Biases in Variance of Decomposed Portfolio Returns

Vitali Alexeev[♯,*], Katja Ignatieva[♮]

♯ Finance Discipline Group, UTS Business School, University of Technology Sydney
Sydney, New South Wales 2007, Australia
♮ School of Risk & Actuarial Studies, University of New South Wales,
Sydney, New South Wales 2052, Australia

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Abstract

Significant portfolio variance biases arise when contrasting multi-period portfolio returns based on the assumption of fixed continuously rebalanced portfolio weights as opposed to buy-and-hold weights. Empirical evidence obtained using S&P500 constituents from 2003 to 2011 demonstrates that, compared with a buy-and-hold assumption, applying fixed weights led to decreased estimates of portfolio volatilities during 2003, 2005 and 2010, but caused a significant increase in volatility estimates in the more turbulent 2008 and 2011. This discrepancy distorts assessments of portfolio risk-adjusted performance when inappropriate weight assumptions are employed. Consequently, for individual investors, who in practice often employ buy-and-hold strategy, the portfolio size recommendations required to achieve the most diversification benefits are typically understated.

Keywords: Portfolio diversification, buy-and-hold strategy, portfolio risk, high-frequency data.
JEL: G11, C58, C63

*Corresponding author: Email: vitali.alexeev@uts.edu.au; Phone: +61 2 9514 7781.

1 Introduction

A common approach in the finance literature for calculating multi-period portfolio returns is to adopt a rebalancing strategy that maintains a fixed weight of each asset in a portfolio at any time.[1] In contrast, if a buy-and-hold strategy is assumed, asset weights may result in allocations far from the initial distribution when price fluctuations of some portfolio constituents outperform others. This is especially pertinent for longer investment horizons. To illustrate this, Figure 1 presents portfolio weight dynamics for the fixed weight (continuous rebalancing) and the buy-and-hold strategies. For a selection of stocks in the top panels, both strategies maintain similar allocations over time and any differences in portfolio mean returns and portfolio variances are expected to be negligible. On the contrary, the bottom panels show that for a different selection of stocks in the portfolio, the buy-and-hold strategy leads to a portfolio that is not well diversified (right bottom panel). In fact, one could argue that this portfolio behaves similarly to a two-stock portfolio, particularly towards the end of the period. In this case, large biases in both the average portfolio return and the portfolio variance may be expected.[2] In evaluating portfolio performance using multi-period portfolio returns an appropriate assumption on asset weights must be employed to avoid biases in estimates of the first and second moments of portfolio returns.

Estimates of portfolio average return and risk will depend on whether the assumption of fixed or buy-and-hold weights is employed. Buy-and-hold weights ensure that compounding the decomposed multi-period portfolio returns yields the returns earned by an investor who holds the portfolio. In contrast, studies that employ fixed portfolio weights for simplicity, often inadvertently assume a rebalancing frequency matching that of the data used for analysis. For example, an equally weighted portfolio is often calculated as the arithmetic average of individual stock returns in periods corresponding to the frequency of the return data (monthly, weekly, daily, or at higher frequencies). In practice, rebalancing on daily, or perhaps even weekly or monthly, basis to maintain portfolio weights in equal proportion might not be viable.

By comparing rebalanced returns with decomposed buy-and-hold returns, Liu and Strong [2008] find that rebalancing assumption causes an upward bias to the size premium and a downward bias to the momentum effect. Through empirical exercise, the authors show that the two methods can produce a portfolio return difference of more than 8% per year, and can lead to different statistical inferences.

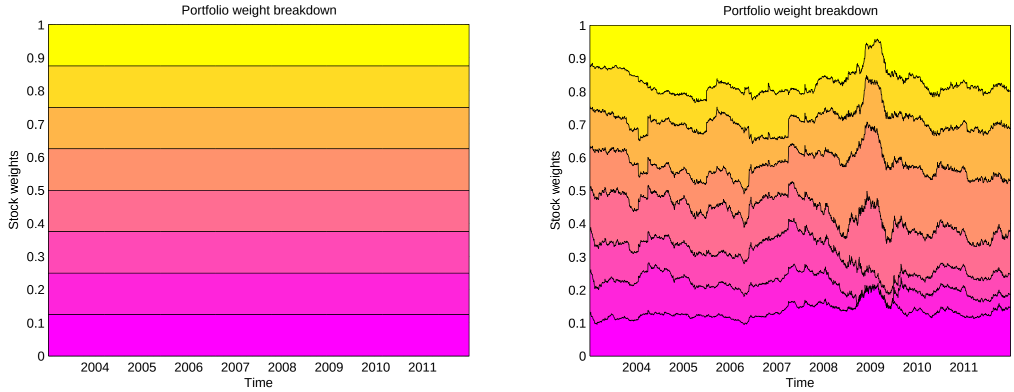
Inspired by this conjecture, we extend the analysis to the second moment and argue that, in the mean-variance framework, considering only the first moment of portfolio return dis-

[1] A few notable works that apply a rebalancing method to calculate multi-period portfolio returns include Fama and French [1996], Carhart [1997], Daniel et al. [1997], Lee and Swaminathan [2000], along with more recent ones of Chan et al. [2002], Ahn et al. [2003], Teo and Woo [2004], Cohen et al. [2005], Nagel [2005], Diether et al. [2009], Huang et al. [2010], Hou et al. [2011].

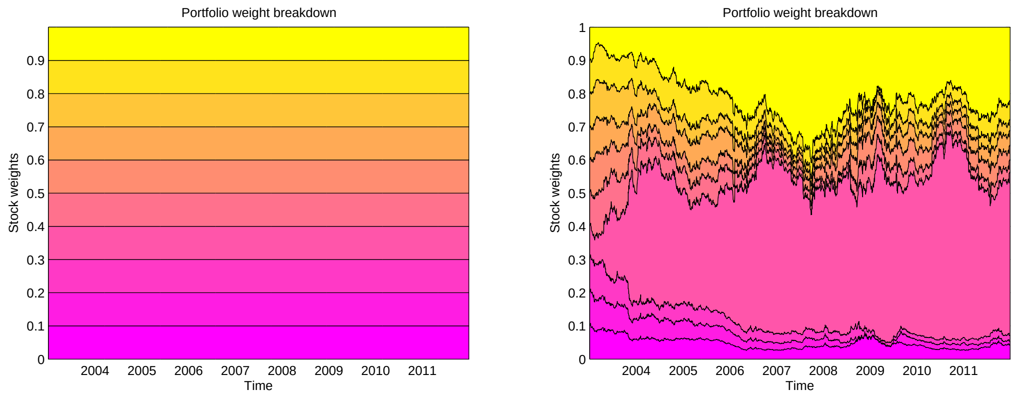
[2] We use the term *bias* to define the discrepancy in portfolio mean return and variance estimates when using the fixed weight (continuous rebalancing) versus the buy-and-hold strategies thus retaining the definition set out in Liu and Strong [2008].

Figure 1: Portfolio weight dynamics for rebalanced (left panels) and buy-and-hold (right panels) strategies.

(a) Small bias example. Portfolio comprises AEP.N, AIG.N, AIV.N, AMGN.OQ, APA.N, APC.N, APH.N, ASH.N (company names associated with the listed RIC codes can be found in supplementary appendix). Provided they had been active for the full period from January 2003 to December 2011, stocks were chosen in alphabetical order. Biases in portfolio mean return and portfolio variance are expected to be negligible since both strategies maintain similar portfolio composition throughout the period.



(b) Large bias example. AKAM.OQ and ATI.N are added to the list of 8 stocks in the panel above. The buy-and-hold portfolio (on the right) is not as well-diversified as the rebalanced portfolio (on the left). Large biases in portfolio mean return and portfolio variance are expected.

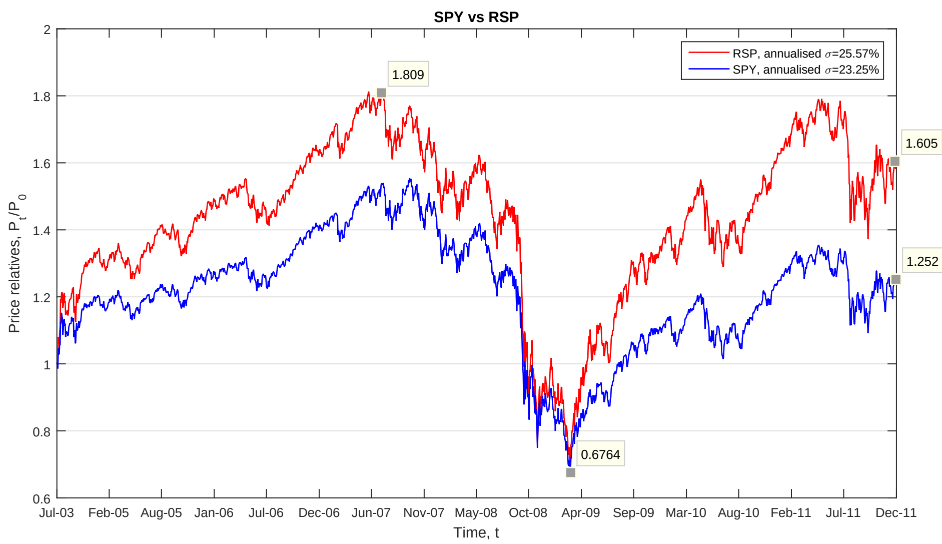


tribution may result in incomplete judgement with respect to portfolio risk and investment decisions. We show that employing a wrong assumption on weights in decomposition of portfolio returns leads to significant discrepancies in portfolio variance estimates. In turn, this may result in erroneous inference about portfolio risk or risk-adjusted portfolio performance. The importance of biases in portfolio variance cannot be overstated. In the last two decades, empirical studies began to rely on data at higher frequencies in evaluating portfolio performance or testing asset pricing models; the portfolio rebalancing frequency may be catching up but at a much slower pace.[3]

[3] Data at higher frequencies demonstrably improves the estimation of risk. A range of efficient estimators has been developed offering a more accurate estimation of financial risk (see McAleer and Medeiros [2008] for a survey on realized estimators). This is also in line with Fama and French [1998, p.1977] who argue that the "annual returns suffice for estimating expected returns, but tests of asset pricing models (which also require second moments) are hopelessly imprecise unless returns for shorter intervals are used".

A rebalancing strategy avoids concentrating in a few stocks and should not increase portfolio risk, but if it does, it should produce higher risk-adjusted returns. This strategy would be the most appropriate for an index or a fund that aims to maintain constant positions, avoiding over-concentration in well-performing stocks and maintaining well-diversified portfolios. Consider, for example, the Guggenheim S&P 500 Equal Weight (NYSEArca: RSP) - an exchange-traded fund (ETF) that tracks the performance of the S&P 500 Equal Weighted USD TR Index. The RSP weighs portfolio holdings equally and rebalances quarterly. Compared to its market-capitalization-weighted counterpart,[4] the S&P 500 ETF (NYSEArca: SPY), the RSP has returned about 35 percent more in total and, despite higher volatility, delivered higher risk-adjusted returns over the period from 2003 to 2011 (see Figure 2). However, a steady outperformance of RSP relative to SPY is not sufficient to adopt a ubiquitous rebalancing strategy. A rebalancing strategy may be tax inefficient and perform poorly in trending markets (see, for example, Perold and Sharpe, 1988). In addition, frequent rebalancing is impractical due to prohibitive transaction costs and few investors would consider rebalancing portfolios on a daily or even a monthly basis. It is common, following significant declines in the financial markets, for investors to question the benefits of rebalancing. Understandably, during the recent financial crisis in 2008, poor investment performance coupled with considerable uncertainty about the future made it seem counterintuitive for investors to rebalance their portfolios by selling their best-performing stocks and committing more capital to underperforming stocks.

Figure 2: S&P 500 ETF (SPY) and Guggenheim S&P 500 Equal Weight ETF (RSP). Relative daily cumulative wealth of the SPY and the RSP from initial investment on July 7, 2003. The RSP is the equally weighted counterpart of the SPY. It has the same constituents as the capitalization weighted S&P 500, but each company is allocated a fixed weight of 0.2 percent quarterly. Since the RSP's inception on April 24, 2003, the reliable trade data for this ETF is traced back only to July 7, 2003. We also adjust the RSP data for the four-for-one split on April 24, 2006.



