Average Futures:

Anti-manipulation Effects, Types, and Volatility

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Abstract

An average futures contract is first introduced by Yoo (2015) and characterized by an expiration price given as the average of its underlying asset prices on multiple reference dates. We revisit the effects of anti-manipulation and volatility reduction of the futures, which are more precisely termed fixed referencing futures here, explore another average futures or variable referencing futures, compare the ways in which their volatilities decrease, discuss which futures contract better serves hedgers and speculators in average futures, and find out which determines the volatility of average futures in general. It is not the number of reference dates but the weight of the remaining reference dates at a given time that determines the volatility of any kind of average futures at any future time.

Average futures; Volatility; Reference dates; Manipulation

JEL classification: G13, G14

1. Introduction

Stock prices can skyrocket or plummet on or around the expiration day of a stock index futures or option. So-called “expiration-day effects” are, to a certain extent, common in the U.S., U.K., Germany, Canada, Spain, India, Taiwan, and Norway (Stoll and Whaley, 1987; Chamberlain et al., 1989; Pope and Yadav, 1992; Swindler et al., 1994; Schlag, 1996; Vipul, 2005; Illueca and Lafuente, 2006; Hsieh, 2009). The effects are caused by index arbitrage and price manipulation (Stoll and Whaley, 1997; Alkebäck and Hagelin, 2004; Hsieh, 2009; Hsieh and Ma, 2009). Because index arbitrage is likely to persist for some time and price manipulation is difficult to prevent, stock markets will not be free of expiration-day effects in the foreseeable future, although it may be possible to manage them to some degree. The effects can be more pronounced in emerging markets as these markets include more noisy traders, and are usually smaller, less regulated, and easier to influence than developed markets such as those in the US.

The easier it is to manipulate the expiration price of an index futures contract, the worse the expiration-date effects can be. Yoo (2015) introduced average (price) futures contracts to alleviate these effects, defining an average futures or forward contract as one with an expiry settlement price that is the average of its underlying asset prices on multiple reference dates. He showed how this tool can reduce expiration-day effects and provided critical features, including volatility, of the futures.

Although Yoo (2015) first introduced average futures and analyzed their critical features theoretically, it may have limitations because, in general, the first-ever paper on any topic in any area rarely satisfies everything such as depth of research. In this paper, we revisit the anti-manipulation effects of average futures and show that they can be

smaller than suggested by Yoo (2015), depending on manipulators' behavior. Second, we introduce another type of average futures, *variable referencing futures*, in an attempt to get a more effective average futures contract than that in Yoo (2015). We also find that variable referencing futures not only better serve hedgers and speculators, who need to hedge against or bet on a periodic average price, but also have much stronger volatility-reduction effects than do the average futures in Yoo (2015), which are more precisely termed *fixed referencing futures* in this paper. Third, we find that, in the case of variable referencing futures, volatility does not necessarily decrease with the number of reference dates. Fourth, we show that the weight of the remaining reference dates at a given time, rather than the number of reference dates, determines the volatility of average futures whether they are variable or fixed referencing ones.

Specifically, Yoo (2015) showed that the profit from manipulating the underlying stock price of an average futures at expiry will be times that of a plain vanilla futures price, with representing the number of reference dates. We find that, if the manipulation is performed not only on the expiration date but also on other reference dates, the manipulation profit will be larger and the anti-manipulation effects of average futures will be smaller. Second, Yoo (2015) showed that the expected volatility of an average futures contract at any future time decreases with its number of reference dates. However, we conclude that, if the rule to set reference dates changes as in variable referencing futures, the relationship between the reference dates and the volatility of the futures can change, too. Based on an investigation of why the previously established relationship does not hold for all kinds of average futures, we propose a new relationship under general assumptions.

Although monthly average futures contracts on eight non-ferrous metals have been traded on the London Metal Exchange for many years, little theoretical or empirical research has been conducted on average futures other than that reported by Yoo (2015). This study will shed light on the depth and breadth of average futures research. Because average futures offer several distinctive advantages as investment choices for hedgers and speculators compared with plain vanilla futures, we expect that they can one day be traded in over-the-counter markets or on exchanges.

The remainder of the paper is organized as follows. In Section 2, we show how the anti-manipulation effects of average future, as defined by Yoo (2015), can be reduced by a manipulator's profit-maximizing behavior. In Section 3, we establish the volatility of

arithmetic average futures contract, which also captures, and is not limited to, the variances of (fixed-referencing) average futures by Yoo (2015). In Section 4, we introduce variable referencing futures, compare volatility reduction effects of both fixed and variable referencing futures, and discuss which one is more desirable for investors in principle and practice. In section 5, we explore the relationship between reference dates and the volatility of average futures of any kind. We find that, in general, the volatility of an average futures contract is determined solely by the weight of its remaining reference dates. Section 6 concludes the paper.

