A linear-rational multi-curve term structure model with stochastic spread*

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Abstract

This study develops a linear-rational multi-curve term structure model based on the Wishart affine process. The model allows for a stochastic correlation between the curves whilst the pricing of swaptions remains at part in terms of numerical complexity with caps and floors. We also show how the constant maturity swap (CMS) and the CMS spread option can be priced. We provide swaption and CMS spread option price approximations that are fast to evaluate and accurate. These approximations heavily rely on the affine property of the Wishart process. We illustrate how the model performs on real data by rolling a calibration using a 3-month long sample of at-the-money swaption data. We find that the estimated parameters are remarkably stable and the calibration procedure is robust. In particular, thanks to the specific Wishart properties the model can handle the stochastic correlation between the OIS term structure and the Euribor-OIS spread term structure.

JEL Classification: G12; G13; C61

Keywords: Interest rate model, Multi-Curve, Wishart process, Stochastic spread, Swaption market, CMS derivatives.

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1 Introduction

Following the global financial crisis, interest rate models were revisited to take into account the widening of the spread between the overnight interest swap (OIS) term structure given by Eonia swaps in the European interest rate market and the Euribor term structure. This led to what is commonly named nowadays the multi-curve models. These models are more challenging as they need to specify not only the dynamic of each curve but also the correlation between these curves making the problem a multidimensional problem that is naturally more complicated. Interest rate derivatives such as caps/floors and swaptions become more difficult to price and manage due to this additional complexity while exotic derivatives are even more challenging but remain the most adequate instruments to reveal the implied correlation structure of the market.

Following the works of Rogers (1997) and Filipović et al. (2017), we propose a three-factor linear-rational multi-curve term structure model based on the Wishart process. Within this framework, the zero-coupon bond price, whose value depends on the OIS curve, and the spread between the Euribor and OIS curves are linear-rational functions of our first two factors given by the diagonal terms of a 2×2 Wishart process. The model enables a stochastic correlation – our third factor – between these two curves, as the Wishart process allows a non trivial correlation between its diagonal terms governed by its off-diagonal component. Thus the model captures the dependency between the OIS curve and the spread that is prevalent in the EUR interest rate derivatives market, while continuing to provide simple and efficient valuation formulas, even in the case of complex products such as the swaption.

Rogers (1997) shows how standard interest rate models fit into the framework of the potential approach. By standard interest rate models we are referring to the exponential affine framework that builds upon Duffie and Kan (1996) and constitutes the dominant part of the interest rate literature.[1] Roughly speaking, the potential approach amounts to conveniently choose a stochastic process to model the underlying risk factors and a function to define a pricing kernel. The author also shows how to generate new interest rate models. Among those new models, the linear-rational model, which owes its name to the fact that the zero-coupon bond price is a linear-rational function of the state variable, is of particular interest as the pricing of swaptions is extremely simple and at par, in terms of computational difficulty, with caps/floors.

Several works investigated the linear-rational interest rate framework, Nakamura and Yu (2000) and Macrina (2014) in the single-curve case and for the multi-curve case Nguyen and Seifried (2015), Macrina and Mahomed (2018) and Filipović et al. (2017) to name a few. However, among multi-curve works, those performing an empirical analysis of the swaption market are much fewer. To the best of our knowledge, such kind of results can be found only in Nguyen and Seifried (2015) who calibrate the model using 1 day of ATM swaption quotes, Crépey et al. (2015b) who calibrate the model using 4 days of swaption quotes (with different strikes, so not only ATM swaptions) and Filipović et al. (2017) who calibrate simultaneously 866 weekly ATM swaption quotes.[2]

Regarding specifically the correlation between the curves, Crépey et al. (2015b) and Nguyen and Seifried (2015) obtain positive correlations with EUR swaption data. Filipović et al. (2017) work with a linear-rational vector affine model and US swaption data. As the US interest rate data do not exhibit any correlation between the curves (Filipović and Trolle, 2013), the limitations of the standard affine model in terms of correlation between the components of the process as explained in Duffie et al. (2003), does not impair the model ability to handle swaption data. However, building a multi-curve linear-rational model using the affine framework that can handle the correlation between the curves, as observed in the EUR interest rate derivatives market, requires to look beyond the vector affine process and consider the Wishart process which is an affine matrix process. Ideally, the correlation should be extracted from the swaption (EUR) derivatives market, which highlights the importance of the linear-rational framework in order to fully exploit the swaption market to calibrate the model.