Indisputably, a buy-and-hold assumption would be more practical for individual in-

[4] If the initial portfolio weights are set proportional to market value, without further adjustments, the buy-and-hold strategy would be equivalent to the market-value-weighted scheme.

vestors.[5] On the other hand, fixed portfolio weight assumption is dominant in portfolio literature. Due to simplicity and tractability of the approach, authors adopting this strategy seem to ignore the associated biases when evaluating buy-and-hold portfolios. Approximate results from a rebalancing strategy "...may suffice for a quick and coarse comparison of investment performance across many assets, but for finer calculations in which the volatility of returns plays an important role ... the approximation may break down." [Campbell et al., 1997, p.10]. Starting from the earlier studies by Roll [1984], Blume and Stambaugh [1983] and Conrad and Kaul [1993] that outline the presence of market microstructure biases and recommend the use of buy-and-hold returns, the recent work by Liu and Strong [2008] discusses in detail the existence of biases resulting from applying the rebalancing method to buy-and-hold portfolios in the U.S. equity market. The authors analyse portfolio returns over a multi-period holding horizon and compute the bias of the *portfolio mean return* in each month as the difference between the average rebalanced return and the decomposed buy-and-hold return. Liu and Strong [2008] demonstrate that rebalancing can lead to spurious statistical inference (the two methods produce a difference in returns of 8% p.a.), and document that rebalancing overstates the size and book-to-market effects, and understates the momentum effect. A more recent empirical study by Gray [2014] uses Australian equities to support the evidence documented in Liu and Strong [2008]. In particular, Gray [2014] shows that the constant weight approach produces significant biases into estimated returns, which, depending on the characteristics of stocks, can approach 1.5% per month.

As outlined above, the existing literature underlines the importance of investigating biases in portfolio mean returns as these can become significant, leading to incorrect inference and investment decisions. Surprisingly, however, the existing works seem to ignore biases that arise in portfolio's higher moments, such as the portfolio variances. Inspired by the work of Liu and Strong [2008] who document significance of the bias in *portfolio mean returns,* we investigate biases that arise in *portfolio variance*s. We define the bias in portfolio variances as the difference between portfolio variance resulting from using the rebalancing strategy and the buy-and-hold approach.

Using large-scale portfolio simulation we confirm our conjecture that, similarly to the bias in portfolio mean returns, biases in portfolio variance exist and can become significant. We test the robustness of our results by constructing a large number of portfolios of randomly selected stocks from the S&P 500 constituent list during the 2003-2011 period. We analyse how often our constructed portfolios differ in terms of estimated portfolio mean returns

[5] As a matter of fact, in 2009, Warren Buffett told PBS "I read a book, what is it, almost 60 years ago roughly, called *The Intelligent Investor* and I really learned all I needed to know about investing from that book, and in particular chapters 8 and 20. . . I haven't changed anything since". Chapter 8 of Benjamin Graham's *The Intelligent Investor* discusses the benefits of a buy-and-hold approach. It reads "...The true investor scarcely ever is forced to sell his shares, and at all other times he is free to disregard the current price quotation. He need pay attention to it and act upon it only to the extent that it suits his book, and no more. Thus the investor who permits himself to be stampeded or unduly worried by unjustified market declines in his holdings is perversely transforming his basic advantage into a basic disadvantage. That man would be better off if his stocks had no market quotation at all, for he would be spared the mental anguish caused him by other persons' mistakes of judgement." (Graham and Zweig, 2003, pp.106-107).

and variances when the fixed weight rebalancing strategy is used instead of the buy-and-hold approach. We show that the portfolio variance bias approaches an asymptotic value as the number of assets in the portfolios increases, indicating the systematic nature of the bias. We find that, compared to the buy-and-hold strategy, rebalancing led to a decrease in volatility of portfolios during 2003, 2005 and 2010, but caused a significant increase in volatility in the more turbulent 2008 and 2011, following the GFC and the European sovereign debt crisis. This is because maintaining equal portfolio weights would require an investor to adopt a buying "losers" and selling "winners" strategy, which would result in a portfolio with elevated volatility due to a large number of "losers" during bear markets and, subsequently, positive variance biases compared to a buy-and-hold portfolio.

Our results indicate that one should exercise caution when assuming multi-period rebalanced portfolio returns, as resulting biases can lead to spurious results when analysing investment strategies or testing asset pricing models. We add to the literature by challenging the equal weight rebalancing strategy that is widely adopted in the literature and demonstrate that continuous rebalancing will not lead to the optimal outcomes in periods of high volatility. The existence of portfolio variance biases during these periods have important implications not only when evaluating the risk of such portfolios, but also when assessing their performance by means of the coefficient of variation, the Sharpe ratio or the signal-to-noise ratio. In addition, we show that for individual investors, who in practice often employ buy-and-hold strategy, the portfolio sizes required to achieve the most diversification benefits are understated.

We emphasise that the results presented in this paper condone neither of the considered strategies. We are not providing a uniform recommendation to market participants on which strategy to adopt. We do find, however, that outcomes from rebalancing strategies appear to be favourable during good economic conditions, but display larger portfolio volatility during economic downturns compared to outcomes of buy-and-hold portfolios. We point out that investors should be consistent when considering decomposed portfolio returns. For a buy-and-hold investor, it would be misleading to calculate portfolio returns using fixed weights and, thus, incorrectly assume periodic rebalancing. For an investor pursuing a fixed weight rebalancing strategy, a constant weight applied to portfolio holdings at the beginning of every period is the only correct approach. Portfolio returns constructed based on the wrong underlying portfolio strategy may lead to biases in portfolio mean returns and portfolio variances. In other words, imposing fixed initial weights when calculating decomposed buy-and-hold portfolio returns may result in biased estimates and incorrect performance metrics.

To summarise, our motivation to compare both, the rebalancing and the buy-and-hold investment strategies, and the resulting portfolio risk, draws on the conclusions from previous academic literature, which suggests that a simple averaging approach introduces significant estimation error. In fact, the estimated returns fail to capture correctly the wealth effects for an investor holding the portfolio, and leads to incorrect statistical inferences in relation to investment strategies. The issues and the results discussed in this paper emphasise the importance of examining portfolio characteristics carefully, and deciding on the investment strategy, knowing possible consequences. This paper will be of interest to researchers testing asset pricing models and practitioners evaluating the performance of investment strategies.

The remainder of the paper is organised as follows. In Section 2 we derive variance bias in multi-period portfolio returns by contrasting the buy-and-hold and the rebalancing methods. In Section 3 we form buy-and-hold and rebalancing portfolios by selecting stocks randomly from the S&P 500 constituent list and analyse the biases empirically. Section 4 contrasts optimal portfolio sizes of well-diversified portfolios under the two portfolio construction methods. In Section 5 we draw conclusions and provide final remarks.

2 Derivations

This section derives biases in portfolio mean returns and portfolio variances. We define bias as the difference between portfolio risk estimates constructed using rebalanced returns and the decomposed buy-and-hold returns. The constructed biases will be analysed in Section 3 when comparing the performance of investors' portfolios constructed using the equally weighted approach and the buy-and-hold strategy.

We begin by focusing on a distinction between decomposed buy-and-hold portfolio returns and rebalanced portfolio returns, assuming that rebalancing is performed every period according to the data sampling frequency.[6] Denoting $P_{i,\tau}$ the price of the ith stock ($i = 1, ..., N$) in a holding period $\tau = 1, ..., T$ where T is the end of the investment horizon, we define individual stock i's simple return for the period τ by $r_{i,\tau} = \frac{P_{i,\tau} - P_{i,\tau-1}}{P_{i,\tau-1}}$. We assume that the investor holds a portfolio of N stocks. When constructing our rebalanced portfolios we adopt the most popular approach - an equally weighted portfolio, choosing the weight of stock i to be $w_i = 1/N$ at the beginning of each holding period τ. We note, however, that the results derived below can be generalised to arbitrary weights w_i with $\sum_{i=1}^{N} w_i = 1$.[7]

For the *rebalanced* portfolios, returns in each holding period τ can be computed as an average of the individual stock returns in that period, that is,

$$r_{reb,\tau} = \frac{1}{N} \sum_{i=1}^{N} r_{i,\tau}. \tag{1}$$

The rebalancing method is inaccurate in reflecting investors' wealth over a multi-period holding horizon, unless the portfolio is rebalanced back to the initial weights at the beginning of each period τ. This, however, appears unrealistic from an investor's perspective, since revisions of portfolio weights are unlikely to occur at regular intervals, especially when taking into account the prohibitive transaction costs associated with frequent periodic rebalancing.

[6] The portfolio is rebalanced every month when using monthly data, every week when using weekly data, and every day when using daily data, etc.

[7] The $1/N$ strategy is often used in practice and its out-performance across a wide range of different asset allocation strategies is well documented in the literature (e.g., DeMiguel et al., 2009).

In practice, new information flow will determine when revisions of weights should take place if at all.[8]

For the *buy-and-hold* portfolio approach, which is a standard and accurate method for measuring the investment performance of the buy-and-hold investor, the return in the first period, $\tau = 1$, can be computed as

$$r_{bh,1} = \frac{1}{N} \sum_{i=1}^{N} r_{i,1}. \tag{2}$$

The returns in any subsequent period,$\tau = 2, ..., T$, are given by

$$r_{bh,\tau} = \frac{1}{\sum_{j=1}^{N} \prod_{t=1}^{\tau-1} (1 + r_{j,t})} \sum_{i=1}^{N} \prod_{t=1}^{\tau-1} (1 + r_{i,t}) \, r_{i,\tau}. \tag{3}$$

Thus, in the first period the buy-and-hold portfolio return corresponds to the average of the individual stock returns in this period, and is equivalent to the return on the rebalanced portfolio. For periods $\tau = 2, ..., T$, the buy-and-hold portfolio returns are computed as the weighted average of period τ stock returns with weights determined by the performance over previous periods. Under the assumption of no auto- and cross-autocorrelation in individual stock returns, the returns for rebalanced portfolios in any two periods are independent, whereas for the buy-and-hold portfolios the returns are dependent in any two periods.

2.1 *Bias in portfolio mean returns*

We denote the average return on the rebalanced portfolio by

$$\bar{r}_\tau = \frac{1}{N} \sum_{i=1}^{N} r_{i,\tau} \tag{4}$$

and thus, the expected return of the rebalanced portfolio is given by

$$E\left(r_{reb,\tau}\right) = E\left[\frac{1}{N} \sum_{i=1}^{N} r_{i,\tau}\right] = E\left[\bar{r}_\tau\right]. \tag{5}$$

First, we derive the return bias for $\tau = 2$, and then generalise it to an arbitrary τ.[9] We use the approximation $1/(1+\bar{r}_\tau) \approx 1 - \bar{r}_\tau$, ignoring higher order terms in the Taylor series expansion. The bias between the expected return of the rebalanced and the buy-and-hold portfolios is given by

$$Bias_2^E = E(r_{reb,2}) - E(r_{bh,2}), \tag{6}$$

and using Eq. (5) and Eq. (3) for $\tau = 2$, we can write

[8] Graham and Zweig [2003] argue that investors are better off adopting a buy-and-hold approach.

[9] We note that there is no bias if the holding period corresponds to a single period ($\tau = 1$). However, one would not consider an investment strategy based on a single period as it is unattractive due to transaction costs; or simply not adequate for constructing a sufficient sample of decomposed portfolio returns for testing asset pricing models.

$$Bias_2^E = E\left[\bar{r}_1\bar{r}_2\right] - \frac{1}{N}\sum_{i=1}^{N} E\left[\left(1-\bar{r}_1\right) r_{i,1} r_{i,2}\right]. \tag{7}$$

The result in Eq. (7) has been documented in Liu and Strong [2008], its derivations appear in Supplementary Appendix A to make this study self-contained.

Assuming that $\bar{r}_1$ is uncorrelated with individual returns $r_{i,1}$ and $r_{i,2}$, as proposed in Liu and Strong [2008], Eq. (7) can further be decomposed as

$$\begin{aligned} Bias_2^E &= E(\bar{r}_1)E(\bar{r}_2) + \underbrace{Cov(\bar{r}_1,\bar{r}_2)}_{>0} - \underbrace{E\left(1-\bar{r}_1\right)}_{>0} \frac{1}{N}\sum_{i=1}^{N} E(r_{i,1})E(r_{i,2}) \\ &\quad + \underbrace{\left[- \underbrace{E\left(1-\bar{r}_1\right)}_{>0} \frac{1}{N}\sum_{i=1}^{N} \underbrace{Cov(r_{i,1}, r_{i,2})}_{<0} \right]}_{>0}. \end{aligned} \tag{8}$$

Eq. (8) indicates that even if returns are independent, the return bias is non-zero. The bias depends on the expected average portfolio return of the rebalanced portfolio, expected individual stock returns, the autocovariance of the portfolio returns and autocovariances of individual stock returns. Following the empirical evidence documented in Lo and Mackinlay [1990] and Mech [1993], portfolio returns are positively autocorrelated, that is, $Cov(\bar{r}_1,\bar{r}_2) > 0$ for the rebalanced portfolio, contributing positively to a bias.[10] Conversely, individual returns are negatively autocorrelated, that is, $Cov(r_{i,1}, r_{i,2}) < 0$, see Fisher [1966], Roll [1984], Lo and Mackinlay [1990], Jegadeesh and Titman [1995].[11] This negative autocorrelation is more pronounced in small and low-price stocks, see Lo and Mackinlay [1990]. Hence, in portfolios comprised of small and low-price stocks, one would expect to observe a positive bias.[12] On the other hand, Kaul and Nimalendran [1990] document positive autocorrelation between stock returns once the bid-ask spread is extracted; which may lead to negative bias in portfolios comprised of large and high-price stocks.