2. Revisiting anti-manipulation effects

Throughout this paper, "the futures" refer to average futures contracts, and continuous compounding is used. The basic notation used in this paper is as follows.

1 : the number of reference dates to determine the settlement price of

an average futures at expiry (n = 1, 2, 3, ...)

2 (=); the expiry date of all average futures contracts

3 , ,..., : reference dates for the futures, where < < ⋯ < <

4 : risk-free interest rate per annum (assumed to be constant)

5 : the market price at time of the stock underlying the futures

6 : dividend yield per annum of the underlying asset (assumed to be constant)

7 , : the (no-arbitrage) market price of an average futures contract at time with reference dates, where , is the price of a plain vanilla futures contract.

8 : the weight of the th reference date, , for an average price futures contract, where ∑ = 1,and > 0,∀

Yoo (2015) defined average futures contracts and proved Proposition 1.

Definition 1 (Yoo, 2015) (Average futures with n reference dates):

A futures contract with the final settlement price at expiry being ∑ · .

Proposition 1 (Yoo, 2015) (Theoretical price of an average futures at *t*):

=

,

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) · , h =

) · + · · ( ) , h ≤ < , 1 ≤ ≤ − 1

) · · ( ) , h 0 ≤ <

The following assumptions are made to show that an average futures contract can alleviate possible price manipulation on its expiration day.

First, a representative stock, *S*, exists along with its average futures contract, . Second, = · · + · · , where and are the (ex-dividend) mean and the standard deviation of the continuously compounded returns of , and is standard Brownian motion.[1] Third,the contract size and expiration date are *L* shares of *S* and , respectively. Fourth, there is a (male) risk-neutral, representative price manipulator, M. Fifth, if M takes a position in at any time before − Δt, he can manipulate at time − Δt, where Δt is the shortest time to execute an order in the stock market.

Sixth, for convenience, it is assumed that M takes a long position in a futures contract at the price of $_{,}$ at time τ, where 0 ≤ τ < − Δt. Seventh, M can buy up the shares at time − Δt to push up by at the expense of an irrecoverable cost, · ( ), where ( ) is the average irrecoverable cost of \$1 of manipulation of given and $_{\Delta}$. [2] Eighth, $^{()}$ > 0 is assumed because the cost of \$1 of

[1] This can be replaced with = (−) · · + · · , where = + . [2] M can make a profit from his futures position by buying up shares of and pushing up to + . However, these shares cannot be re-sold at + later because no one else wants to buy them at the price artificially raised by M, and M can only resell the shares at the bid prices available in

manipulation increases as increases *ceteris paribus*. For convenience, ( ) = c · is assumed, where > 0. Finally, let = ,∀ .

Suppose that the share price to be realized at without M's manipulation is .[3]

The conditional mean of at − Δt, is $_{\Delta} = _{\Delta} \cdot {}^{\cdot\Delta}$. Given a (plain vanilla) futures contract , M can push up to + by spending · ( ) = · , and his net profit at will be $_{,} - _{,} - c \cdot = + - _{,} - c \cdot =$ $\cdot + - _{,} - c \cdot$. His expected net profit at expiry (Π) is therefore:

$$_{\Delta}[\Pi] = _{\Delta} \cdot + - _{,} - c \cdot$$

$= \cdot {}_{\Delta} \cdot {}^{\cdot\Delta} + - {}_{,} - c \cdot .$

The relevant *incremental* cash flow due to the price manipulation is $\cdot - c \cdot$. To maximize this result, the first-order condition (FOC) with respect to is $- 2c =$ 0 or $^{*} =$. The maximized *incremental* cash flow for M due to the price manipulation by * is therefore:

$$^{*} = {}^{*} - c \cdot ({}^{*}) = 2 - 4 = 4 > 0.$$

Next, given an average futures contract , where ≥ 2, M similarly manipulates by spending $\cdot ( ) = \cdot$. Then ${}_{,} = \Sigma$, and M's net profit is:

the limit order book (LOB) of this order-driven market, prices that are lower than . For example, suppose the current share price is \$100. The currently available bid (ask) prices are \$99, \$98, and \$97 (\$101, \$102, and \$103), and the size of each bid or ask price is 1 share. To push the price up to \$103, M should pay \$306 for the three shares on the ask side. To resell them, M can only do so at the bid prices available in the LOB, which are \$99, \$98, and \$97. Because M paid \$306 and receives \$294 in this round-trip transaction, the loss is \$12. Similarly, it can be shown that the loss will be \$6 or \$2 if M wants to manipulate the price by \$2 or \$1. As is clear in this example, some losses will be involved when manipulating the share price. Therefore, M's net profit from price manipulation is his gain in futures trading minus his loss in stock trading. [3] The calculation is abbreviated here. For more details, refer to Yoo (2015).