[1] We refer the reader to Da Fonseca et al. (2013), Moreni and Pallavicini (2014), Morino and Runggaldier (2014), Crépey et al. (2015a), Grbac et al. (2015), Grasselli and Miglietta (2016), Cuchiero et al. (2016), Cuchiero et al. (2019) or Alfeus et al. (2020) just to name a few.

[2] The data used in Crépey et al. (2015b) are also used in Crépey et al. (2015a) but they need to rely on Singleton and Umantsev (2002) to price swaptions as the model is of the standard exponential affine type.

As a first result of our three-factor linear-rational multi-curve term structure model based on the Wishart process, we derive a pricing formula for swaptions whose numerical cost is at par with caps and floors. Swaptions are important interest rate derivatives both in terms of transaction volume and as key elements in the pricing process of any sophisticated interest rate derivatives. Indeed, according to Skantzos and Garston (2019), as of June 2018, the monthly trading volume of the interest rate options market is approximately 1.5 trillion USD, two thirds of which comes from swaption trades and a further 125 billion USD from the cap/floor market. As such, the swaption market is a major component of the interest rate derivatives market. Further to this, exotic interest rate derivatives need to be priced with a model that has to be calibrated on the swaption market. Therefore, building a model that can be calibrated easily on swaption data so that its performance can be analyzed is a crucial first step.

Notice that even when single-curve models were the standard in the interest rate derivatives industry, swaptions were challenging to price as they are a kind of product that is intrinsically multidimensional. Even if there are some approximation formulas for the swaption price, see for example Collin-Dufresne and Goldstein (2002), Singleton and Umantsev (2002) or Schrager and Pelsser (2006), the numerical difficulty is such that only very few empirical studies on the swaption market are available in the literature (see Trolle and Schwartz, 2014). This is in sharp contrast with the equity derivatives literature where comparisons between different model specifications were extensively performed. As a consequence, it should not come as a surprise that in the multi-curve case, which is more challenging numerically, the swaption market is barely analyzed. With regards to the correlation between the curves, it is problematic as swaptions should be used to estimate that correlation.

As a second result, we show that exotic interest rate derivatives such as the constant maturity swap (CMS) and CMS spread options can be priced in our framework. We then develop approximations, which enable a fast and accurate pricing, by adjusting Collin-Dufresne and Goldstein (2002)'s methodology. The approximation technique is very flexible and applies, with equal performance, to swaptions, CMS and CMS spread options. The approximation crucially relies on the affine property of the Wishart process. As such, the model enables an efficient and fast pricing of exotic interest rate derivatives, it is definitively an important second step.

Finally, we perform a rolling calibration over a 3-month sample of daily ATM swaption prices. The calibrated parameters are extremely stable thereby showing the ability of the model to handle the daily fluctuation of the data. We show how information regarding the correlation between the two curves can be extracted from swaptions as well as the advantages of the Wishart process compared to the affine vector process to manage this dependency. Compared with the multi-curve model proposed by Filipović et al. (2017) on the US market, which by construction cannot generate a significant correlation between the curves, we show that the correlation factor explains more than 90% of the implied correlation between the OIS curve and the spread on the European market whatever the maturity. Finally, using the calibrated model, we show that the approximation formulas, for both swaptions and CMS/CMS spread options, are very accurate. The results convincingly demonstrate the need to account for correlations between the OIS curve and the spread as well as the performance of the Wishart process as a modeling tool for interest rate derivatives.

The structure of the paper is as follows. Section 2 reviews the main analytical properties of the Wishart process. In section 3, the interest rate model is specified and we make explicit the pricing formulas for different interest rate products. Section 4 presents the data and illustrates how well the model performs in practice. Section 5 concludes the paper while proofs and tables are gathered in the appendix.