Using Eq. (3) and Eq. (5), we can express the bias in portfolio mean returns for $\tau = 2, ..., T$ as

[10] In fact, transaction costs cause the portfolio return autocorrelation by delaying price adjustment.

[11] The negative autocorrelation in individual returns is caused by nonsynchronous trading (Fisher, 1966), transaction costs and bid-ask spreads (Roll, 1984, Jegadeesh and Titman, 1995).

[12] For instance, Liu [2006] documents high correlation between the returns of infrequently traded stocks and size, as well as the bid-ask spread; and Branch and Freed [1977], Conrad and Kaul [1993] find a negative relationship between price and the bid-ask spread.

$$
\begin{aligned}
Bias^E_\tau &= E(r_{reb,\tau}) - E(r_{bh,\tau}) \\
&= \sum_{i=1}^{N}\left[\frac{1}{N}E(r_{i,\tau}) - E\left(\frac{1}{\sum_{j=1}^{N}\prod_{t=1}^{\tau-1}(1+r_{j,t})}\sum_{i=1}^{N}\prod_{t=1}^{\tau-1}(1+r_{i,t})\,r_{i,\tau}\right)\right].
\end{aligned}
\tag{9}
$$

Generalising the discussion above to an arbitrary τ, we concur that positive bias in portfolio mean returns is most likely to occur in small and low-price stock portfolios, and negative bias is expected in large and high-price stock portfolios. Liu and Strong [2008] note that negative bias can arise when expected stock returns are constant over time but vary cross-sectionally, that is, when high (low) expected returns are associated with higher (lower) expected weights in the buy-and-hold portfolios (second term of Eq. (9)). Rebalancing reverses this effect (first term of Eq. (9)).

The two approximations employed by Liu and Strong [2008] to arrive at a closed form solution in Eq. (7) are restrictive. It is well- known that the distribution of individual securities is negatively skewed. As a result, this approximation may not be valid. In our empirical analysis we avoid using the closed form solution and rely on real data with no assumptions on return distributions.[13]

2.2 *Bias in portfolio variance*

Similar to the computation of the bias in portfolio mean returns, we derive bias in portfolio variance for $\tau = 2$, and generalise it to an arbitrary τ. The variance bias between the rebalanced and the buy-and-hold portfolio is given by

$$
Bias^V_2 = Var(r_{reb,2}) - Var(r_{bh,2}), \tag{10}
$$

where the variance of the rebalanced portfolio is determined by

$$
Var(r_{reb,2}) = Var\left[\bar{r}_2\right] = E\left[\bar{r}_2^2\right] - E\left[\bar{r}_2\right]^2, \tag{11}
$$

and the variance of the buy-and-hold portfolio can be written (see Supplementary Appendix A for details) as

$$
Var(r_{bh,2}) \quad = \quad Var\left[(1-\bar{r}_1)(\bar{r}_2 + \frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2})\right]. \tag{12}
$$

Continuing with the assumption that portfolio returns $\bar{r}_1$ and $\bar{r}_2$ are uncorrelated with

[13] The following result employs the second order Taylor expansion around zero: $1/(1+r) \approx 1 - r + r^2$ for $r \to 0$. Using daily return data for the S&P500 constituents, we estimate, on average, less than 0.01% difference between approximations using the first vs the second order Taylor series.

individual stock returns, $r_{i,1}$ and $r_{i,2}$, and using the approximation $1/(1+\bar{r}_\tau) \approx 1 - \bar{r}_\tau$ as before,[14] $Bias_2^V$ for the variance reduces to

$$\begin{aligned} Bias_2^V &= Var(r_{reb,2}) - Var(r_{bh,2}) \\ &= 2Cov(\bar{r}_2, \bar{r}_1\bar{r}_2) - \underbrace{Var(\bar{r}_1\bar{r}_2)}_{>0} - \underbrace{\frac{1}{N^2} Var\left(\sum_{i=1}^{N} r_{i,1}r_{i,2}\right)}_{>0} - \underbrace{\frac{1}{N^2} Var\left(\bar{r}_1 \sum_{i=1}^{N} r_{i,1}r_{i,2}\right)}_{>0}. \end{aligned} \quad (13)$$

From Eq. (13) we observe that the bias in portfolio variance is not zero; it depends on the autocovariance of portfolio returns, the autocovariance of individual returns, as well as the variance of the sum of product of individual and portfolio returns. Similarly to the bias in portfolio mean return, the bias in portfolio variance can take either positive or negative values, depending on the time-series properties of portfolio returns and individual stock returns. Using the same argument as above, portfolio returns are more likely to be positively autocorrelated, that is $Cov(\bar{r}_1, \bar{r}_2) > 0$, for the rebalanced portfolio (see, for example, Lo and Mackinlay, 1990, Mech, 1993). Hence, positive autocovariance in portfolio returns will contribute to a positive bias in variance.

Eq. (13) can be generalised for $\tau = 2, ..., T$ as follows:

$$\begin{aligned} Bias_\tau^V &= Var(r_{reb,\tau}) - Var(r_{bh,\tau}) \\ &= Var(\bar{r}_\tau) - Var\left[\frac{1}{\sum_{j=1}^{N} \prod_{t=1}^{\tau-1} (1 + r_{j,t})} \sum_{i=1}^{N} \prod_{t=1}^{\tau-1} (1 + r_{i,t})\, r_{i,\tau}\right]. \end{aligned} \quad (14)$$

Relative to a buy-and-hold approach, a negative bias indicates that a rebalancing strategy underestimates the average portfolio return or variance, while a positive bias shows that a rebalancing strategy overstates these estimates.

3 Empirical analysis

In this section we put our theoretical results derived in Section 2 to the test. We first discuss data used for construction of rebalanced and decomposed buy-and-hold portfolio returns, followed by the estimation methodology, results and implications for portfolio construction and investment decisions.

3.1 Data

We construct equally weighted rebalanced and buy-and-hold portfolios of various sizes from S&P 500 constituents over a nine-year sample period from January 2, 2003 to December

[14] We tested the simplifying assumption, $1/(1+\bar{r}_\tau) \approx 1 - \bar{r}_\tau$, by investigating the difference between empirical portfolio return and variance biases in Eqs. (6) and (10), respectively, and biases calculated based on the decomposed terms under simplifying assumption in Eqs. (8) and (13). These results show that the difference between the two bias estimates is negligible.

30, 2011. We let the number of stocks in each portfolio vary between 1 and 80, and select stocks randomly without replacement. The period under consideration includes the global financial crisis (GFC) associated with the bankruptcy of Lehman Brothers in September 2008 and the subsequent period of turmoil in the U.S. and international financial markets. The underlying data are 5 minute, daily, weekly and monthly observations on prices for 501 stocks drawn from the constituent list of the S&P 500 index during the sample period, obtained from SIRCA Thompson Reuters Tick History. This data set was constructed by Dungey et al. [2012] and does not contain all the stocks listed in the S&P 500 index, but has drawn from that population to select those with sufficient coverage and data availability for high frequency time series analysis. The original dataset of over 900 stocks was taken from the 0#.SPX mnemonic provided by SIRCA. This included several stocks that were traded OTC and on alternative exchanges. The stocks that changed the currency in which they were traded during the period under consideration were excluded from the analysis. We adjusted the dataset for changes in RIC codes[15] resulted from mergers and acquisitions, stock splits and trading halts. We removed stocks with insufficient number of observations. We force the inclusion of Lehman Brothers until their bankruptcy in September 2008, but drop Fannie Mae and Freddie Mac from the analysis. The data handling process is documented in the web-appendix to Dungey et al. [2012]. The final data set contains 501 individual stocks. Our estimation methods are summarised in the next subsection and detailed in Supplementary Appendix B. The full list of stocks including Reuters Identification Codes (RICs) is provided in Appendix C.

3.2 *Estimation method and results*

We allow for the diversification effect in portfolios, that is, the relationship between the decreasing risk in portfolios when the number of securities in these portfolios increases.[16] Figure 3 represents the variance bias in portfolios by year (2003-2011).[17] To calculate biases, for each simulated portfolio, we retain the same draw of stocks from the S&P 500 constituents list when contrasting rebalanced and buy-and-hold approaches. The number of stocks, $N = 1, ..., 80$, is shown on the x-axis.[18] Stocks in the simulated portfolios are chosen randomly without replacements with the number of stocks in portfolios varying from 1 to 80.[19] We perform 10,000 random draws and compute the median variance bias (blue solid line in Figure 3), the mean variance bias (blue dotted line) and the 90% confidence band (shaded region between the 5th and the 95th percentile of estimated biases based on 10,000 draws

[15] A Reuters instrument code, or RIC, is a ticker-like code used by Thomson Reuters to identify financial instruments and indices.

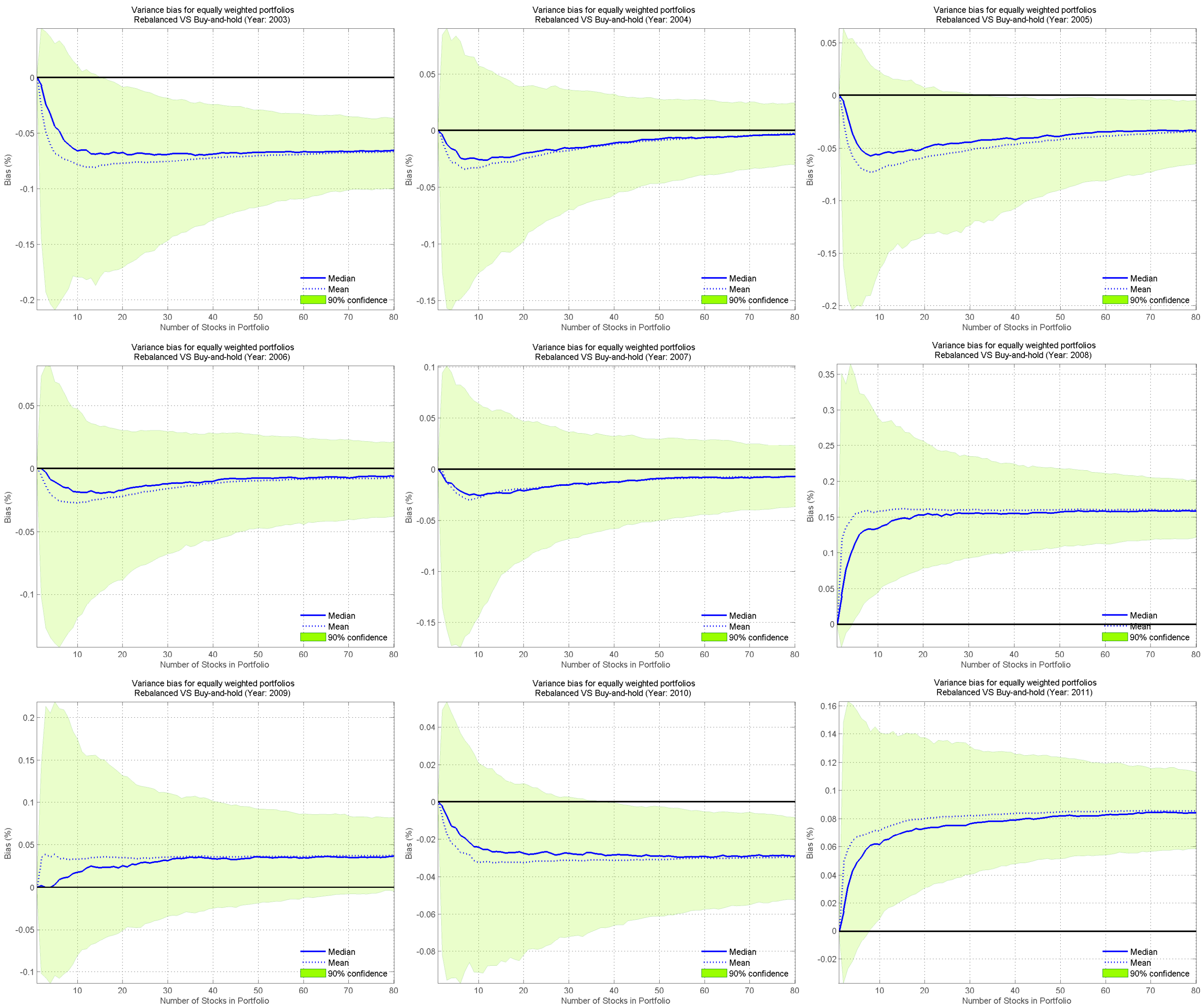
[16] We note that one can obtain most of the benefits of diversification by holding a relatively small number of stocks; see, e.g., Elton and Gruber [1977].

[17] In the finance literature, measuring risk is more contentious than measuring return. With different sampling frequencies, our risk measures, even when annualised, may differ. To help in our comparison of biases in portfolio variance across different data sampling frequencies, we find it practical to focus on relative measures for presentation purposes, and define bias in portfolio variance in our empirical section as $Bias^V/Var(r_{bh})$.

[18] The portfolio variance bias for a single stock portfolio is always zero and provides a natural starting point in the Figure.