$\cdot {}_{,} - {}_{,} - c \cdot = \cdot \Sigma - {}_{,} - c \cdot ,$

and the expected net profit at expiry is:

$${}_{\Delta}[\Pi] = {}_{\Delta}$$

$= \cdot_{\Delta} \cdot \cdot \Delta + \Sigma - {}_{,} - c \cdot .$

The *incremental* cash flow due to the manipulation is − c · . The FOC with respect to is − 2c = 0 or ** = · = · *, and the maximized incremental profit (**) for M is

$$^{**} = \cdot \qquad \cdot\, ^{*} > 0. \quad ^{**}$$

Accordingly, the size of the manipulation of $_{,}$ will be

$$+ ^{**} + \sum - + \sum = \qquad = \frac{1}{2} \cdot \cdot$$

$$^{**} = \frac{1}{2} \cdot \cdot$$

Consequently, the optimal size of manipulation decreases to ** = · *. The nature of average futures contracts means that the impact of $1 of manipulation of on the average futures' price at expiry decreases to times that of a plain vanilla futures. The resulting manipulation size of $_{,}$ therefore shrinks to × = times that of $_{,}$, and so does M's manipulation profit, or ** = · * as shown in Table 1 in Yoo (2015). Yoo's analysis ends here.

Table 1. (Yoo, 2015) (Price Manipulation Risk of Two Futures Contracts)

Type of futures ( )	Manipulation of	Manipulation of	M's profit
Plain vanilla (= 1)			
Average (≥ 2) · · ·			

However, we find that M can *increase* profits by manipulating not just on the expiration date, but also on any other reference date, where = 2, 3, ..., . This is because the futures' final settlement price, = ∑ · , will be moved not just by a change in but also by a change in any . In addition, each of = 1,2,..., } affects equally, so any is exactly as meaningful as for M. Accordingly, there is no reason why M should treat any differently than . Second, when manipulating an , M will face the same circumstances as when manipulating : All but are either constants or uncontrollable variables at time $_{\Delta}$, just as all but are constants at time $_{\Delta}$. As a result, the optimal size of manipulation of any will equal that of . Third, the optimal profit from manipulating any equals that from manipulating . From all these observations, Proposition 1 (summarized in Table 1) can be established. Proofs of all propositions or lemmas are provided in the Appendix.

Proposition 1 (Price manipulation effect of average futures):

i) The manipulation size of an average futures shrinks not to times but

times that of a comparable plain vanilla futures contract.

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ii) The manipulation profit from an average futures for M shrinks not to times

but times that from a comparable plain vanilla futures contract.

iii) Not only on the expiration date but all the other values on the other

reference dates can be manipulated by · .

Table 2. (Price Manipulation Risk of Two Futures Contracts)[4]

Type of futures ( )	Manipulation of	Manipulation of	M's profit
Plain vanilla (= = 1)			
Average (= 1,2,...,)	·	·	·

3. The volatility of average futures

In this section, we explore the volatility of average futures of any kind, including the ones by Yoo (2015), which are more precisely termed *fixed-referencing futures*.[5] The expiration date (time) of an average futures contract with reference dates is denoted by , not by , as it should be a fixed point of time regardless of the size of , where

$\geq \geq \cdots \geq > 0$. The additional notation is as follows, where *variable referencing futures*, the other type of average futures, will be introduced and explained in the next section. Note that no special restrictions are placed on { , ,..., } for (general) average futures (except $\geq \geq \cdots \geq$) but that some restrictions are imposed on

[4] The notation " , " in Yoo (2015) has been changed to " ." [5] We also interchangeably use forward and futures contracts, as both are essentially the same when interest rates are constant, which we assumed to be the case.

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{ ( ), = 1,2,..., } for variable referencing futures.

9 { , ,..., }:
reference dates for an average futures, where $\geq \geq \cdots \geq > 0$

10 ( )| ( ) = + (−), = 1,2,..., :
reference dates for a *variable* referencing futures, where $0 \leq <$

11 ( ): the weight of remaining reference dates at *t* for an average futures contract, where ( ) = ∑

12 { ,..., }: the price at *t* of an average futures with reference dates , ,...,

13 , , : the price at *t* of an average futures with reference dates ( ) = + (−), where = 1,2,...,

14 : the price at *t* of an average futures contract with ( )

The no-arbitrage price of the futures can be expressed succinctly as:[6]

$$\{ ,..., \} = 1 = 1 \; ()() + 1 \;,$$

where [·] is the expected value under the risk-neutral measure at time *t*, and is the indicator function satisfying

$$( ) = \begin{cases} 1, \in \\ 0, h \end{cases}.$$

Second, as in Black-Scholes (1973), follows an Ito process:

[6] For convenience and later use, ${ ,..., } = \sum$,where > .

$$= \cdot \cdot + \cdot \cdot , (1)$$

where μ and σ are the mean and the standard deviation of the continuously compounded annual returns of , respectively, and B_t is a standard Brownian motion. The variance of arithmetic average futures contract is determined by Proposition 2, which also summarizes, and is not limited to, Yoo (2015)'s findings on variances of (fixed-referencing) average futures contracts. Proofs of all propositions or lemmas are provided in the Appendix.