2 The Wishart process

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ we denote by $\mathbb{E}[\,\cdot\,]$ (resp. $\mathbb{E}_t[\,\cdot\,] := \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$) the expectation (resp. conditional expectation) under the probability measure $\mathbb{P}$. The Wishart process, proposed in Bru (1991) and introduced in finance in Gouriéroux and Sufana (2010), satisfies the matrix stochastic differential equation

$$dx_t = (\omega + mx_t + x_t m^\top)dt + \sqrt{x_t}dw_t\sigma + \sigma^\top dw_t^\top \sqrt{x_t}\,, \tag{1}$$

where x_t is an $n \times n$ matrix that belongs to the set of positive definite matrices denoted $\mathbb{S}_n^{++}$, m, σ belong to the set of $n \times n$ real matrices denoted $\mathsf{M}(n)$, $\{w_t; t \geq 0\}$ is a matrix Brownian motion of dimension $n \times n$ (*i.e.*, a matrix of n^2 independent scalar Brownian motions) under the probability measure $\mathbb{P}$ and $\cdot^\top$ stands for the matrix transposition.[3] The matrix $\omega \in \mathbb{S}_n^{++}$ satisfies certain constraints involving $\sigma^\top \sigma$ to ensure the positiveness of the matrix process x_t. Note that the transpositions in Eq. (1) are necessary to preserve the symmetry of the solution. The quantity $\sqrt{x_t}$ is well defined since $x_t \in \mathbb{S}_n^{++}$. The matrix m is such that $\{\Re(\lambda_i^m) < 0; i = 1, \ldots, n\}$ where $\lambda_i^m \in Spec(m)$ for $i = 1, \ldots, n$ and $Spec(m)$ is the spectrum of the matrix m while $\Re(\cdot)$ stands for the real part. The matrix σ belongs to $\mathsf{GL}_n(\mathbb{R})$ the general linear group over $\mathbb{R}$ (*i.e.*, the set of real invertible matrices). Thanks to the invariance of the law of the Brownian motion to rotations and the polar decomposition of σ, we can assume that $\sigma \in \mathbb{S}_n^{++}$.

The infinitesimal generator of the Wishart process is given by (Bru, 1991):

$$\mathcal{G} = \mathrm{tr}[(\omega + mx + xm^\top)D + 2xD\sigma^2 D]\,, \tag{2}$$

where D is the $(n \times n)$ matrix operator $D_{ij} := \partial_{x_{ij}}$.

Using Eq. (1), one can establish the following relations for the quadratic covariations of the components of the Wishart process:

$$d\langle x_{11,\cdot}, x_{11,\cdot}\rangle_t = 4x_{11,t}(\sigma_{11}^2 + \sigma_{12}^2)dt\,, \tag{3}$$
$$d\langle x_{22,\cdot}, x_{22,\cdot}\rangle_t = 4x_{22,t}(\sigma_{12}^2 + \sigma_{22}^2)dt\,, \tag{4}$$
$$d\langle x_{12,\cdot}, x_{12,\cdot}\rangle_t = x_{11,t}(\sigma_{12}^2 + \sigma_{22}^2)dt + 2x_{12,t}(\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22})dt + x_{22,t}(\sigma_{11}^2 + \sigma_{12}^2)dt\,, \tag{5}$$
$$d\langle x_{11,\cdot}, x_{12,\cdot}\rangle_t = 2x_{11,t}(\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22})dt + 2x_{12,t}(\sigma_{11}^2 + \sigma_{12}^2)dt\,, \tag{6}$$
$$d\langle x_{12,\cdot}, x_{22,\cdot}\rangle_t = 2x_{12,t}(\sigma_{12}^2 + \sigma_{22}^2)dt + 2x_{22,t}(\sigma_{11} + \sigma_{22})\sigma_{12}dt\,, \tag{7}$$
$$d\langle x_{11,\cdot}, x_{22,\cdot}\rangle_t = 4x_{12,t}(\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22})dt\,. \tag{8}$$

Bru (1991) showed that the Wishart process is affine, that is the moment generating function is exponentially affine in the state variable. More precisely, the moment generating function is given by

$$\Phi(t, \theta_1, \theta_2, x) := \mathbb{E}\left[\exp\left(\mathrm{tr}[\theta_1 x_t] + \int_0^t \mathrm{tr}[\theta_2 x_u]du\right)\right]\,, \tag{9}$$

where θ_1, θ_2 belong to $\mathbb{S}_n$ the set of real $n \times n$ symmetric matrices, $\mathrm{tr}[\cdot]$ stands for the trace of a matrix and $\mathbb{E}[\cdot] := \mathbb{E}[\cdot | x_0 = x]$.