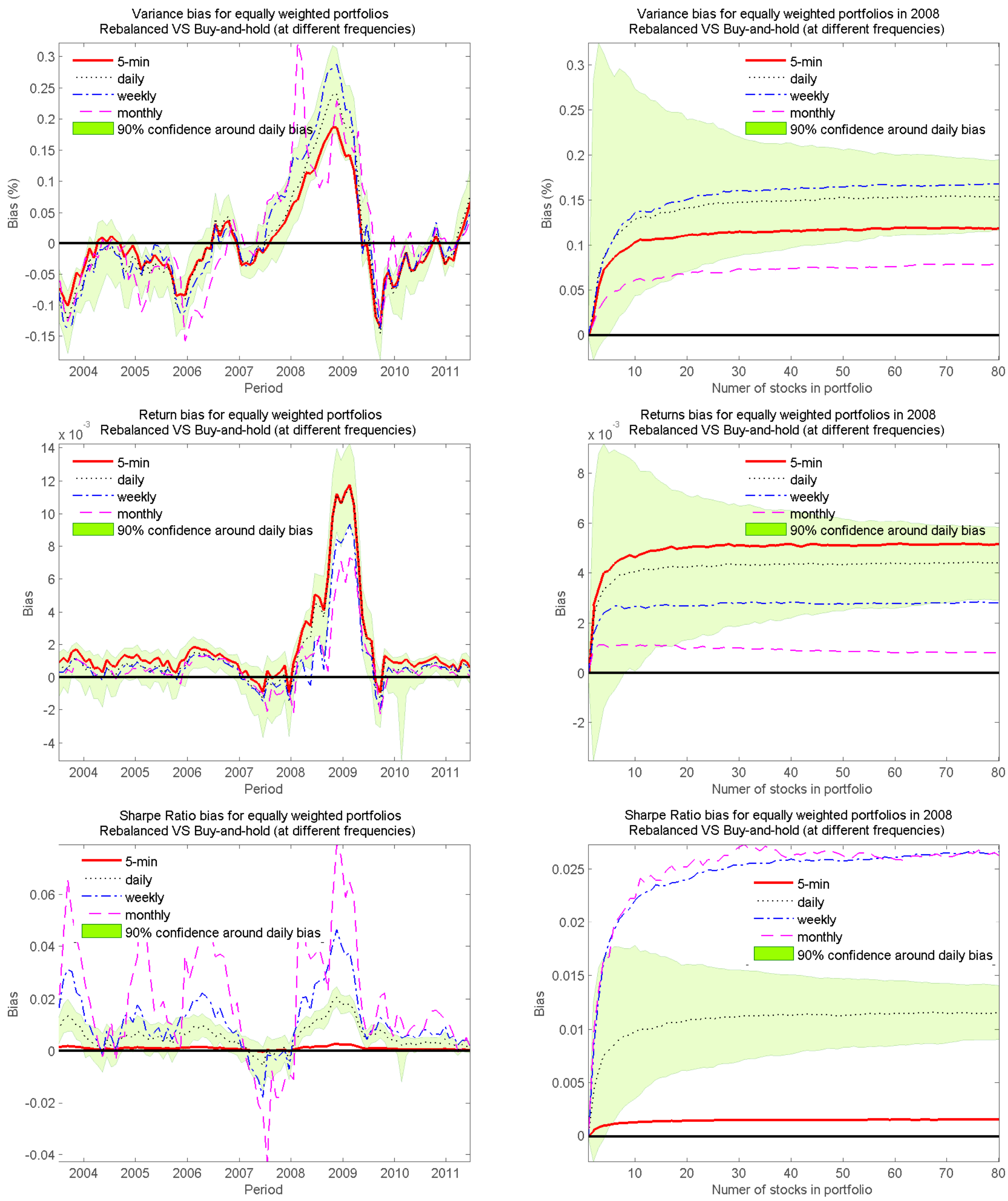
[19] Alexeev and Tapon [2014] show that 80 stocks are sufficient to get most of the diversification benefits even for conservative investors. Alexeev and Dungey [2015] reconfirm this result using high-frequency data.

Figure 3: VARIANCE BIAS IN PORTFOLIOS BY YEAR.



Variance bias (mean, median and confidence bands) for the rebalanced vs. the buy-and-hold portfolio by year (2003-2011). The shaded region represents the area between the 5th and the 95th percentiles of estimated biases for 10,000 random draws. To construct portfolios, N stocks are selected randomly, where $N = 1, ..., 80$ and the number of possible distinct portfolio combinations is $\frac{501!}{N!(501-N)!}$, which is $\gg 10,000$ for any $2 < N < 500$. The portfolio variance bias for single stock portfolios is always zero and provides a natural starting point.

Figure 4: Bias in portfolios at different frequencies.



Variance bias (top panel), average return bias (middle panel) and signal-to-noise ratio bias (bottom panel) across time (left panels) and for the selected year 2008 (right panels). All quantities are constructed based on returns of randomly selected portfolios of 50 assets using the past one year of data. At the end of each month in the period from 2003 to 2011 we obtain portfolio bias estimates using one year of past data (panels on the left). Using 5-minute, daily, weekly and monthly sampling frequencies, we estimate variance biases for 2008 across randomly selected portfolios of sizes $N = 1, ..., 80$ stocks (panels on the right). The shaded region represents the area between the 5th and the 95th percentile of estimated daily biases for 10,000 random draws. To construct portfolios N stocks are selected randomly based on daily data where $\frac{501!}{N!(501-N)!} \gg 10,000$. We use the same dataset but sample at different frequencies. We keep the same sample of assets in each simulated portfolio across estimations for different frequencies. Overnight returns for the 5-minute data have been included.

for each portfolio size). We observe that the sign of the variance bias depends on the year under consideration. For example, during the turbulent 2008 associated with the start of the GFC, variance bias is positive and significant, which shows that the rebalancing approach has exacerbated the estimates of variance. This is because maintaining equal portfolio weights will require an investor to adopt the buying "losers" and selling "winners" strategy, which will result in a portfolio with increased volatility (due to a large number of "losers" during the GFC) and, subsequently, positive variance biases compared to a buy-and-hold portfolio. This is in contrast to the results obtained in calm periods, e.g., 2003-2007, when the variance bias is significant and negative. The results indicate that rebalancing during bad economic times leads to elevation of portfolio volatility and thus, might not be optimal in bear markets. In addition, we observe that variance biases, in most cases, approach an asymptotic value for portfolios in excess of 50 assets depending on the year considered, which points out a systematic nature of the biases.

For brevity we choose to analyse well-diversified portfolios comprised of 50 stocks, where stocks are selected randomly without replacement. The number of random draws remains 10,000.[20] Every month, in the period from 2003 to 2011, we trace biases in portfolio mean returns, portfolio variances and signal-to-noise ratios[21] using one year of past data. These are shown in the left panels of Figure 4 for the variance bias (top left), mean return bias (middle left) and signal-to-noise ratio bias (bottom left). Although our focus in this section is on the analysis of *daily* returns, we apply 5-minute, weekly and monthly sampling frequencies as robustness checks when estimating variance biases.[22] We use the same dataset and the same random selection of assets in each simulated portfolio but sample prices at different frequencies to reconstruct multi-period returns. Overnight returns for 5-minute data have been included to ensure that the terminal wealth of a buy-and-hold investor is accounted for correctly.[23] Panels on the right (Figure 4) show biases for 2008[24] across randomly selected portfolios of size $N = 1, ..., 80$ stocks. The shaded region represents the 90% confidence interval around daily biases. As expected, the higher the frequency of the data, and thus, the frequency of rebalancing to maintain equal weights in the portfolio, the larger the bias in returns (middle right panel). Confirming the results reported in Figure 3, we observe from the left panel of Figure 4 that for portfolios of 50 assets, significant negative biases occur during 2003, 2005 and 2010. This indicates that rebalancing of portfolios in these years leads to a lower variance than the buy-and-hold approach, and thus, indicates that the rebalancing strategy underestimates portfolio variance. Significantly positive biases, occurring in the more

[20]The total number of possible combinations of 50 stocks out of 501 is $\frac{501!}{50!(501-50)!} = 2.57 \times 10^{69}$.

[21]The bias in the signal-to-noise ratio will be similar to the bias in the Sharpe ratio if the risk free rate remains constant throughout the entire holding period. For small infrequent changes in the risk free rate, the two biases will be approximately equal. The difference between the signal-to-noise and the Sharpe ratio corresponds to $\triangle r_f \left(\frac{\sigma_{bh} - \sigma_{reb}}{\sigma_{bh}\sigma_{reb}}\right)$.

[22]We showed that the bias in portfolio variance is influenced by the intertemporal dependencies in returns. Much of the autocovariance in individual stock returns can be attributed to the bid-ask bounce and the short-term autocovariance in portfolio returns to non-synchronous trading. Increasing the sampling frequency of our data will make these effects more pronounced.

[23]We also estimate the bias in portfolio variance when overnight returns are excluded and observe that the sign and significance of the portfolio variance biases in every period remain the same.

[24]This year corresponds to the beginning of the GFC, and was selected as a prominent example of high volatility in the financial markets.

turbulent 2008 and 2011, indicate that the rebalancing strategy overshoots the buy-and-hold strategy. This confirms our previous results that the rebalancing appears to be a favourable strategy during good economic conditions, but results in larger portfolio volatility during economic downturns.

In Table 1 we present a summary of the results for equally weighted rebalanced and buy-and-hold portfolios, obtained using daily data and 10,000 randomly constructed portfolios of 50 stocks. We assume that from the first trading day of the year the investor either follows a rebalancing strategy and calculates portfolio returns using Eq. (1), or adheres to a buy-and-hold strategy using Eq. (3). For each given year we estimate portfolio mean returns (columns 1 and 4), standard deviations (columns 2 and 5) and signal-to-noise ratios[25] (columns 3 and 6) based on daily returns within that year. The results are reported in annualised terms.[26] For the bias results in columns (7) through (9), "*" denotes significance at 10% significance level, that is, when the range from the 5th percentile to the 95th percentile of estimated biases in portfolio statistics for a given year does not contain zero. We emphasise that the bias statistics are computed based on matched pairs of rebalanced and buy-and-hold portfolio returns, i.e., for the same draw of stocks in the same period. We compute biases at the end of a year for each of the 10,000 portfolios, and then average them across these portfolios. We notice that portfolio mean returns in 2006, and especially in 2008-2009, were overstated by the rebalancing approach. This overstatement of portfolio returns have been observed in at least 90% of the 10,000 randomly constructed portfolios. On the other hand, the variance has been significantly understated in 2003, 2005, and 2010. The largest bias in portfolio variance has occurred in 2008, with another significant exaggeration in 2011. This confirms our previous results from Figure 3 and Figure 4. The overstatement of the signal-to-noise ratio by the rebalancing strategy occurred in 2003, 2006, 2008 and 2009.

Table 2 reports the ten largest positive (negative) biases in portfolio mean returns and portfolio variances in panel A (panel B). We observe that the largest significant biases in portfolio mean returns occur during the most turbulent 2008-2009, confirming previous results. The results for the largest significant biases in portfolio variance are mixed; however 6 out of 9 significant biases occur between November 2007 and August 2011. This period corresponds to the turbulent period of the financial crisis, followed by the global recession. We confirm previous results that the rebalancing method exacerbates expected returns and variances of buy-and-hold portfolios during that period. The results for the lowest biases indicate that none of the return biases are significant at the 90% level; however all the portfolio variance biases are significant, with the largest significant biases occurring in 2009. Our results indicate that researchers and portfolio managers may mistakenly measure portfolio performance when relying only on biases for the portfolio mean returns while ignoring second moments.

[25] We avoid the sample size bias in the signal-to-noise ratio discussed in Miller and Gehr [1978] since the number of observations is the same for both the buy-and-hold and the rebalanced portfolios in each simulation.

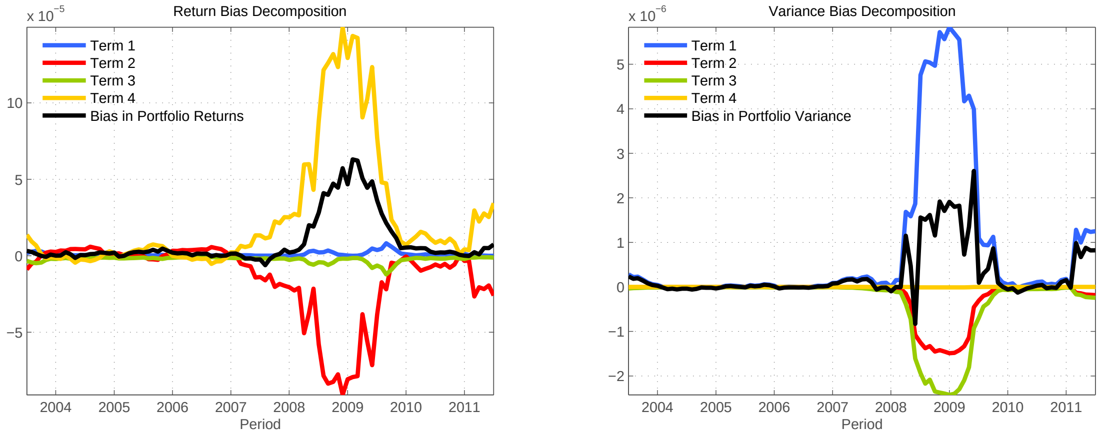
[26] Daily estimates have been annualised using a factor of 250 for mean returns, and $\sqrt{250}$ for standard deviation and signal-to-noise ratio.