Proposition 2 (Variances of average futures prices):

Given $\geq \geq \cdots \geq$ and $\geq$, i) and ii) hold.

i) var ${ ,..., }$

$$= , , ( ) , , , , - 1 ,$$

,

,

where , = min(,) − and , = max(− ,0).

ii) var ${ ,..., }$ > var ${ ,..., }$, where = 2,3,..., ≥ 0 and ≥ .

4. Variable referencing futures

In this section, we introduce another kind of average futures, variable referencing futures, compare volatility reduction effects of both fixed and variable referencing futures, discuss which one is more desirable for investors in principle and practice, and

examine whether the volatility reduction rule for fixed referencing futures is also applicable to variable referencing futures.

4.1 Rationale for variable referencing futures

Proposition 2 (ii) appears to imply that an average futures contract with more reference dates will have a smaller variance than one with fewer reference dates. However, strictly speaking, the actual variance will also depend on several assumptions. Yoo (2015) demonstrated that the implied rule is true under two conditions. First, an average futures with reference dates has (− 1) reference dates in common with one with (− 1) reference dates, which are , ,..., . Second, , the other reference date, is the closest to time 0, or < min{ , ,..., }. In other words, as the number of reference dates of an average futures contract increases incrementally, its existing reference dates remain intact and a new one is added, which is closer to time 0 than any of the other existing reference dates.

However, a superior rule of addition for reference dates may be available in light of the investors' needs. Suppose investors want an average futures contract, which, given , adequately represents the average of its underlying asset prices for the futures' *entire life*, . Then reference dates should be set as follows, with ( ) denoting the th reference date given , where = 1, 2,..., . If = 2, they should be { (2), (2)} =

, . If = 3, they should be { (3), (3), (3)} = , , . In general, given = N, they should be { ( ), ( ),..., ( ), ( )} =

, ,..., , . As is clear, the th reference date of this futures contract with reference dates, ( ), changes with . That is, ( ) = ≠

= ( ) where ≤ min{ , }, and ≠ . However, in Yoo (2015),

( ) = ( ) = , where ≤ min{ , } and ≠ . For example, given = 3 and 4, two sets of reference dates in Yoo (2015) can be { , , } = , , and

{ , , , } = , , , . In general, they are { , , } =

, , and { , , , } = , , , , where is any positive integer greater than 3. In other words, the time interval between reference dates never changes even if does, and the existing reference dates remain intact as increases. Consequently, the th reference date of this futures with reference dates,

( ), remains unchanged as grows. Therefore, ( ) = ( ) = for all ≤

min{ , }.

To distinguish the futures by Yoo (2015) and the other futures with reference dates { ( )} introduced here, we assigned them the labels of *fixed* referencing and *variable* referencing futures, respectively. The former's existing reference dates do not change as grows, whereas the latter's do. Additionally, given , the latter change in such a way that they become more meaningful when averaged compared with the former. Variable referencing futures may therefore be able to better serve hedgers and speculators, who typically try to hedge against or bet on a periodic average price. For example, suppose = 0, = 1 (year), = 3 and 4, and one year has 240 trading days. The two sets of reference dates for the variable referencing futures contract are *always* , , =

, , and , , , = , , , , whereas those for the fixed one can be any two sets such as , , and , , , .

In this sense, variable referencing futures may be able to better represent a yearly average than can fixed ones.

4.2 Volatility of variable referencing futures

Here we determine the variance of variable referencing futures and compare it with that of fixed referencing futures in Yoo (2015). The general rule of setting reference dates for variable referencing futures is:

( ): ( ) = + (−): = 1,2,..., (2)

Equation (2) indicates that reference dates *evenly* divide the time period between and . For example, if = 0, they divide the whole period of into sub-periods of length . Let , , denote the time-*t* price of a variable referencing futures contract with such reference dates. Then, by (i) of Proposition 2, the variance at (<) of , , is

var , , = ∑ ( ) (,) ( ) − 1 (,) ( ) , . (3)

As we now know two formulas of the variances of both fixed and variable referencing futures, we can compare both variances in at least two ways. First, we examine how the

variance of an average futures contract, fixed or variable referencing, changes as its number of reference dates, , increases. Second, we compare two variances of a fixed and a variable referencing futures contract with the same number of reference dates.