Following Grasselli and Tebaldi (2008), it is possible to prove that

$$\Phi(t, \theta_1, \theta_2, x_0) = \exp\left(\mathrm{tr}[a(t, \theta_1, \theta_2)x_0] + b(t, \theta_1, \theta_2)\right)\,, \tag{10}$$

with the deterministic functions $(a(t, \theta_1, \theta_2), b(t, \theta_1, \theta_2))$, where $a(t, \theta_1, \theta_2)$ is an $n \times n$ matrix function and $b(t, \theta_1, \theta_2)$ a scalar function, satisfying the system

$$a' = am + m^\top a + 2a\sigma^2 a + \theta_2\,, \tag{11}$$
$$b' = \mathrm{tr}[\omega a]\,, \tag{12}$$

with initial conditions $a(0, \theta_1, \theta_2) = \theta_1$ and $b(0, \theta_1, \theta_2) = 0$. As usual $\cdot'$ denotes the time derivative.

Eq. (11) is a Matrix Riccati ordinary differential equation (ODE) whose solution is given by

$$a(t, \theta_1, \theta_2) = (\theta_1 A_{12}(t) + A_{22}(t))^{-1}(\theta_1 A_{11}(t) + A_{21}(t))\,, \tag{13}$$

[3] By definition, w_t is an $(n \times n)$ matrix Brownian motion if and only if $\forall u, v \in \mathbb{R}^n$, $(w_t u, w_t v)$ is a vector Brownian motion with covariance structure $\mathrm{cov}_t\,[dw_t u, dw_t v] = u^\top v I_n dt$ with I_n the $n \times n$ identity matrix.

where

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} := \exp\left\{ t \begin{pmatrix} m & -2\sigma^2 \\ \theta_2 & -m^\top \end{pmatrix} \right\} . \tag{14}$$

Eq. (12), along with the corresponding initial condition, leads to $b(t)$ after integration.

We denote by e_{ij} the basis of $\mathsf{M}(n)$, it is the $n \times n$ matrix with 1 in the (i, j) place and zero elsewhere, so that $x_{ij,t} = \mathrm{tr}[e_{ij} x_t]$. Then,

$$d\mathbb{E}[x_t] = (\omega + m\mathbb{E}[x_t] + \mathbb{E}[x_t]m^\top)dt\,, \tag{15}$$

that leads if $n = 2$ and m is diagonal to the ODEs

$$d\mathbb{E}[x_{11,t}] = (\omega_{11} + 2m_{11}\mathbb{E}[x_{11,t}])dt\,, \tag{16}$$

$$d\mathbb{E}[x_{22,t}] = (\omega_{22} + 2m_{22}\mathbb{E}[x_{22,t}])dt\,, \tag{17}$$

and we conclude that $\mathbb{E}[x_{11,t}]$ only depends on ω_{11}, m_{11} and $x_{11,0}$ and not on $x_{12,0}$. Similarly, $\mathbb{E}[x_{22,t}]$ only depends on ω_{22}, m_{22} and $x_{22,0}$ and not on $x_{12,0}$. For $x_{12,t}$, we get

$$d\mathbb{E}[x_{12,t}] = (\omega_{12} + (m_{11} + m_{22})\mathbb{E}[x_{12,t}])dt\,. \tag{18}$$

It implies that even if $x_{12,0} = 0$, we can have $\mathbb{E}[x_{12,t}] \neq 0$ for $t > 0$ if $\omega_{12} \neq 0$.

If the process $(x_t)_{t\geq 0}$ is stationary then $\bar{x}_\infty = \lim_{t\to+\infty} \mathbb{E}[x_t]$ satisfies the matrix equation

$$m\bar{x}_\infty + \bar{x}_\infty m^\top = -\omega\,. \tag{19}$$

The Wishart process was initially defined and analyzed in Bru (1991) under the assumption that $\omega = \beta\sigma^2$ with $\beta \in \mathbb{R}_+$ such that $\beta \geq n+1$ to ensure that $x_t \in \mathbb{S}_n^{++}$. Hereafter, this specification will be referred to as the Bru case. It was later extended in Mayerhofer et al. (2011) (see also Cuchiero et al. 2011) to the case $\omega \in \mathbb{S}_n^{++}$ and proved that if

$$\omega \succeq \beta\sigma^2\,, \tag{20}$$

with $\beta \geq n+1$ (where Eq. (20) means that $\omega - \beta\sigma^2 \in \mathbb{S}_n^{++}$) then $x_t \in \mathbb{S}_n^{++}$. From a financial modeling point of view, the advantage of having ω not so tightly related to the volatility matrix σ is that they are naturally estimated using different financial products, which gives the model a flexibility that is often necessary in the applications.