Table 1: MEAN PORTFOLIO RETURN, VARIANCE AND SIGNAL-TO-NOISE RATIO FOR REBALANCED AND BUY-AND-HOLD PORTFOLIOS USING DAILY DATA.

	Rebalanced portfolio			Buy-and-hold portfolio			Bias		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Year	Avg. return (%)	St.Dev (%)	Signal-to-noise	Avg. return (%)	St.Dev (%)	Signal-to-noise	$Bias^E$	$\frac{Bias^V}{Var(r_{bh})}$ (%)	$Bias^S$
2003	33.5	17.3	1.94	33.2	17.9	1.85	0.36	-6.97*	0.09*
2004	16.3	13.2	1.24	15.6	13.2	1.18	0.66	-0.87	0.06
2005	7.2	12.2	0.59	6.2	12.5	0.49	0.97	-4.26*	0.09
2006	12.6	12.4	1.02	11.1	12.4	0.90	1.43*	-0.99	0.12*
2007	2.9	17.0	0.17	4.8	17.1	0.28	-1.85	-1.10	-0.10
2008	-36.1	45.1	-0.80	-41.6	41.8	-0.99	5.43*	16.35*	0.19*
2009	48.9	35.5	1.38	45.3	34.8	1.30	3.57*	3.72	0.08*
2010	21.0	21.3	0.99	20.5	21.6	0.95	0.51	-2.88*	0.04
2011	-1.3	27.3	-0.04	-1.8	26.3	-0.07	0.48	8.43*	0.02

Using daily frequency, we compute annualised averages of portfolio returns (columns 1 and 4), standard deviation (columns 2 and 5) and signal-to-noise ratios (columns 3 and 6) based on 10,000 randomly constructed portfolios of 50 stocks that are equally weighted at the beginning of each year. Daily estimates have been annualised using a factor of 250 for average returns; $\sqrt{250}$ for standard deviation and signal-to-noise ratio. For the bias in average portfolio returns (columns 7), portfolio variance (column 8), and signal-to-noise ratio (column 9), "*" denotes significance at 10% significance level. For presentation purposes, bias in variance (column 8) is presented as a percentage.

Figure 5: Bias Decomposition.



Return bias decomposition $Bias_2^E$ derived in Eq. (8) (left panel) and variance bias decomposition $Bias_2^V$ derived in Eq. (13) (right panel). Namely, $Bias_2^E = \underbrace{E(\bar{r}_1)E(\bar{r}_2)}_{Term\,1} + \underbrace{Cov(\bar{r}_1,\bar{r}_2)}_{Term\,2} \underbrace{-E\left(1-\bar{r}_1\right)\frac{1}{N}\sum_{i=1}^{N}E(r_{i,1})E(r_{i,2})}_{Term\,3} \underbrace{-E\left(1-\bar{r}_1\right)\frac{1}{N}\sum_{i=1}^{N}Cov(r_{i,1},r_{i,2})}_{Term\,4}$ and $Bias_2^V = \underbrace{2\,Cov(\bar{r}_2,\bar{r}_1\bar{r}_2)}_{Term\,1} \underbrace{-Var(\bar{r}_1\bar{r}_2)}_{Term\,2} \underbrace{-\frac{1}{N^2}Var\left(\sum_{i=1}^{N}r_{i,1}r_{i,2}\right)}_{Term\,3} \underbrace{-\frac{1}{N^2}Var\left(\bar{r}_1\sum_{i=1}^{N}r_{i,1}r_{i,2}\right)}_{Term\,4}$. Portfolios are constructed from randomly selected 50 assets. At the end of each month, in the period from 2003 to 2011, the bias estimates and the decomposed terms were obtained using one year of past data. Decomposed terms and biases are the averages of 10,000 randomly drawn portfolios each with 50 assets.

This is especially pertinent for the situation at hand: although biases in portfolio mean returns appear insignificant due to increased volatility, biases in the variance of portfolios are significant for all years under consideration.

Liu and Strong [2008] and Canina et al. [1998] discuss common time-series characteristics of both the portfolio and individual stocks returns and the implication that these characteristics may exhibit on the portfolio return bias. However, Canina et al. [1998] do not derive the portfolio mean return bias but instead use regression analysis to explain the bias using a set of time-series characteristics of the underlying portfolios and stocks. They calculate the cross-sectional average autocorrelations of each stock, the autocorrelations for the equally weighted rebalanced portfolio, and the cross-sectional variance of the average returns. Lo and Mackinlay [1990] document that average daily autocorrelations in returns are mostly negative. The empirical literature shows that individual stock returns are negatively autocorrelated because of non-synchronous trading (e.g., Fisher [1966]) or bid-ask spreads (e.g., Roll [1984], Jegadeesh and Titman [1995]). Our evidence precludes us from drawing the same conclusion. Given that our sample comprises the largest 501 stocks in the U.S. financial markets, non-synchronous trading or bid-ask spreads might not be an issue, at least for daily or lower frequencies. Consistent with the previous literature (Lo and Mackinlay [1990], Mech [1993], Canina et al. [1998]) we observe, on average, positive first-order autocorrelations[27] in

[27]The results for autocorrelations are not reported here, but are available from the authors upon request.

Table 2: 20 LARGEST BIASES.

Rank	Year	Month	$Bias^E$	Rank	Year	Month	$\frac{Bias^V}{Var(r_{bh})}$ (%)
			Panel A: Months with highest bias				
1	2008	October	15.84*	1	2008	November	8.51*
2	2009	March	13.17*	2	2008	October	7.51*
3	2008	November	11.36*	3	2009	January	4.47*
4	2008	December	6.31	4	2006	July	4.40*
5	2008	September	4.29	5	2009	February	4.20*
6	2008	July	3.49*	6	2004	July	4.01*
7	2009	February	3.27	7	2008	September	3.80
8	2009	May	3.06	8	2006	June	3.32*
9	2008	January	3.02	9	2011	August	3.15*
10	2009	January	2.91	10	2007	November	2.93*
			Panel B: Months with lowest bias				
108	2009	April	-4.47	108	2009	April	-10.53*
107	2008	June	-2.63	107	2009	May	-5.53*
106	2009	August	-2.36	106	2009	August	-4.91*
105	2011	September	-1.68	105	2004	January	-3.57*
104	2006	January	-0.89	104	2008	August	-3.26*
103	2003	April	-0.81	103	2003	August	-3.16*
102	2003	May	-0.79	102	2003	October	-2.83*
101	2009	December	-0.75	101	2010	April	-2.34*
100	2007	December	-0.74	100	2003	July	-2.29*
99	2004	April	-0.70	99	2011	October	-2.29*

The largest positive (negative) biases in portfolio returns and portfolio variances are reported in panel A (panel B). We observe that the largest significant biases in portfolio returns occur during the turbulent years 2008-2009. The results for the largest significant biases in portfolio variance are mixed; however 6 out of 9 significant biases occur between November 2007 and August 2011. "*" denotes significance at 10% significance level, that is, when the range from the 5th percentile to the 95th percentile of estimated biases in portfolio statistics for a given year does not contain zero. For presentation purposes, bias in variance is presented as a percentage.

portfolios for the first half of our sample.[28] However, following the financial crisis associated with the bankruptcy of Lehman Brothers in September 2008 and the subsequent period of turmoil in the U.S., we observe negative first-order autocorrelations in portfolio returns. The second- and third-order autocorrelations in portfolios are negative on average, which is in line with the results reported in Canina et al. [1998]. The cross-sectional variance of average returns is stable for the first half of our sample, and becomes volatile starting from 2007, which corresponds to the start of the GFC and subsequent period of global recession.

Figure 5 shows bias in the portfolio mean return, $Bias^E$, and bias in the portfolio variance, $Bias^V$, respectively, decomposed into its components as defined by Eq. (8) and Eq. (13). The terms that impact the mean return bias (left panel) include the autocovariance of average portfolio returns (Term 2, red line) and the term involving autocovariance of individual stock returns (Term 4, yellow line). In contrast, average portfolio returns (Term 1, blue line) and the

[28] Transaction costs cause portfolio returns to be autocorrelated by delaying price adjustment.

term involving average individual stock returns (Term 3, green line) are negligible. The terms that impact bias in portfolio variance (right panel) include the covariance between the average portfolio returns $\bar{r}_2$ and the product $\bar{r}_1\bar{r}_2$ (Term 1, blue line), the variance of the product of portfolio returns $\bar{r}_1\bar{r}_2$ (Term 2, red line) and the term that depends on the variance of the sum of the product of individual stock returns $\sum_{i=1}^{N} r_{i,1}r_{i,2}$ (Term 3, green line). Conversely, the variance of the product of the average portfolio return and the sum of individual returns $\bar{r}_1 \sum_{i=1}^{N} r_{i,1}r_{i,2}$ (Term 4, yellow line) can be neglected.

Transactions costs have important implications for the bias in portfolio mean returns, considering the amount of trading involved in the case of rebalanced portfolios. Liu and Strong [2008] discuss this issue and document four alternative estimates of transactions costs. In fact, from Eq. (8) we can directly observe the relationship between bias in the portfolio mean return and expected returns on both the portfolio and individual stocks. On the contrary, the portfolio variance bias in Eq. (13) relies solely on variances of the portfolio and individual stock returns, as well as intertemporal dependencies in these returns, with transactions costs having minimal influence on the bias.

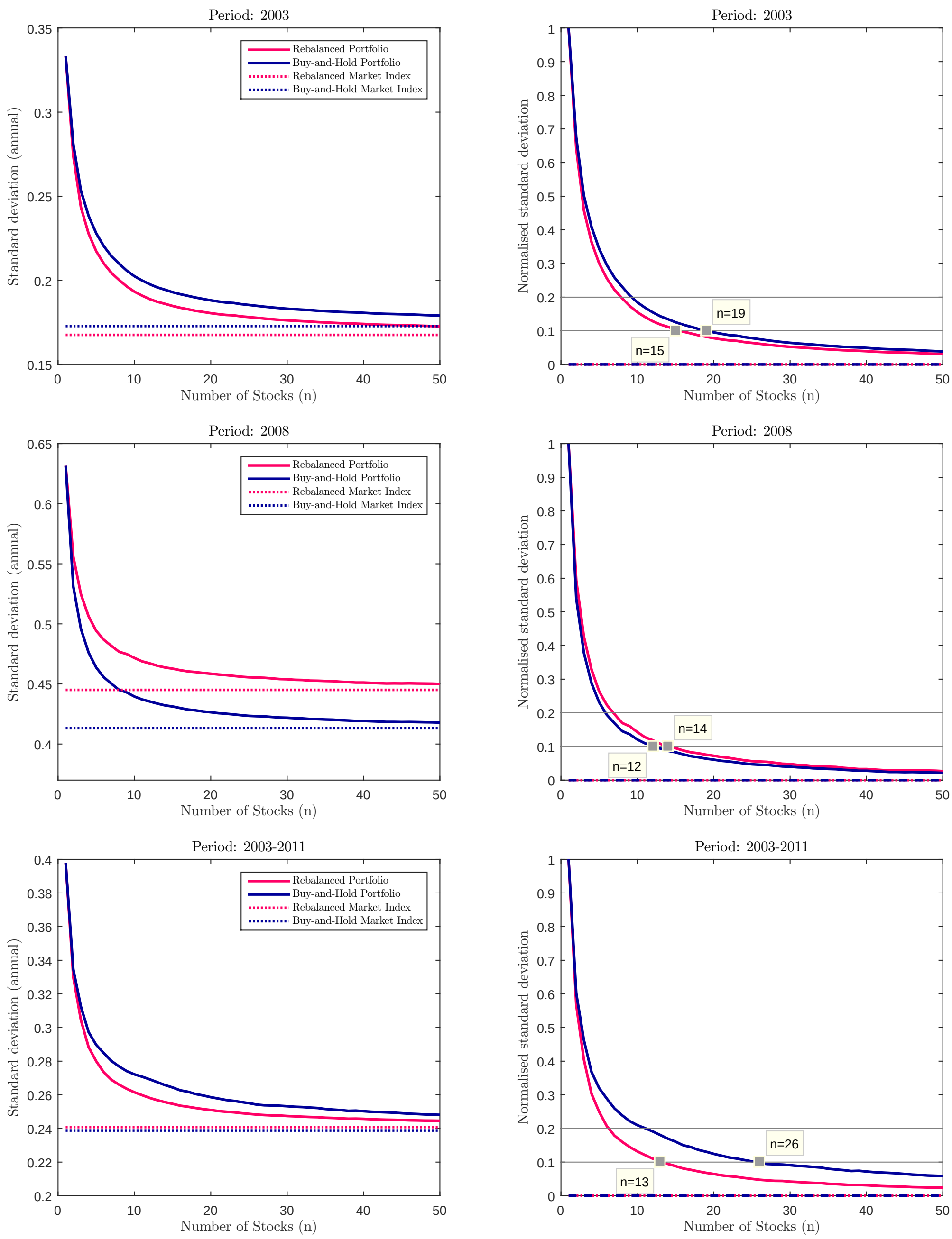
In the next section we demonstrate that the optimal portfolio size required to obtain a well-diversified portfolio is heavily reliant on portfolio construction method especially for longer investment horizons.