Given that = 0.08, = 0.02, = 0, = 0.4, = 0.25,and − = , **Figure 1**

(**Figure 2**) shows how the variances of a fixed (variable) referencing futures contract decreases as its reference dates increase from 1 to 5. X-axis and y-axis refer to the passage of time and the variance of the fixed referencing futures with being 1 to 5, respectively.

Figure 1 The variance and the number of reference dates of fixed referencing futures

Figure 2 The variance and the number of reference dates of variable referencing futures

From these graphs, we find the following features of the two types of average futures. First, the volatility of fixed referencing futures does not decrease much even if their reference dates increase. Given that the time to expiration is 0.25, the (expected) variance at expiry of the fixed referencing futures only decreases by 0.0023 to 0.0026 as increases by 1, and the variance falls to 0.0330 when reaches 5. In contrast, the volatility of variable referencing futures does decrease fast as their reference dates increase. Given that the time to expiration is 0.25 and is only 2, the (expected) variance at expiry of the variable referencing futures falls down to 0.0262, which is even lower than 0.0330, the variance of fixed referencing futures with being 5. When is

5, the variance of variable referencing futures is as low as 0.0183. Third, given the same , the variance of fixed referencing futures never becomes smaller that that of variable referencing futures no matter how large is. This is because the newly added reference date for fixed referencing futures never gets closer to time 0 than that for variable referencing futures does, which is due to the reference date setting rules of both futures. All these results imply that, if an average futures contract is to be used to minimize excessive expiration day volatility, variable referencing futures are much more effective than are fixed referencing futures by Yoo (2015).

4.3 Volatility and reference dates for variable referencing futures

Here we revisit the relationship between the volatility and the reference dates of (fixed referencing) average futures by Yoo (2015), show that it does not hold for variable referencing futures, and propose a new relationship for variable referencing futures. Because Yoo (2015) analyzes only fixed referencing futures, his conclusions about the volatility of average futures may be limited to that kind of average futures. He concludes that the variance of the price of an average futures contract at any future time *t* decreases with the number of its reference dates. We investigate whether this is true for variable referencing futures, find that it is not, and show why it is not.

Specifically, we show, with a counterexample, that the variance of a variable referencing futures contract does not always decrease as increases. Suppose the stochastic process of is (1), with , , , and being 10%, 30%, 0, and 1, respectively. Then the variances of ' ' and ' ' at $t = 0.75$ are:

$$. ' ' \approx 0.00150 \times (\, _{.} \,) \text{ ,and}$$

$$. ' ' \approx 0.00151 \times (\, _{.} \,) \, .$$

Because var . ' ' > var . ' ', an increase in does not always reduce the variance of average futures contracts. It does reduce the variance of fixed referencing futures, but not necessarily that of variable referencing futures.

To determine why this is the case, the two variances at = 0.75 can be represented as:

$$\text{var} \; . ' ' = \text{var} \; . \; ^{1}4 \text{ ,and var} \; . ' ' = \text{var} \; . \; ^{1}5 \; . + \; ^{1}5 \; .$$

As increases from 4 to 5 given = 0.75, two things happen simultaneously. First, the weight on each reference date decreases from to . Second, the sum of the weights increases from (of = 1) to + of = and = , making the variance of

' ' larger than that of ' ' . This reversal does not always occur at every value of

but does so sometimes. However, the reversal or increase in the sum of weights with an

increase of given never occurs with fixed referencing futures contracts because, for

fixed referencing futures, the remaining reference dates at , or the set of

{ | > , = 1,2,..., } given , are always fixed, even if increases. This feature can

be generalized as Proposition 3.

Proposition 3 (Variances of variable referencing futures' prices):

i) In general, var ' ' does not always decrease in .

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ii) var ' ' ≥ var ' ' ,

where ≤ , is a divisor of , and and are natural numbers

iii) lim → var ' ' = 2∫ ∫ ( )( ) (( )) −

1 ( ) .

If = ,

lim → var ' ' = 2 (− 1) − (3 − 4 + 1) − (− 1)

(+)(2 +) '

where = (−), and = (−).

iv) var ' ' ≥ lim → var ' ' , where is a natural number.

Let us elaborate on Proposition 3(ii). As in (i), var ' ' is not always a decreasing function of given . But if changes from to in such a way that is a

divisor of , then var ' ' ≥ var ' ' . Recalling what happens to var . , , =

var ' ' when changes from 4 to 5,

var ' ' = var < var ' ' = var . + .