The moment generating function Eq. (10) gives the Laplace transform of the process x_t as the following proposition shows.

Proposition 2.1. *Define*

$$\Xi_t := -\frac{1}{2}\int_0^t e^{(t-s)m}(-2\sigma^2)e^{(t-s)m^\top} ds\,, \tag{21}$$

$$\Lambda_t := \Xi_t^{-1} e^{mt} x_0 e^{m^\top t}\,, \tag{22}$$

then the Laplace transform of x_t in the Bru case (i.e., $\omega = \beta\sigma^2$) rewrites as

$$\mathbb{E}_{x_0}\left[\mathrm{etr}(-\theta_1 x_t)\right] = \det\left(I + 2\Xi_t\theta_1\right)^{-\beta/2} \mathrm{etr}\left(-\frac{\Lambda_t^\top}{2} 2\Xi_t\theta_1(I + 2\Xi_t\theta_1)^{-1}\right)\,, \tag{23}$$

for $\theta_1 \in \mathbb{S}_n^{++}$.

The Laplace transform Eq. (23) is known to be associated with the density of a non-central Wishart distribution. Indeed, if X is a random variable with non-central Wishart distribution, it takes values in $\mathbb{S}_n^{++}$ and its law is denoted by $W_n(\beta, \Xi, \Lambda)$, with $\beta \geq n$, $\Xi \in \mathbb{S}_n^{++}$ and $\Lambda \in \mathsf{M}(n)$. The density of X, reported in Gupta and Nagar (2000, Eq. 3.5.1 p. 114) for example, is given by

$$f(X) = \frac{2^{-\frac{n\beta}{2}}}{\Gamma_n(\beta/2)} \det(\Xi)^{-\frac{\beta}{2}} \operatorname{etr}\left(-\frac{\Lambda}{2} - \frac{\Xi^{-1}X}{2}\right) \det(X)^{\frac{\beta-n-1}{2}} {}_0F_1\left(\frac{\beta}{2}; \frac{1}{4}\Lambda\Xi^{-1}X\right), \tag{24}$$

with $X \in \mathbb{S}_n^{++}$, $\Gamma_n(z)$ with $z \in \mathbb{C}$ the multivariate gamma function defined in Gupta and Nagar (2000, Eq. 1.4.5 p. 18) and ${}_0F_1(a; Z)$ with $a \in \mathbb{C}$ and $Z \in \mathsf{M}(n)$ is the hypergeometric function of matrix argument (see Gupta and Nagar, 2000, p. 34 for a definition). According to Gupta and Nagar (2000, Theorem 1.4.1 p. 19) the following relation between the multivariate gamma and the standard gamma function (of scalar argument) holds $\Gamma_n(z) = \pi^{\frac{1}{4}n(n-1)} \prod_{i=1}^{n} \Gamma(z - (i-1)/2)$. In Gupta and Nagar (2000), $\beta \in \mathbb{N}$ while one consequence of Bru (1991) is to extend to the case $\beta \in \mathbb{R}$ with $\beta \geq n+1$ (see Mayerhofer 2019 and references therein). An efficient numerical algorithm to compute hypergeometric function of matrix argument appears in Koev and Edelman (2006) and its first use in quantitative finance can be found in Kang et al. (2017).