4 Applications to Portfolio Diversification

Previous literature often employs a simplified portfolio return decomposition, involving rebalancing the portfolio at the beginning of each time period back to the initial weights (see, Evans and Archer, 1968, Wagner and Lau, 1971, Tang, 2004, Kryzanowski and Singh, 2010, Chong and Phillips, 2013, among others). These studies underestimate the number of stocks required to achieve a certain level of portfolio diversification in buy-and-hold portfolios commonly held by individual investors. Intuitively, this can be inferred from Figure 1 where a buy-and-hold portfolio initially comprised of 10 stocks, eventually resembles characteristics of a two-stock portfolio. In this section, using portfolio simulation we formally investigate the optimal size of portfolios based on the two approaches: fixed weight and buy-and-hold strategies. Figure 6 shows the reduction in diversifiable risk as the number of holdings in a portfolio increases. We perform the analysis year-by-year (exemplar years 2003 and 2008 are included in the top and middle panels, respectively) and over the entire period (bottom panel). In the left panels of Figure 6 the vertical axis depicts portfolio risk measured by the annualised standard deviation, σ_n where $n = 1, ..., N$.[29] Diversifiable risk is defined as the difference between portfolio risk σ_n (solid line) and the market risk σ_N (dashed horizontal line) for rebalanced (red lines) and buy-and-hold (blue lines) portfolios, $\sigma_n - \sigma_N$. The largest difference is observed in a more volatile year 2008. However, the degree of portfolio diversification (the marginal reduction in the standard deviation of adding an extra security) as

[29]Similar figures can be observed in seminal works by Evans and Archer [1968] and Solnik [1974].

Figure 6: Bias in optimal portfolio size.



Left panels depict portfolio risk (σ_n, solid lines) and the market risk (σ_N, dotted horizontal lines) against the number of portfolio holdings. Right panels show the portfolio diversifiable risk as a percentage of the total diversifiable risk, $\Omega_n = (\sigma_n - \sigma_N)/(\sigma_1 - \sigma_N)$. We contrast the results obtained using rebalanced strategy (red lines) with the results from buy-and-hold approach (blue lines). Horizontal grey lines at 0.2 and 0.1 represent 80% and 90% reduction in diversifiable risk, respectively.

the portfolio risk approaches the market risk asymptote is difficult to compare across years, and portfolio construction strategies. To facilitate comparison among different periods and across two different portfolio construction methods, we normalise the portfolio diversifiable risk measures. As in Alexeev and Dungey [2015], we define a normalised risk measure that takes values between zero, for fully diversified portfolios, and one, for single-stock portfolios, as

$$\Omega_n = \frac{\sigma_n - \sigma_N}{\sigma_1 - \sigma_N},$$

where σ_1 represents a single-stock portfolio risk, σ_N is the market risk computed using all N stocks. The results are reported in the right panel of Figure 6. Although the difference in the number of stocks required to achieve diversification appears insignificant when comparing fixed weight and buy-and-hold strategies in individual years (top and middle panels), this difference becomes highly pronounced when investigating the entire period (bottom panel). In contrast to previous literature that employs fixed weight approach and suggests that between 10 and 15 stocks are enough to provide adequate diversification, we find that in buy-and-hold portfolios the recommended number of stocks is substantially higher. For example, during the 2003-2011 period, to achieve 90% diversification using fixed weight strategy investors should hold 13 stocks, while for investors adopting a buy-and-hold approach the required number of stocks increases to 26. This result reconfirms our initial intuition that in order to maintain well-diversified portfolios without periodic rebalancing, buy-and-hold investors will require a larger number of holdings compared to investors who rebalance frequently. Overall, one would expect that the longer the period under consideration the larger the difference in the number of stocks required will be to achieve diversification under the two approaches.

5 Conclusion

Rebalancing is an essential component of the portfolio management process. The assumption of continuous rebalancing, although impractical, has gained popularity among researchers due to its simplicity and tractability. Significant biases in portfolio mean returns and portfolio variances arise as a consequence of incorrectly adopting a rebalancing assumption in estimating multi-period portfolio returns for buy-and-hold investors. Although such misalignments are rare in the finance literature, adaptation of rebalancing frequency matching that of the underlying data is quite frequent. In analysing bias in decomposed portfolio mean returns, Liu and Strong [2008] define return bias as the difference between mean returns of portfolios constructed using the rebalanced and buy-and-hold assumptions. They show that return bias can have detrimental consequences and results in misleading conclusions on momentum profits. Focusing on the second moment of return distributions, we compute variance bias as the difference between the variance of portfolios constructed using rebalanced returns and the decomposed buy-and-hold returns. We show that variance bias is significant and systematic. We demonstrate that the assumption of continuously rebalanced versus buy-and-hold weights have direct implications for the optimal number of portfolio holdings recommendation in diversified portfolios. In light of the evidence on existence of bias in portfolio mean returns as well as variance, evaluation of portfolio risk-adjusted performance may be affected.

In our empirical exercise we avoid restrictive assumptions used to derive the closed form solution and examine biases arising in the means and variances of portfolios using equally weighted rebalanced and buy-and-hold portfolios of various sizes constructed from S&P 500 constituents over the nine-year sample period ranging from January 2, 2003 to December 30, 2011. Allowing the number of stocks in portfolios to vary between 1 and 80, we find that the bias in portfolio variance approaches an asymptotic value for portfolios in excess of 50 assets, pointing to a systematic nature of the bias. Our results (the sign of the bias and its significance) depend on the period under consideration with its specific time-series properties for both the portfolio returns and the individual stock returns. In particular, we find that negative variance biases tend to occur during 2003, 2005 and 2010 indicating that rebalancing of portfolios understates portfolio variances. Significantly positive biases are attributed to more turbulent 2008 and 2011, indicating that the rebalancing strategy overstates the buy-and-hold strategy during these times. This result is not surprising since to maintain equal portfolio weights for stocks in a portfolio at each time, an investor will have to adopt a buying "losers" and selling "winners" strategy, resulting in a portfolio with elevated volatility and, subsequently, positive variance biases compared to a buy-and-hold portfolio. We observe the largest significant biases in portfolio returns between 2007 and 2011, corresponding to the turbulent period of the GFC, followed by the global recession. The existence of large biases in the variance of portfolios during adverse economic conditions suggests that rebalancing might not be an optimal investment strategy in crisis (and perhaps post-crisis) periods, and a buy-an-hold strategy should be considered as a viable alternative during these times.

When explaining bias in portfolio mean returns and variances, we find that higher cross-sectional variability in average portfolio returns results in higher biases. Other variables contributing to the explanation of these biases include autocovariances of average portfolio returns and autocovariances of individual stock returns. Furthermore, when analysing bias decomposition, we observe that the autocovariance of average portfolio returns and the autocovariance of individual stock returns impact the return bias; whereas the bias in portfolio variance is influenced by the intertemporal dependencies in both the portfolio and individual stock returns.

Overall, the existence of portfolio variance biases have important implications not only when evaluating portfolio risk, but also in measuring portfolio performance. Our results indicate that one should exercise caution and apply correct assumption on portfolio weight dynamics when dealing with multi-period portfolio returns, as biases in portfolio variances and portfolio mean returns can lead to spurious results when analysing investment strategies or testing asset pricing models. We emphasise that researchers might fall into a methodological trap when observing (possibly insignificant) biases in portfolio returns and ignoring second moments, that may, in fact, include large biases. The existence of portfolio variance biases, particularly during the turbulent periods of financial crises and global recessions, might have important implications when evaluating portfolio risk-adjusted performance. We show that for buy-and-hold investors the recommended portfolio sizes to achieve well-diversified portfolios should be substantially larger than previously recommended in the finance literature based on results that employ fixed weight portfolio construction approach.

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Supplementary Appendix

A. Derivations

Bias in portfolio returns

Using Eq. (5) and Eq. (3) for $\tau = 2$, we can express the bias between the expected return of the rebalanced and the buy-and-hold portfolio as follows:

$$\begin{aligned}
Bias_2^E &= E(r_{reb,2}) - E(r_{bh,2}) \\
&= E\left[\bar{r}_2\right] - E\left[\frac{1}{\sum_{j=1}^{N}\left(1+r_{j,1}\right)}\sum_{i=1}^{N}\left(1+r_{i,1}\right)r_{i,2}\right] \qquad \text{(A.1)} \\
&= E\left[\bar{r}_2\right] - E\left[\frac{1}{N\left(1+\bar{r}_1\right)}\left\{\sum_{i=1}^{N} r_{i,2} + \sum_{i=1}^{N} r_{i,1}r_{i,2}\right\}\right] \\
&\approx E\left[\bar{r}_2\right] - E\left[\left(1-\bar{r}_1\right)\frac{1}{N}\left\{\sum_{i=1}^{N} r_{i,2} + \sum_{i=1}^{N} r_{i,1}r_{i,2}\right\}\right] \\
&= E\left[\bar{r}_2\right] - E\left[\frac{1}{N}\sum_{i=1}^{N} r_{i,2} + \frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2} - \bar{r}_1\frac{1}{N}\sum_{i=1}^{N} r_{i,2} - \bar{r}_1\frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2}\right] \\
&= E\left[\bar{r}_2\right] - E\left[\bar{r}_2 + \frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2} - \bar{r}_1\bar{r}_2 - \bar{r}_1\frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2}\right] \\
&= E\left[\bar{r}_1\bar{r}_2 - \left(1-\bar{r}_1\right)\frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2}\right] \\
&= E\left[\bar{r}_1\bar{r}_2\right] - \frac{1}{N}\sum_{i=1}^{N} E\left[\left(1-\bar{r}_1\right)r_{i,1}r_{i,2}\right]. \qquad \text{(A.2)}
\end{aligned}$$

Assuming that $\bar{r}_1$ is uncorrelated with individual returns $r_{i,1}$ and $r_{i,2}$, Eq. (A.2) can be expressed as

$$\begin{aligned}
Bias_2^E &= E(\bar{r}_1)E(\bar{r}_2) + Cov(\bar{r}_1,\bar{r}_2) - \frac{1}{N}\sum_{i=1}^{N} E\left(1-\bar{r}_1\right)\left[E(r_{i,1})E(r_{i,2}) + Cov(r_{i,1},r_{i,2})\right] \\
&= E(\bar{r}_1)E(\bar{r}_2) + \underbrace{Cov(\bar{r}_1,\bar{r}_2)}_{>0} - \underbrace{E\left(1-\bar{r}_1\right)}_{>0}\frac{1}{N}\sum_{i=1}^{N} E(r_{i,1})E(r_{i,2}) \\
&\quad + \underbrace{\left[-\underbrace{E\left(1-\bar{r}_1\right)}_{>0}\frac{1}{N}\sum_{i=1}^{N}\underbrace{Cov(r_{i,1},r_{i,2})}_{<0}\right]}_{>0}. \qquad \text{(A.3)}
\end{aligned}$$

Bias in portfolio variance

The variance bias between the rebalanced portfolio and the buy-and-hold portfolio is given by

$$Bias_2^V = Var(r_{reb,2}) - Var(r_{bh,2}), \tag{A.4}$$

where the variance of the rebalanced portfolio is

$$Var(r_{reb,2}) = Var\left[\bar{r}_2\right] = E\left[\bar{r}_2^2\right] - E\left[\bar{r}_2\right]^2 \tag{A.5}$$

and the variance of the buy-and-hold portfolio can be written as

$$\begin{aligned}
Var(r_{bh,2}) &= Var\left[\frac{1}{\sum_{j=1}^{N}\left(1+r_{j,1}\right)}\sum_{i=1}^{N}\left(1+r_{i,1}\right)r_{i,2}\right] \\
&= Var\left[\frac{1}{N\left(1+\bar{r}_1\right)}\left\{\sum_{i=1}^{N} r_{i,2} + \sum_{i=1}^{N} r_{i,1}r_{i,2}\right\}\right] \\
&\approx Var\left[\left(1-\bar{r}_1\right)\frac{1}{N}\left\{\sum_{i=1}^{N} r_{i,2} + \sum_{i=1}^{N} r_{i,1}r_{i,2}\right\}\right] \\
&= Var\left[\frac{1}{N}\sum_{i=1}^{N} r_{i,2} + \frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2} - \bar{r}_1\frac{1}{N}\sum_{i=1}^{N} r_{i,2} - \bar{r}_1\frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2}\right] \\
&= Var\left[\bar{r}_2 + \frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2} - \bar{r}_1\bar{r}_2 - \bar{r}_1\frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2}\right] \\
&= Var\left[\bar{r}_2(1-\bar{r}_1) + (1-\bar{r}_1)\frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2}\right] \\
&= Var\left[(1-\bar{r}_1)(\bar{r}_2 + \frac{1}{N}\sum_{i=1}^{N} r_{i,1}r_{i,2})\right].
\end{aligned} \tag{A.6}$$