However, if changes from 4 to 8 or any other multiple of 4, it can be shown that

var ' ' = var $^1{}_4$ > var ' ' = var $^1{}_8$ + $^1{}_8$.

Furthermore, if > , for example, = , then

var ' ' = var $^1{}_4$ > var ' ' = var $^1{}_8$.

The primary reason for the first inequality is that, while the weight of remaining

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reference dates is = 0.25 in both the left-hand side (LHS) and the right-hand side (RHS), half of it applies to in the RHS, the variance of which is smaller than that of . Further, the main reason for the second inequality is that the weight of remaining reference dates in the RHS is , which is just the half of the in the LHS. In either inequality, the variance of the futures always decreases with if increases this way. Additionally, (iii) and (iv) of Proposition 3 mean that the variance of the futures decreases in the limit as goes to infinity.

5. General effects of reference dates on volatility

In this section, we explore the relationship between reference dates and the volatility of average futures of *any* kind, including fixed and variable referencing futures. In general, there are no restrictions on how to set reference dates or their weights. For example, we can include an average futures with = 3, in which { , , } = , ,1 and { , , } = {0.17,0.56, 0.27}. To this end we define the sum of the weights of the remaining reference dates or "sum-of-weights" for short and provide a notation related to it.

Definition (sum-of-weights function):

( ) = ∑ : sum of weights (of remaining reference dates) at ,

where

0 ≤ t ≤ ,

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= the *i*th reference date (set without any restrictions, and =).

(≥ 0) = the weight of .

= · ()() + ·

= the time-*t* price of an average futures, with *W* as its sum of weights.

Here we analyze not just variable or fixed referencing futures but *any possible* kind of average futures. Given that { , ,..., } is the set of reference dates chosen freely, where > > > ⋯ > , an average futures can be identified with its sum of weights, ( ) = ∑ , by

⎧ 0 | | ( ) = ⎨ +

+ + ⋮ | | ⎩≤ ≤ < ≤ < ≤ <

⋮

= 1 0 ≤ <

.

As shown above, can be regarded as a weight-setting function for an average futures. And, because an average futures can be identified with its , becomes the time price of an average futures, with ( ) being its weight setting function. We introduced mainly because a variance of *any* kind of average futures is a function of ( ).

Proposition 4 (Comparison of variances of average futures):

The variance of the price of an average futures contract increases with the sum of weights of its remaining reference dates. That is,

$$\text{var} \geq \text{var} \;,$$ [7]

where ,(−) ≥ 0, < , and ( ) ≥ ( ) for all > .

Proposition 4 demonstrates that we can determine which variance of two average futures is larger by comparing their sum of weights, whether they are fixed or variable referencing futures. In addition, Proposition 2 (ii) and Proposition 3 (ii) can be verified by Proposition 4, combined with the following lemma.

Lemma 1

(1) Suppose denotes the sum of weights for a fixed referencing futures

{ ,..., }. Then ( ) ≥ ( ),∀ , where < .

(2) Suppose denotes the sum of weights for a variable referencing futures

, , . Then ( ) ≥ ( ),∀ , where is a divisor of , and both are natural numbers.

Figure 3 confirms Lemma 1 and (i) of Proposition 3. Panel 1 shows two fixed referencing futures with = 2 and 3, whose expiration prices are + and

+ + , respectively. As in Lemma 1, ( ) ≥ ( ). Panel 2 shows two variable referencing futures with = 2 and 3, whose expiration prices are

+ and + + , respectively. As shown clearly, ( ) ≥ ( )

[7] Recall that and are time *U*-conditional expectation of final payoffs of underlying assets. Therefore they are well-defined even if *U* is after the maturities of the futures.

1 A. Fixed referencing futures

n=2 n=3

0.5[0] 0 3/12 6/12 9/12 12/12 1 B. Variable referencing futures

n=2 n=3

0.50

0 2/12 4/12 6/12 8/12 10/12 12/12

1 C. Variable referencing futures

n=2 n=4

0.5[0] 0 3/12 6/12 9/12 12/12 **Figure 3** Sum of weights of fixed and variable referencing futures

does not always hold for all values of . However, if the two values of are a divisor and

a multiple, ( ) ≥ ( ) holds, as in Panel 3.

6. Conclusion

Our research is three-fold: we revisit the anti-manipulation effects of average futures by Yoo (2015), explore variable referencing futures, and determine the general relationship between the volatility and reference dates of any kind of average futures contract. For the second issue, we introduce variable referencing futures, explore their distinctive features, compare their volatility with that of fixed referencing futures, and discuss which futures contract would better serve hedgers and speculators in average futures in theory and in practice.