3 A multicurve model

3.1 The OIS and Euribor-OIS term structure curves

We follow Filipović et al. (2017), who build upon the potential approach proposed by Rogers (1997), in order to develop a two-curve model based on a 2×2 Wishart process (*i.e.*, $n = 2$). First, we define a pricing kernel as:[4]

$$\zeta_t := e^{-\alpha t}(1 + x_{11,t}), \tag{25}$$

with $\alpha \in \mathbb{R}_+$ and $(x_{11,t})_{t\geq 0}$ is the $(1,1)^{th}$ element of a Wishart process $(x_t)_{t\geq 0}$. According to Rogers (1997), the pricing kernel can be rewritten as follows. Define the positive function $f : \mathbb{S}_2^{++} \to \mathbb{R}^+$ such that $f(x) := 1 + \operatorname{tr}[e_{11}x]$. Define $g(x) := (\alpha - \mathcal{G})f(x)$, which is a positive function for sufficiently large α (*i.e.*, $\alpha > \operatorname{tr}[\omega]$).

The pricing kernel allows us to compute the time t value of a collateralized zero-coupon bond with maturity T, denoted $P(t,T)$, that is given by

$$P(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right] = \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}\right], \tag{26}$$

$$= e^{-\alpha(T-t)} \frac{1 + \mathbb{E}_t[x_{11,T}]}{1 + x_{11,t}}, \tag{27}$$

with $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ the (conditional) expectation under the risk neutral probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which zero-coupon bond prices are martingale.

The expectation in Eq. (27) can be explicitly computed thanks to the affine property of the Wishart process and for the particular specification adopted here is very simple as the following proposition shows.

Proposition 3.1. *Under the assumption that the matrix m in Eq.* (1) *is diagonal, Eq.* (16) *holds and the zero-coupon bond is given by*

$$P(t,T) = e^{-\alpha(T-t)} \frac{b_1(T-t) + a_1(T-t)x_{11,t}}{1 + x_{11,t}}, \tag{28}$$

with $b_1(t) := 1 + \frac{\omega_{11}}{2m_{11}}(e^{2m_{11}t} - 1)$ and $a_1(t) := e^{2m_{11}t}$.

[4]Note that Filipović et al. (2017) suggests to consider a function of the form $e^{-\alpha t}(a_0 + a_1 x_{11,t})$ with $a_0 > 0$ and $a_1 > 0$ but for identification reasons, clearly explained in Filipović et al. (2017, Theorem 5), one needs to impose $a_0 = 1$ and $a_1 = 1$.

Note that the zero-coupon bond price only depends on $x_{11,t}$, which is a consequence of the diagonal form chosen for m. This assumption is for convenience of the exposition only and can be relaxed at the expense of more cumbersome formulas. One striking property of linear-rational models, whether they are built upon the standard affine process of Duffie and Kan (1996) or the Wishart process, is that the bond price does not depend on the volatility structure of the process. This has a strong consequence in terms of model implementation as it enables the calibration of the parameters ω_{11}, m_{11} and $x_{11,0}$, from the bond yield curve only.

According to Rogers (1997, Eq. 2.4), the short rate is given by

$$r_t = \frac{(\alpha - \mathcal{G})f}{f}\,, \tag{29}$$

$$= \alpha - \frac{\omega_{11} + 2m_{11}x_{11,t}}{1 + x_{11,t}}\,, \tag{30}$$

and is positive by construction as α is such that $g(x)$ is positive. Also, as m has negative eigenvalues then $x_{11,t}$ is stationary and it is straightforward to check from Eq. (28) the following result

$$\lim_{T\to+\infty} -\frac{1}{T-t}\ln P(t,T) = \alpha\,, \tag{31}$$

so that α is the infinite-maturity zero-coupon bond yield as in Filipović et al. (2017). It gives a very simple way to estimate the parameter α from the zero-coupon bond price.

The discount factor $P(T, T+\delta)$ is related to the time T overnight indexed swap (OIS) rate with maturity $T+\delta$ by the formula

$$\mathrm{OIS}(T, T+\delta) = \frac{1}{\delta}\frac{1 - P(T, T+\delta)}{P(T, T+\delta)}. \tag{32}$$

The above formula holds for an OIS with maturity less than one year.