In the third equality we applied an approximation $1/(1+\bar{r}_\tau) \approx 1-\bar{r}_\tau$, as before, ignoring higher order terms in the Taylor series expansion. We can further rewrite Eq. (A.6) as follows:

$$\begin{aligned}
Var_{bh,2} = \; & Var(\bar{r}_2) + \frac{1}{N^2}Var\left(\sum_{i=1}^{N} r_{i,1}r_{i,2}\right) + Var(\bar{r}_1\bar{r}_2) + \frac{1}{N^2}Var\left(\bar{r}_1\sum_{i=1}^{N} r_{i,1}r_{i,2}\right) \\
& + \frac{2}{N}Cov\left(\bar{r}_2, \sum_{i=1}^{N} r_{i,1}r_{i,2}\right) - 2Cov(\bar{r}_2, \bar{r}_1\bar{r}_2) - \frac{2}{N}Cov\left(\bar{r}_2, \bar{r}_1\sum_{i=1}^{N} r_{i,1}r_{i,2}\right) \\
& - \frac{2}{N}Cov\left(\bar{r}_1\bar{r}_2, \sum_{i=1}^{N} r_{i,1}r_{i,2}\right) - \frac{1}{N^2}Cov\left(\sum_{i=1}^{N} r_{i,1}r_{i,2}, \bar{r}_1\sum_{i=1}^{N} r_{i,1}r_{i,2}\right) \\
& + \frac{2}{N}Cov\left(\bar{r}_1\bar{r}_2, \bar{r}_1\sum_{i=1}^{N} r_{i,1}r_{i,2}\right).
\end{aligned} \tag{A.7}$$

Continuing with the assumption that portfolio returns are uncorrelated with the individual stock returns, $Bias_2^V$ for the variance reduces to

$$
\begin{aligned}
Bias_2^V &= Var(r_{reb,2}) - Var(r_{bh,2}) \\
&= Var\,(\bar{r}_2) - Var(\bar{r}_2) - \frac{1}{N^2} Var\left(\sum_{i=1}^{N} r_{i,1} r_{i,2}\right) - Var(\bar{r}_1 \bar{r}_2) \\
&\quad - \frac{1}{N^2} Var\left(\bar{r}_1 \sum_{i=1}^{N} r_{i,1} r_{i,2}\right) + 2Cov(\bar{r}_2, \bar{r}_1 \bar{r}_2) \\
&= 2Cov(\bar{r}_2, \bar{r}_1 \bar{r}_2) - \underbrace{Var(\bar{r}_1 \bar{r}_2)}_{>0} - \underbrace{\frac{1}{N^2} Var\left(\sum_{i=1}^{N} r_{i,1} r_{i,2}\right)}_{>0} - \underbrace{\frac{1}{N^2} Var\left(\bar{r}_1 \sum_{i=1}^{N} r_{i,1} r_{i,2}\right)}_{>0} \quad \text{(A.8)}
\end{aligned}
$$

Further simplification of Eq. (A.8) is possible only with additional restrictive assumptions, e.g., serial independence and normality.

B. Estimation and simulation algorithm

Algorithm 1 CONSTRUCTING SIMULATED PORTFOLIOS AND OBTAINING RESULTS.

1. Randomly select N stocks out of all available stocks without replacement.
2. Given daily prices, $P_{i,\tau}$, $i = 1, ..., N$, $\tau = 0, ..., T$ calculate simple return for each stock selected in Step (1), $r_{i,\tau}$, $\tau = 1, ..., T$.
3. Given selection in Step (1) and using Eqs. (1) and (3), calculate decomposed portfolio returns for rebalancing and buy-and-hold approaches, respectively.
4. Find expected value and variance for the two portfolios obtained in Step (3) and calculate associated biases using Eqs. (9) and (14) for $\tau = T$.
5. Repeat Steps (1)-(4) $10,000$ times.
6. Based on results of Step (5) obtain mean, median, 5th and 95th percentiles for return and variance biases calculated in Step (4).
7. Repeat Steps (1)-(6) for the next period applying:
   (a) (overlapping one year rolling windows) by moving a one-year estimation window one month at a time; used for Figures 4 and 5.
   (b) (non-overlapping annual windows) by selecting price quotes from the last trading day of a previous year and to the last trading day of the current year for which the analysis is performed; used for Figure 3 and Table 1.
   (c) (non-overlapping monthly windows) by selecting price quotes from the last trading day of a previous month and to the last trading day of the current month for which the analysis is performed; used for Table 2.
8. Repeat Steps (2)-(7) for other data frequencies (but track the selection of stocks in Step (1) in portfolios to avoid sample selection bias).
9. Repeat Steps 1-8 for each $N = 1, ..., 80$.

C. *List of stocks*

RIC Code	Company Name	RIC Code	Company Name
A.N	Agilent Technologies Inc	AA.N	Alcoa Inc
AAPL.OQ	Apple Inc	ABC.N	AmerisourceBergen Corporation
ABT.N	Abbott Laboratories	ACAS.OQ	American Capital Ltd
ACE.N	ACE Limited	ACN.N	Accenture plc
ADBE.OQ	Adobe Systems Inc	ADI.N	Analog Devices Inc
ADM.N	Archer Daniels Midland Company	ADP.OQ	Automatic Data Processing Inc
ADSK.OQ	Autodesk Inc	AEE.N	Ameren Corporation
AEP.N	American Electric Power Co Inc	AES.N	The AES Corporation
AET.N	Aetna Inc	AFL.N	AFLAC Inc
AGN.N	Allergan Inc	AIG.N	American International Group Inc
AIV.N	Apartment Investment & Management Co	AIZ.N	Assurant Inc
AKAM.Oq	Akamai Technologies Inc	AKS.N	AK Steel Holding Corporation
ALL. N	The Allstate Corporation	ALTR.OQ	Altera Corp
AM.N	American Greetings Corp	AMAT.OQ	Applied Materials Inc
AMCC.OQ	Applied Micro Circuits Corp	AMD.N	Advanced Micro Devices Inc
AMGN.OQ	Amgen Inc	AMT.N	American Tower Corporation
AMZN.OQ	Amazoncom Inc	AN.N	AutoNation Inc
ANF.N	Abercrombie & Fitch Co	APA.N	Apache Corp
APC.N	Anadarko Petroleum Corporation	APD.N	Air Products & Chemicals Inc
APH.N	Amphenol Corporation	APOL.OQ	Apollo Group Inc
ARG.N	Airgas Inc	ASH.N	Ashland Inc
ATI.N	Allegheny Technologies Inc	AVB.N	Avalonbay Communities Inc
AVP.N	Avon Products Inc	AVY.N	Avery Dennison Corporation
AXP.N	American Express Company	AZO.N	AutoZone Inc
BA.N	Boeing Co	BAC.N	Bank of America Corporation
BAX.N	Baxter International Inc	BBBY.OQ	Bed Bath & Beyond Inc
BBT.N	BB&T Corporation	BBY.N	Best Buy Co Inc
BC.N	Brunswick Corporation	BCR.N	CR Bard Inc
BDX.N	Becton Dickinson and Company	BEN.N	Franklin Resources Inc
BHI.N	Baker Hughes Incorporated	BIIB.OQ	Biogen Idec Inc
BK.N	The Bank of New York Mellon Corporation	BLK.N	BlackRock Inc
BLL.N	Ball Corporation	BMC.OQ	BMC Software Inc
BMS.N	Bemis Company Inc	BMY.N	Bristol-Myers Squibb Company
BRCM.OQ	Broadcom Corp	BSX.N	Boston Scientific Corporation
BUT.N	Peabody Energy Corp	BWA.N	BorgWarner Inc
BXP.N	Boston Properties Inc	C.N	Citigroup Inc
CA.OQ	CA Technologies	CAG.N	ConAgra Foods Inc
CAH.N	Cardinal Health Inc	CAM.N	Cameron International Corporation
CAT.N	Caterpillar Inc	CB.N	The Chubb Corporation
CBE.N	Cooper Industries plc	CBG.N	CBRE Group Inc
CCE.N	Coca-Cola Enterprises Inc	CCL.N	Carnival Corporation
CEG.N	Constellation Energy Group Inc	CELG.OQ	Celgene Corporation
CERN.OQ	Cerner Corporation	CHK.N	Chesapeake Energy Corporation
CHRQ.OQ	CH Robinson Worldwide Inc	CI.N	Cigna Corp
CIEN.OQ	CIENA Corp	CINF.OQ	Cincinnati Financial Corp
CL.N	Colgate-Palmolive Co	CLF.N	Cliffs Natural Resources Inc
CLX.N	The Clorox Company	CMA.N	Comerica Incorporated
CME.OQ	Comcast Corporation	CMI.N	CME Group Inc
CMS.N	Cummins Inc	CMSCSA.OQ	CMS Energy Corp
CNP.N	CenterPoint Energy Inc	CNX.N	CONSOL Energy Inc
COF.N	Capital One Financial Corp	COG.N	Cabot Oil & Gas Corporation
COH.N	Coach Inc	COL.N	Rockwell Collins Inc
COP.N	ConocoPhillips	COST.OQ	Costco Wholesale Corporation
CPB.N	Campbell Soup Co	CPWR.OQ	Compuware Corporation
CR.N	Crane Co	CRM.N	Salesforcecom
CSC.N	Computer Sciences Corporation	CSCO.OQ	Cisco Systems Inc
CSX.N	CSX Corp	CTAS.OQ	Cintas Corporation
CTB.N	Cooper Tire & Rubber Co	CTL.N	CenturyLink Inc
CTSH.OQ	Cognizant Technology Solutions Corporation	CTXS.OQ	Citrix Systems Inc
CVC.N	Cablevision Systems Corporation	CVG.N	Convergys Corporation
CVH.N	Coventry Health Care Inc	CVS.N	CVS Caremark Corporation
CVX.N	Chevron Corporation	D.N	Dominion Resources Inc
DD.N	E I du Pont de Nemours and Company	DDR.N	DDR Corp
DDS.N	Dillards Inc	DE.N	Deere & Company
DELL.OQ	Dell Inc	DF.N	Dean Foods Company
DGX.N	Quest Diagnostics Inc	DHI.N	DR Horton Inc
DHR.N	Danaher Corp	DIS.N	Walt Disney Co
DLTR.OQ	Dollar Tree Inc	DLX.N	Deluxe Corp
DNB.N	Dun & Bradstreet Corp	DNR.N	Denbury Resources Inc
DO.N	Diamond Offshore Drilling Inc	DOV.N	Dover Corp

