p}_n\big], \tag{C.9}$$

$$\Sigma_n^{(3)} := \mathbb{E}\big[\big(\hat{s}_n - \hat{p}_n\big)(\tilde{v} - \hat{p}_n)\big] = \Sigma_n^{(2)}, \tag{C.10}$$

where the last equality follows from iterated expectations. The starting values are:

$$\Sigma_0^{(1)} = \sigma_{\tilde{v}}^2, \quad \Sigma_0^{(2)} = \mathbb{V}\Big[\rho \frac{\sigma_{\tilde{v}}}{\sigma_{\tilde{a}}}\tilde{a}\Big] = \rho^2 \sigma_{\tilde{v}}^2. \tag{C.11}$$

Furthermore, $\Sigma_n^{(1)} \geq \Sigma_n^{(2)}$ because we have

$$0 \leq \mathbb{V}\big[\tilde{v} - \hat{s}_n\big] = \mathbb{V}\big[\tilde{v} - \hat{p}_n + \hat{p}_n - \hat{s}_n\big] = \Sigma_n^{(1)} + \Sigma_n^{(2)} - 2\Sigma_n^{(3)} = \Sigma_n^{(1)} - \Sigma_n^{(2)}. \tag{C.12}$$

The filter dynamics are given by:

$$\begin{aligned}
\Sigma_n^{(1)} &= \mathbb{V}\big[\tilde{v} - \hat{p}_{n-1} - \Delta\hat{p}_n\big] \\
&= \mathbb{V}\big[\tilde{v} - \hat{p}_{n-1} - \lambda_n(\beta_n^I(\tilde{v} - \hat{p}_{n-1}) + \beta_n^L(\hat{s}_{n-1} - \hat{p}_{n-1}) + \Delta w_n)\big] \qquad (C.13) \\
&= (1 - \lambda_n\beta_n^I)^2\Sigma_{n-1}^{(1)} + (\lambda_n\beta_n^L)^2\Sigma_{n-1}^{(2)} - 2\lambda_n\beta_n^L(1 - \lambda_n\beta_n^I)\Sigma_{n-1}^{(3)} + \lambda_n^2\Delta\sigma_w^2, \\
\Sigma_n^{(2)} &= \mathbb{V}\big[\hat{s}_{n-1} + \Delta\hat{s}_n - (\hat{p}_{n-1} + \Delta\hat{p}_n)\big] \\
&= \mathbb{V}\big[\hat{s}_{n-1} + \phi_n(\beta_n^I(\tilde{v} - \hat{p}_{n-1} + \hat{p}_{n-1} - \hat{s}_{n-1}) + \Delta w_n) \\
&\qquad - \hat{p}_{n-1} - \lambda_n(\beta_n^I(\tilde{v} - \hat{p}_{n-1}) + \beta_n^L(\hat{s}_{n-1} - \hat{p}_{n-1}) + \Delta w_n)\big] \qquad (C.14) \\
&= (\beta_n^I)^2(\phi_n - \lambda_n)^2\Sigma_{n-1}^{(1)} + (1 - \beta_n^I\phi_n - \beta_n^L\lambda_n)^2\Sigma_{n-1}^{(2)} \\
&\quad + 2\beta_n^I(\phi_n - \lambda_n)(1 - \beta_n^I\phi_n - \beta_n^L\lambda_n)\Sigma_{n-1}^{(3)} + (\phi_n - \lambda_n)^2\Delta\sigma_w^2.
\end{aligned}$$

To find the equations for the constants λ_n and ϕ_n appearing in (C.4) and (C.5) we need the investors' innovation processes. The informed investor (who knows $\tilde{v}$) has innovations defined by

$$\begin{aligned}
z_n^I :&= \hat{y}_n - \Big(\beta_n^I + \beta_n^L \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}\Big)(\tilde{v} - \hat{p}_{n-1}) \\
&= \Delta w_n + \beta_n^L(\hat{s}_{n-1} - \hat{p}_{n-1}) - \beta_n^L \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}(\tilde{v} - \hat{p}_{n-1}).
\end{aligned} \tag{C.15}$$