Our findings are as follows. First, the anti-manipulation effects of average futures (fixed referencing) can be smaller than suggested by Yoo (2015). Second, variable referencing futures not only better serve hedgers and speculators, who need to hedge against or bet on a periodic average price, but also have much stronger volatility-reduction effects than do the fixed referencing futures in Yoo (2015). Third, in the case of variable referencing futures, volatility does not necessarily decrease with the number of reference dates. Fourth, the weight of the remaining reference dates at a given time, rather than the number of reference dates, determines the volatility of average futures whether they are variable or fixed referencing.

To our knowledge, this paper is the second academic work on average futures. We hope that this and further research will help a range of average futures or forwards begin to be traded in over-the-counter markets or, ultimately, on exchanges.

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Appendix

Proof of Proposition 1

This proof is similar to Yoo (2015)'s proof of manipulation of , with the sixth and seventh assumptions replaced with the following generalized assumptions that allow M to manipulate any , not just :

> Sixth, M takes a long position in one futures contract at the price of $_{,}$ at time τ, where $0 \leq \tau < - \Delta t$;
>
> Seventh, M can buy shares at time $- \Delta t$ to push up by at the expense of an irrecoverable cost, · ( ), where $k = 1, 2, ..., n$, and ( ) is the average irrecoverable cost of \$1 of manipulation of given and $_{\Delta}$.

Yoo (2015) proves that, from manipulation, M will receive an expected profit of ·

and that will be moved by · at . Because the expected profit is greater than 0, risk-neutral M will always manipulate at $- \Delta t$. For the same reason, if a

positive expected profit can be realized from manipulation, M will always manipulate . Then the relevant question becomes "is this possible?" The answer is yes, and, in fact, this can be proven in a way similar to manipulation. For convenience, manipulation will not be considered while M manipulates , and, afterwards, both manipulations will be combined to determine the cumulative effects on M's profits and the futures' final settlement price, $_{,}$.

Suppose that the share price to be realized at without M's manipulation in $_2$.

The conditional means of $_2$ and at $-\Delta t$ are $_\Delta = _\Delta \cdot {}^{\cdot\Delta}$ and

$_\Delta = _\Delta \cdot {}^{\cdot(\Delta)}$, respectively. M manipulates $_2$ by by spending

$\cdot ( ) = \cdot$ as in the eighth assumption in Section 1. Then, at time , $_, =$

Σ

, where $\sum = \sum + \sum$. Therefore, his

expected profit (Π) at $-\Delta t$ is

$$_\Delta[\Pi] = _{\Delta,} - _, - c \cdot$$

$$+ + \sum$$

$$= _\Delta \qquad - _, - c \cdot$$

$$= _\Delta + + + \sum - _, - c \cdot$$

$$= {}^\cdot{}_\Delta \cdot {}^{\cdot\Delta\cdot(\Delta)} + {}^\Sigma - _, - c \cdot .$$

The *incremental* cash flow due to the manipulation of by is $- c \cdot$

. The FOC with respect to is $- 2c \cdot = 0$ or $= \cdot$. The maximized

incremental profit ( ) for M is therefore

$$= \cdot - c \cdot ( ) = {}^{\cdot\,\cdot} - c \cdot {}^{1} \cdot {}_2 = {}^{1} \cdot {}_4 > 0.$$

Accordingly, the size of the manipulation of $_,$ will be

$+ + \sum$

Now let us combine manipulation and manipulation. At time $- \Delta t$, M

will manipulate , irrespective of manipulation in the past, exactly the same way

as . As in Yoo (2015), and manipulation are virtually identical. This result is

intuitively appealing as there should be no difference between and in their

meaningfulness in or M's profit-seeking behavior. Specifically, M will spend c $\cdot$

$\cdot$ to push up to $+ \cdot$, and receive a profit of $\cdot$ from it.

Combining these two manipulations leads to the following cumulative effects:

i) Both and are manipulated by and , respectively, and =

$= \cdot =$ **.

ii) M's profit is doubled compared with that from an -only manipulation,

changing from $\cdot$ to $\cdot + \cdot = \cdot$.

iii) Manipulation of , is double that from -only manipulation, changing

from to $+ = \cdot + \cdot = \cdot$.

If M does the same thing to , ,..., and , the results will be:

i) Every is manipulated by $\cdot =$ **, where = 1, 2, ..., .

ii) M's profit is times higher than that from -only manipulation: It changes

from $\cdot$ to $\cdot \cdot = \cdot$. This translates to times, not times, the profit from manipulating a comparable plain vanilla futures contract or

.

iii) The manipulation size of is times larger than that from -only

manipulation, changing from · to · · = · . Again, this

translates into times, not times, the manipulation size of a comparable

plain vanilla futures contract or .