RIC Code	Company Name	RIC Code	Company Name
DOW.N	The Dow Chemical Company	DRI.N	Darden Restaurants Inc
DTE.N	DTE Energy Co	DTV.OQ	DIRECTV Inc
DUK.N	Duke Energy Corporation	DV.N	DeVry Inc
DVA.N	DaVita Inc	DVN.N	Devon Energy Corporation
DYN.N	Dynegy Inc	EA.OQ	Electronic Arts Inc
EBAY.OQ	eBay Inc	ECL.N	Ecolab Inc
ED.N	Consolidated Edison Inc	EFX.N	Equifax Inc
EIX.N	Edison International	EL.N	Estee Lauder Companies Inc
EMC.N	EMC Corporation	EMN.N	Eastman Chemical Co
EMR.N	Emerson Electric Co	EOG.N	EOG Resources Inc
EP.N	El Paso Corp	EQR.N	Equity Residential
EQT.N	EQT Corporation	ESRX.OQ	Express Scripts Inc
ESV.N	Ensco plc	ETFC.OQ	E_TRADE Financial Corporation
ETN.N	Eaton Corporation	ETR.N	Entergy Corporation
EW.N	Edwards Lifesciences Corp	EXC.N	Exelon Corporation
EXPD.OQ	Expeditors International of Washington Inc	EXPE.OQ	Expedia Inc
F.N	Ford Motor Co	FAST.OQ	Fastenal Company
FCX.N	Freeport-McMoRan Copper & Gold Inc	FDO.n	Family Dollar Stores Inc
FDX.N	FedEx Corporation	FE.N	FirstEnergy Corp
FFIV.OQ	F5 Networks Inc	FHN.N	First Horizon National Corporation
FII.N	Federated Investors Inc	FISV.OQ	Fiserv Inc
FITB.OQ	Fifth Third Bancorp	FLIR.OQ	FLIR Systems Inc
FLR.N	Fluor Corporation	FLS.N	Flowserve Corp
FMC.N	FMC Corp	FMCC.OB	Federal Home Loan Mtg
FNMA.OB	Fannie Mae	FRX.N	Forest Laboratories Inc
FTI.N	FMC Technologies Inc	GAS.N	AGL Resources Inc
GCI.N	Gannett Co Inc	GD.N	General Dynamics Corp
GE.N	General Electric Company	GGP.N	Gilead Sciences Inc
GILD.OQ	General Mills Inc	GIS.N	Corning Inc
GLW.N	GameStop Corp	GME.N	Genworth Financial Inc
GNW.N	Google Inc	GPC.N	Genuine Parts Company
GPS.N	Gap Inc	GR.N	Goodrich Corp
GS.N	The Goldman Sachs Group Inc	GT.N	Goodyear Tire & Rubber Co
GWW.N	WW Grainger Inc	HAL.N	Halliburton Company
HAR.N	Harman International Industries Inc	HAS.O	Hasbro Inc
HBAN.OQ	Huntington Bancshares Incorporated	HCBK.OQ	Hudson City Bancorp Inc
HCN.N	Health Care REIT Inc	HCP.N	HCP Inc
HD.N	The Home Depot Inc	HIG.N	Hartford Financial Services Group Inc
HMA.N	Health Management Associates Inc	HNZ.N	H J Heinz Company
HON.N	Honeywell International Inc	HOT.N	Starwood Hotels & Resorts Worldwide Inc
HP.N	Helmerich & Payne Inc	HPQ.N	Hewlett-Packard Company
HRB.N	H&R Block Inc	HRL.N	Hormel Foods Corp
HRS.N	Harris Corp	HSP.N	Hospira Inc
HSY.N	Hershey Co	HUM.N	Humana Inc
IACI.O	IAC_InterActiveCorp	IBM.N	International Business Machines Corp
IFF.N	International Flavors & Fragrances Inc	IGT.N	International Game Technology
INTC.OQ	Intel Corporation	INTU.OQ	Intuit Inc
IP.N	International Paper Co	IPG.N	The Interpublic Group of Companies Inc
IR.N	Ingersoll-Rand Plc	IRM.N	Iron Mountain Inc
ISRG.OQ	Intuitive Surgical Inc	ITT.N	ITT Corporation
ITW.N	Illinois Tool Works Inc	JBL.N	Jabil Circuit Inc
JCI.N	Johnson Controls Inc	JCP.N	J C Penney Company Inc
JDSU.OQ	JDS Uniphase Corporation	JEC.N	Jacobs Engineering Group Inc
JNJ.N	Johnson & Johnson	JNPR.K	Juniper Networks Inc
JNS.N	Janus Capital Group Inc	JNY.N	The Jones Group Inc
JOY	Joy Global Inc	JPM.N	JPMorgan Chase & Co
JWN.N	Nordstrom Inc	K.N	Kellogg Company
KBH.N	KB Home	KEY.N	KeyCorp
KFT.N	Kraft Foods Inc	KIM.N	Kimco Realty Corporation
KLAC.OQ	KLA-Tencor Corporation	KMB.N	Kimberly-Clark Corporation
KMX.N	CarMax Inc	KO.N	The Coca-Cola Company
KR.N	The Kroger Co	KSS.N	Kohls Corp
L.N	Loews Corporation	LEG.N	Leggett & Platt Incorporated
LEH.N	Lehman Brothers	LEN.N	Lennar Corp
LH.N	Laboratory Corp of America Holdings	LIFE.OQ	Life Technologies Corporation
LIZ.N	Liz Claiborne Inc	LLL.N	L-3 Communications Holdings Inc
LLTC.OQ	Linear Technology Corp	LLY.N	Eli Lilly & Co
LM.N	Legg Mason Inc	LMT.N	Lockheed Martin Corporation
LNC.N	Lincoln National Corp	LOW.N	Lowes Companies Inc
LPX.N	Louisiana-Pacific Corp	LSI.N	LSI Corporation
LTD.N	Limited Brands Inc	LUK.N	Leucadia National Corp
LUV.N	Southwest Airlines Co	LXK.N	Lexmark International Inc

RIC Code	Company Name	RIC Code	Company Name
MAR.N	Marriott International Inc	MAS.N	Masco Corporation
MAT.O	Mattel Inc	MBI.N	MBIA Inc
MCD.N	McDonalds Corp	MCHP.OQ	Microchip Technology Inc
MCK.N	McKesson Corporation	MCO.N	Moodys Corp
MDP.N	Meredith Corp	MDT.N	Medtronic Inc
MET.N	MetLife Inc	MHP.N	The McGraw-Hill Companies Inc
MHS.N	Medco Health Solutions Inc	MKC.N	McCormick & Co Inc
MMC.N	Marsh & McLennan Companies Inc	MMM.N	3M Co
MO.N	Altria Group Inc	MOLX.OQ	Molex Inc
MON.N	Monsanto Co	MOS.N	The Mosaic Company
MRK.N	Merck & Co Inc	MRO.N	Marathon Oil Corporation
MS.N	Morgan Stanley	MSFT.OQ	Microsoft Corporation
MTB.N	M&T Bank Corporation	MTG.N	MGIC Investment Corp
MTW.N	Manitowoc Co Inc	MU.OQ	Micron Technology Inc
MUR.N	Murphy Oil Corporation	MWV.N	MeadWestvaco Corporation
MWW	Monster Worldwide Inc	MYL.OQ	Mylan Inc
NBL.N	Noble Energy Inc	NBR.N	Nabors Industries Ltd
NCR.N	NCR Corp	NDAQ.OQ	Nasdaq OMX Group Inc
NE.N	Noble Corp	NEM.N	Newmont Mining Corp
NFLX.OQ	Netflix Inc	NFX.N	Newfield Exploration Co
NI.N	NiSource Inc	NKE.N	Nike Inc
NOC.N	Northrop Grumman Corporation	NOV.N	National Oilwell Varco Inc
NRG.N	NRG Energy Inc	NSC.N	Norfolk Southern Corp
NTAP.OQ	NetApp Inc	NTRS.OQ	Northern Trust Corporation
NU.N	Northeast Utilities	NUE.N	Nucor Corporation
NVDA.OQ	NVIDIA Corporation	NVLS.OQ	Novellus Systems Inc
NWL.N	Newell Rubbermaid Inc	NWSA.O	News Corp
NYT.N	The New York Times Company	ODP.N	Office Depot Inc
OI.N	Owens-Illinois Inc	OKE.N	ONEOK Inc
OMC.N	Omnicom Group Inc	OMX.N	OfficeMax Incorporated
ORCL.OQ	Oracle Corporation	ORLY.OQ	OReilly Automotive Inc
OXY.N	Occidental Petroleum Corporation	PAYX.OQ	Paychex Inc
PBCT.OQ	Peoples United Financial Inc	PBI.N	Pitney Bowes Inc
PCAR.OQ	PACCAR Inc	PCG.N	PG&E Corp
PCL.N	Plum Creek Timber Co Inc	PCLN.OQ	pricelinecom Incorporated
PCP.N	Precision Castparts Corp	PDCO.OQ	Patterson Companies Inc
PEG.N	Public Service Enterprise Group Inc	PEP.N	Pepsico Inc
PFE.N	Pfizer Inc	PFG.N	Principal Financial Group Inc
PG.N	Procter & Gamble Co	PGN.N	Progress Energy Inc
PGR.N	Progressive Corp	PH.N	Parker Hannifin Corporation
PHM.N	PulteGroup Inc	PKI.N	PerkinElmer Inc
PLD.N	Prologis Inc	PLL.N	Pall Corp
PMCS.OQ	PMC-Sierra Inc	PMTC.OQ	Parametric Technology Corporation
PNC.N	PNC Financial Services Group Inc	PNW.N	Pinnacle West Capital Corporation
POM.N	Pepco Holdings Inc	PPG.N	PPG Industries Inc
PPL.N	PPL Corporation	PRGO.OQ	Perrigo Co
PRU.N	Prudential Financial Inc	PSA.N	Public Storage
PWER.OQ	Power-One Inc	PWR.N	Quanta Services Inc
PX.N	Praxair Inc	PXD.N	Pioneer Natural Resources Co
QCOM.OQ	QUALCOMM Incorporated	QLGC.OQ	QLogic Corp
R.N	Ryder System Inc	RAI.N	Reynolds American Inc
RDC.N	Rowan Companies Inc	RF.N	Regions Financial Corp
RHI.N	Robert Half International Inc	RIG.N	Transocean Ltd
RL.N	Ralph Lauren Corporation	ROK.N	Rockwell Automation Inc
ROP.N	Roper Industries Inc	ROST.OQ	Ross Stores Inc
RRC.N	Range Resources Corporation	RRD.OQ	RR Donnelley & Sons Company
RSG.N	Republic Services Inc	RSH.N	RadioShack Corp
RTN.N	Raytheon Co	S.N	Sprint Nextel Corp
SANM.OQ	Sanmina-SCI Corp	SBUX.OQ	Starbucks Corporation
SCG.N	SCANA Corp	SE.N	Spectra Energy Corp
SEE.N	Sealed Air Corporation	SHLD.OQ	Sears Holdings Corporation
SHW.N	The Sherwin-Williams Company	SIAL.OQ	Sigma-Aldrich Corporation
SJM.N	The J M Smucker Company	SLB.N	Schlumberger Limited
SLE.N	Sara Lee Corp	SLM.O	SLM Corporation
SNA.N	Snap-on Inc	SNDK.OQ	SanDisk Corp
SNV.N	Synovus Financial Corp	SO.N	Southern Company
SPG.N	Simon Property Group Inc	SPLS.OQ	Staples Inc
SRCL.OQ	Stericycle Inc	SRE.N	Sempra Energy
SSP.N	The E W Scripps Company	STI.N	SunTrust Banks Inc
STJ.N	St Jude Medical Inc	STR.N	Questar Corporation
STT.N	State Street Corp	STZ.N	Constellation Brands Inc
SUN.N	Sunoco Inc	SVU.N	SUPERVALU Inc

RIC Code	Company Name	RIC Code	Company Name
SWK.N	Stanley Black & Decker Inc	SWN.N	Southwestern Energy Co
SWY.N	Safeway Inc	SYK.N	Stryker Corp
SYMC.OQ	Symantec Corporation	SYY.N	Sysco Corp
T.N	AT&T Inc	TAP.N	Molson Coors Brewing Company
TE.N	TECO Energy Inc	TER.N	Teradyne Inc
TEX.N	Terex Corp	TGT.N	Target Corp
THC.N	Tenet Healthcare Corp	TIE.N	Titanium Metals Corporation
TIF.N	Tiffany & Co	TIN.N	Temple-Inland Inc
TJX.N	The TJX Companies Inc	TLAB.OQ	Tellabs Inc
TMK.N	Torchmark Corp	TMO.N	Thermo Fisher Scientific Inc
TNB.N	Thomas & Betts Corp	TROW.OQ	T Rowe Price Group Inc
TSN.N	Tyson Foods Inc	TSO.N	Tesoro Corporation
TSS.N	Total System Services Inc	TUP.N	Tupperware Brands Corporation
TWX.N	Time Warner Inc	TXN.N	Texas Instruments Inc
TXT.N	Textron Inc	TYC.N	Tyco International Ltd
UIS.N	Unisys Corporation	UNH.N	Unitedhealth Group Inc
UNM.N	Unum Group	UNP.N	Union Pacific Corporation
UPS.N	United Parcel Service Inc	URBN.OQ	Urban Outfitters Inc
USB.N	U.S. Bancorp	UTX.N	United Technologies Corp
VAR.N	Varian Medical Systems Inc	VFC.N	VF Corporation
VLO.N	Valero Energy Corporation	VMC.N	Vulcan Materials Company
VNO.N	Vornado Realty Trust	VRSN.OQ	VeriSign Inc
VTR.N	Ventas Inc	VZ.N	Verizon Communications Inc
WAG.N	Walgreen Co	WAT.N	Waters Corp
WDC.N	Western Digital Corp	WEC.N	Wisconsin Energy Corp
WFC.N	Wells Fargo & Company	WFR.N	MEMC Electronic Materials Inc
WFT.N	Weatherford International Ltd	WHR.N	Whirlpool Corp
WLP.N	WellPoint Inc	WM.N	Waste Management Inc
WMB.N	Williams Companies Inc	WMT.N	Wal-Mart Stores Inc
WOR.N	Worthington Industries Inc	WPI.N	Watson Pharmaceuticals Inc
WPO.N	The Washington Post Company	WY.N	Weyerhaeuser Co
WYNN.OQ	Wynn Resorts Ltd	X.N	United States Steel Corp
XEL.N	Xcel Energy Inc	XL.N	XL Group plc
XLNX.OQ	Xilinx Inc	XOM.N	Exxon Mobil Corporation
XRAY.OQ	DENTSPLY International Inc	XRX.N	Xerox Corp
YHOO.OQ	Yahoo! Inc	YUM.N	Yum! Brands Inc
ZION.OQ	Zions Bancorp	ZMH.N	Zimmer Holdings Inc