The less-informed investor (who knows $\tilde{a}$) learns about $\tilde{v}$ over time by filtering the aggregate order flow process to construct the estimate process s_n given by (C.7). His innovations are defined by

$$\begin{aligned}
z_n^L :&= \hat{y}_n - (\beta_n^I + \beta_n^L)(\hat{s}_{n-1} - \hat{p}_{n-1}) \\
&= \Delta w_n + \beta_n^I(\tilde{v} - \hat{s}_{n-1}).
\end{aligned} \tag{C.16}$$

Finally, the market makers' innovations are defined by

$$z_n^M := \hat{y}_n, \tag{C.17}$$

because all trades of the forms (C.1) and (C.2) are unpredictable for the market makers. Based on the requirement (C.6), we can use (C.17) to obtain the representation

$$\Delta \hat{p}_n = \frac{\mathbb{E}[(\tilde{v} - \hat{p}_{n-1}) z_n^M]}{\mathbb{V}[z_n^M]} z_n^M. \tag{C.18}$$

We can then use the projection theorem for multivariate normals to see that the price coefficient in (C.4) is given by

$$\lambda_n = \frac{\beta_n^I \Sigma_{n-1}^{(1)} + \beta_n^L \Sigma_{n-1}^{(2)}}{(\beta_n^I)^2 \Sigma_{n-1}^{(1)} + (\beta_n^L)^2 \Sigma_{n-1}^{(2)} + 2\beta_n^I \beta_n^L \Sigma_{n-1}^{(2)} + \Delta \sigma_w^2}. \tag{C.19}$$

Similarly, we can use the less-informed investor's innovation process (C.16) to re-write (C.7) as

$$\Delta \hat{s}_n = \frac{\mathbb{E}[(\tilde{v} - \hat{s}_{n-1}) z_n^L]}{\mathbb{V}[z_n^L]} z_n^L. \tag{C.20}$$

Consequently, we find the coefficient requirement

$$\phi_n = \frac{\beta_n^I (\Sigma_{n-1}^{(1)} - \Sigma_{n-1}^{(2)})}{(\beta_n^I)^2 (\Sigma_{n-1}^{(1)} - \Sigma_{n-1}^{(2)}) + \Delta \sigma_w^2}. \tag{C.21}$$

Step 2: The price and updating processes as well as the order flow coefficients also need to be consistent with the two informed investors' optimization problems. First, we consider the better-informed investor where the less-informed investor's strategy is fixed to be the conjectured strategy (B.5). Then, for $\Delta \theta_n^I \in \sigma(\tilde{v}, y_1, ..., y_{n-1})$, we have

$$
\begin{aligned}
&\mathbb{E}[(\tilde{v}-p_n)\Delta\theta_n^I|\tilde{v}-\hat{p}_{n-1}]\\
&=\Delta\theta_n^I\mathbb{E}[\tilde{v}-p_{n-1}-\lambda_n\Delta\theta_n^I-\lambda_n\beta_n^L(s_{n-1}-p_{n-1})|\tilde{v}-\hat{p}_{n-1}]\\
&=\Delta\theta_n^I(\tilde{v}-p_{n-1})-\lambda_n(\Delta\theta_n^I)^2\\
&-\lambda_n\beta_n^L\Delta\theta_n^I\mathbb{E}[s_{n-1}+\hat{s}_{n-1}-\hat{s}_{n-1}+\hat{p}_{n-1}-\hat{p}_{n-1}-p_{n-1}|\tilde{v}-\hat{p}_{n-1}]\\
&=\Delta\theta_n^I(\tilde{v}-p_{n-1})-\lambda_n(\Delta\theta_n^I)^2-\lambda_n\beta_n^L\Delta\theta_n^I\Big(s_{n-1}-\hat{s}_{n-1}+\hat{p}_{n-1}-p_{n-1}+\frac{\Sigma_{n-1}^{(2)}}{\Sigma_{n-1}^{(1)}}(\tilde{v}-\hat{p}_{n-1})\Big)\\
&=\Delta\theta_n^I X_{n-1}^{(1)}-\lambda_n(\Delta\theta_n^I)^2-\lambda_n\beta_n^L\Delta\theta_n^I X_{n-1}^{(2)}, \qquad \text{(C.22)}
\end{aligned}
$$

where we have defined the two state-variables:

$$
X_n^{(1)}:=\tilde{v}-p_n, \quad X_n^{(2)}:=s_n-\hat{s}_n+\hat{p}_n-p_n+\frac{\Sigma_n^{(2)}}{\Sigma_n^{(1)}}(\tilde{v}-\hat{p}_n). \qquad \text{(C.23)}
$$