Proof of Proposition 2

As in (i) and given ≤ , it can be shown that

$$\{ ,..., \} = 1 \; ()() + 1$$

$$= ()()() + ()$$

$$= {}_{,}()_{,}$$

where

$$_{,} = \min(,) - {}_{,} = \max(- ,0).$$ Accordingly,

,

$$\{ ,..., \} = {}_{,,}()_{,,}$$

.

Because = (,) + | − |, it can also be shown that

{ ,..., }

$= ()_{, , , , ,}{}^{,}{}_{,}\, |\, _{, ,}|$

,

$= ()_{, , , , ,}{}^{,}{}_{,}$

,

.

Then, given < ≤ ,

,

$\text{var}^{\{ ,..., \}} = ()_{, , , , ,}{}^{,}{}_{,} - 1$

,

,

Accordingly, the variance of the futures at *t* becomes

$\text{var}^{\{ ,..., \}} = {}^{\{ ,..., \}} - {}^{\{ ,..., \}}$

which is Proposition 2 (i).

$= ()_{, , , , ,}{}^{,}{}_{,} - 1 .$

Therefore, (ii) can be proved by Proposition 4 and Lemma 1, which will be introduced later.

Proof of Proposition 3

Proposition 3 (i) is verified by the counterexample in Section 4. Proposition 3 (ii) is a special case of Proposition 4 to be proven later. Proposition 3 (iv) will be proven by

Proposition 4 and Lemma 1, which will be proven later. Here, we prove 3 (iii) only. By Proposition 2, the following holds as approaches infinity.

$\lim_{\rightarrow} \mathrm{var}\ {}^{,\,,}$

$$= {}^{()()(())} - 1_{()}$$

$$+ {}^{()()(())} - 1_{()}$$

$$= 2\,{}^{()()(())} - 1_{()}.$$

For reference, var ' ' converges to the following, depending on the range of . That is, given ≤ ,

$$\lim_{\rightarrow} \mathrm{var}\ {}^{,\,,} = 2\,{}^{()()(())} - 1\,.$$

Given < ≤ ,

$$\lim_{\rightarrow} \mathrm{var}\ {}^{,\,,} = 2\,{}^{()()(())} - 1\,.$$

Proof of Proposition 4

Suppose { ,..., } is the union of two sets of reference dates for the two average futures, with and being their weight-setting functions or sums of weights. We first prove the case of = . The two futures' prices at maturity are and .

Then = ∑ , and = ∑ for some values of , , where

≥ 0, and ≥ 0,∀ . Their variances can be expressed as

,

= , ;

, where , ; =

,and

(, h) > 34

. Here,

. Here,

= , ;

,

, ; is increasing in and as ≥ 0. And and can be

interpreted as the two expected values of , ; . Similarly, and can

be interpreted as two density functions of , ; . Also ( ) ≥ ( ) for all > ,

which implies that larger weights are always assigned to larger values of , ; in

∑ , , ; compared with ∑ , , ; . Hence var ≥

var .

We can now prove the case of < < . Recall = ∑ ()() +

∑ . For example, given ≤ < ,

= ()() + .

This can be interpreted as the price of an average futures contract, for which the reference dates and the weights are { , ,..., , } and

{ , ,..., ,∑ ( )( ) }, respectively. Two sums of weights of remaining reference dates for this futures can therefore be defined as

$$_{,}(\ ) = _{(\ )(\ ,\ )}$$

and

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$$_{,}(\ ) = _{(\ )(\ ,\ )}.$$

$_{,}(\ )$ and $_{,}(\ )$ are decreasing in , and $_{(\ )(\ ,\ )}$ is increasing in .

Here, $_{,}(\ )$and $_{,}(\ )$ can be interpreted as the two expected values of

$^{(\ )(\ ,\ )}$ given . Again ( ) ≥ ( ) (for all >), which implies that larger weights are assigned to larger values of $^{(\ )(\ ,\ )}$ in

$\sum^{(\ )(\ ,\ )}$ than in $\sum^{(\ )(\ ,\ )}$. Therefore,

$_{,}(\ ) \geq _{,}(\ )$.

For convenience, let = and = for all ∈ { + 1,..., }. Similarly, we denote = ∑ ( ) , = ∑ ( ) , = , and =

for all ∈ { + 1,..., }. Then, and can be expressed as

,

= , ;

,

,and

= , ;

where (·) is the same function as defined above. Therefore, ≥ as

, ; is increasing in and and ( ) ≥ ( ) for all > .

Proof of Lemma 1

We prove (1) first. If ≤ < , ( ) = < = ( ), where 1 ≤ ≤ −

1 . If < , ( ) ≤ ( ) = 1. Thus ( ) ≤ ( ),∀ . (2) can be similarly

proven.