The dynamics of the first state-variable are given by

$$
\begin{aligned}
\Delta X_n^{(1)}&=-\Delta p_n\\
&=-\lambda_n\Big(\Delta\theta_n^I+\beta_n^L(s_{n-1}-p_{n-1})+\Delta w_n\Big)\\
&=-\lambda_n\Big(\Delta\theta_n^I+\beta_n^L(s_{n-1}-p_{n-1})+z_n^I-\beta_n^L(\hat{s}_{n-1}-\hat{p}_{n-1})+\beta_n^L\frac{\Sigma_{n-1}^{(2)}}{\Sigma_{n-1}^{(1)}}(\tilde{v}-\hat{p}_{n-1})\Big)\\
&=-\lambda_n\Big(\Delta\theta_n^I+\beta_n^L X_{n-1}^{(2)}+z_n^I\Big). \qquad \text{(C.24)}
\end{aligned}
$$

Similarly, by using (C.13)-(C.14) and (C.19)-(C.21) we find the dynamics of the second state-variable to be:

$$
\Delta X_n^{(2)}=(\phi_n-\lambda_n)\Delta\theta_n^I-(\phi_n\beta_n^I+\lambda_n\beta_n^L)X_{n-1}^{(2)}-\lambda_n\frac{\Sigma_n^{(2)}}{\Sigma_n^{(1)}}z_n^I. \qquad \text{(C.25)}
$$

Second, we consider the less-informed investor and here the fully informed investor's strategy is fixed as in (B.4). Then, for $\Delta\theta_n^L \in \sigma(\tilde{a}, y_1, ..., y_{n-1})$, we have

$$\begin{aligned}
\mathbb{E}[(\tilde{v} - p_n)\Delta\theta_n^L|\tilde{a} - \hat{p}_{n-1}] &= \Delta\theta_n^L \mathbb{E}[(\tilde{v} - p_{n-1} - \lambda_n \Delta\theta_n^L - \lambda_n \beta_n^I(\tilde{v} - p_{n-1})|\tilde{a} - \hat{p}_{n-1}] \\
&= -\lambda_n(\Delta\theta_n^L)^2 + (1 - \lambda_n\beta_n^I)\Delta\theta_n^L \mathbb{E}[\tilde{v} - p_{n-1}|\tilde{a} - \hat{p}_{n-1}] \\
&= -\lambda_n(\Delta\theta_n^L)^2 + (1 - \lambda_n\beta_n^I)\Delta\theta_n^L\Big(\hat{s}_{n-1} - p_{n-1}\Big) \\
&= -\lambda_n(\Delta\theta_n^L)^2 + (1 - \lambda_n\beta_n^I)\Delta\theta_n^L(Y_{n-1}^{(2)} + Y_{n-1}^{(1)}), \qquad \text{(C.26)}
\end{aligned}$$

where we have defined the two state-variables:

$$Y_n^{(1)} := s_n - p_n, \quad Y_n^{(2)} := \hat{s}_n - s_n. \qquad \text{(C.27)}$$

Similarly to the fully informed investor considered before, we find the dynamics

$$\begin{aligned}
\Delta Y_n^{(1)} &= \phi_n\Big(z_n^L + \beta_n^I Y_{n-1}^{(2)} + \Delta\theta_n^L - \beta_n^L Y_{n-1}^{(1)}\Big) - \lambda_n\Big(\beta_n^I(Y_{n-1}^{(1)} + Y_{n-1}^{(2)}) + \Delta\theta_n^L + z_n^L\Big), \\
\Delta Y_n^{(2)} &= -\phi_n\Big(\beta_n^I Y_{n-1}^{(2)} + \Delta\theta_n^L - \beta_n^L Y_{n-1}^{(1)}\Big).
\end{aligned}$$

Based on the above dynamics of the state-variables (C.23) we see that the fully informed investor's problem (C.22) is a quadratic maximization problem. Therefore, subject to second-order conditions, we find optimal orders $\Delta\hat{\theta}_n^I$ which are linear in the state-variables (C.23). Similarly, given the above dynamics of the state-variables (C.27), we see that the less-informed investor's problem (C.26) is also quadratic with linear optimal orders $\Delta\hat{\theta}_n^L$. By inserting the respective optimal linear orders into their respective quadratic optimization problems, we find recursions for the coefficients describing the two quadratic value functions.

C.2 Expected profits for constrained rebalancer strategies

Table 2 below shows the rebalancer's expected trading profits conditional on the target $\tilde{a}$ for each of the constrained strategies (1 through 5) considered in Table 1 and for the rebalancer's equilibrium strategy ("Equilibrium"). Given risk neutrality and the linearity of the prices and informed orders, the rebalancer's value function is quadratic in $\tilde{a}$. We average over $\tilde{v}$ and the noise trader orders for various market parameterizations. The rebalancer's value function based on the equilibrium strategy includes an additional term that reflects the contribution of dynamic trading (versus deterministic trading) when using the equilibrium strategy due to dynamics in sunshine trading (due to the impact of the random aggregate orders y_n through q_n) and endogenous learning (due to the impact of y_n on $\tilde{a} - \theta_n^R - q_n$ through q_n). There are several things to note in Table 2: First, the rebalancer's expected profits when $\tilde{a} \neq 0$ are negative when ρ is zero or sufficiently small. This is because the rebalancer's orders on average push the price away from $\tilde{v}$. The rebalancer's expected profits increase significantly when the rebalancer splits his orders over time relative to just trading once either at date 1 or at an optimally chosen single date. In addition, taking the intraday pattern of price impact into account also has a significant effect. However, the incremental impact of sunshine trading seems small. Lastly, the incremental impact of dynamic (rather than deterministic) trading — which takes both endogenous learning and sunshine trading dynamics into account — also has only a small effect relative to deterministic trading taking just intraday price impact differences into account.

Table 2: Expected rebalancing cost for various strategies (see list in the paper) conditional on target $\tilde{a}$. The parameters are $N := 10$, $\sigma_w^2 := 4$, $\sigma_{\tilde{v}}^2 := 1$, $\rho := 0$ (top panel only) and $\sigma_{\tilde{a}}^2 := 1$ (lower panel only).

Strategy	$\sigma_{\tilde{a}}^2 := 0.48$	$\sigma_{\tilde{a}}^2 := 1$	$\sigma_{\tilde{a}}^2 := 3.7$
1	$-0.4364\,\tilde{a}^2$	$-0.4489\,\tilde{a}^2$	$-0.4830\,\tilde{a}^2$
2	$-0.3669\,\tilde{a}^2$	$-0.3660\,\tilde{a}^2$	$-0.3569\,\tilde{a}^2$
3	$-0.1402\,\tilde{a}^2$	$-0.1403\,\tilde{a}^2$	$-0.1278\,\tilde{a}^2$
4	$-0.1267\,\tilde{a}^2$	$-0.1261\,\tilde{a}^2$	$-0.1142\,\tilde{a}^2$
5	$-0.1267\,\tilde{a}^2$	$-0.1259\,\tilde{a}^2$	$-0.1121\,\tilde{a}^2$
Equilibrium	$0.00008 - 0.1267\,\tilde{a}^2$	$0.00026 - 0.1259\,\tilde{a}^2$	$0.0014 - 0.1121\,\tilde{a}^2$
	$\rho := 0.24$	$\rho := 0.47$	$\rho := 0.86$
1	$-0.2805\,\tilde{a}^2$	$-0.1378\,\tilde{a}^2$	$0.0401\,\tilde{a}^2$
2	$-0.2533\,\tilde{a}^2$	$-0.0991\,\tilde{a}^2$	$0.1251\,\tilde{a}^2$
3	$-0.0114\,\tilde{a}^2$	$0.1029\,\tilde{a}^2$	$0.2783\,\tilde{a}^2$
4	$-0.0033\,\tilde{a}^2$	$0.1187\,\tilde{a}^2$	$0.3111\,\tilde{a}^2$
5	$-0.0029\,\tilde{a}^2$	$0.1194\,\tilde{a}^2$	$0.3120\,\tilde{a}^2$
Equilibrium	$0.0034 - 0.0029\,\tilde{a}^2$	$0.0150 + 0.1194\,\tilde{a}^2$	$0.1031+0.3120\,\tilde{a}^2$