Additionally we consider the Euribor rate $L(T, T+\delta)$, which is the rate at time T for the period $[T, T+\delta]$. Let us denote by $\mathrm{Spread}(T, T+\delta)$, the spread between the Euribor and OIS rates, this is the difference between $L(T, T+\delta)$ and $\mathrm{OIS}(T, T+\delta)$ and is often called the Euribor-OIS spread. Before the global financial crisis, the spread was negligible but after the crisis it widened significantly and a multi-curve interest rate model aims at taking into account that spread and its stochastic evolution. The Euribor-OIS spread is defined by:

$$\mathrm{Spread}(T, T+\delta) := L(T, T+\delta) - \mathrm{OIS}(T, T+\delta)\,, \tag{33}$$

$$= L(T, T+\delta) - \frac{1}{\delta}\left(\frac{1}{P(T, T+\delta)} - 1\right). \tag{34}$$

Similar to the approach in the appendix of Filipović et al. (2017), we specify for the time T deflated value of the Euribor-OIS spread payment at time $T+\delta$ a linear functional of the stochastic process $(x_{22,t})$, the $(2,2)^{th}$ element of the Wishart process. More precisely, the deflated time-T value of the Euribor-OIS spread time-$T+\delta$ payment is defined as[5]

$$\zeta_T P(T, T+\delta)\delta\mathrm{Spread}(T, T+\delta) = e^{-\alpha T}x_{22,T}. \tag{35}$$

Once the deflated value of the Euribor-OIS spread payment at a future date is specified, its expectation gives the value of the spread as a (linear-rational) function of the process as shown in the next proposition.

[5]Filipović et al. (2017, online appendix) suggest to specify the right hand side of Eq. (35) as $e^{-\alpha T}(1 + x_{22,T})$ but we found that specification rather inconvenient as the left hand side of Eq. (35) can be arbitrarily small, if for example the spread is small, and as $x_{22,t}$ is a positive process it can lead to calibration problems. In fact, the specification Eq. (35) matches the one of Rogers (1997, Example 3.7).

Proposition 3.2. *The time-t value of the Euribor-OIS spread payment set at time T and made at time $T+\delta$, simply called the (time-t value) Euribor-OIS spread, is given by:*

$$A(t,T,T+\delta) = \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_T P(T,T+\delta)\delta \text{Spread}(T,T+\delta)\right], \tag{36}$$

$$= \frac{1}{\zeta_t}\mathbb{E}_t\left[e^{-\alpha T} x_{22,T}\right], \tag{37}$$

$$= e^{-\alpha(T-t)}\frac{b_2(T-t)+a_2(T-t)x_{22,t}}{1+x_{11,t}}, \tag{38}$$

with $b_2(t) := \frac{\omega_{22}}{2m_{22}}(e^{2m_{22}t}-1)$ and $a_2(t) := e^{2m_{22}t}$ if we assume that m is diagonal.

Notice that according to Eq. (27) the OIS term structure depends on $x_{11,t}$ whilst the Euribor-OIS spread depends on $x_{22,t}$ and as these two processes are stochastically correlated, the model proposed here is an interest rate multi-curve model with stochastic spread, as Filipović et al. (2017)'s model is, but with the additional important property that the Euribor-OIS spread is correlated with the OIS term structure.[6] What is more, the correlation can take any sign thanks to the property of the Wishart process. Let us now have a closer look at the rich correlation structure generated by the model.

3.2 Correlation structure

In Filipović et al. (2017), the time T deflated value of the Euribor-OIS spread payment at time $T+\delta$ given by Eq. (35) is an affine function of a standard vector affine process that is *independent* of the standard vector affine process that drives the OIS term structure given by Eq. (25). This independence is motivated by the empirical finding in Filipović and Trolle (2013) which states that the OIS term structure and the Libor-OIS spread term structure are not correlated. For the Eonia-Euribor market the empirical correlation between the Eonia and Euribor-Eonia curves is not null (as we shall see later in the empirical section), it is the main motivation to introduce the Wishart process to capture such a dependency.

Let us denote $F(T_1-t, x_{11,t}) := P(t,T_1)$ the bond price with maturity T_1 given by Eq. (28). One can check that

$$\partial_{x_{11}}F = e^{-\alpha(T_1-t)}\left(1-\frac{\omega_{11}}{2m_{11}}\right)\frac{(e^{2m_{11}(T_1-t)}-1)}{(1+x_{11})^2} \leq 0, \tag{39}$$

since $m_{11}<0$, whilst the Euribor-OIS spread $G(T_2-t, x_{11,t}, x_{22,t}) := A(t,T_2,T_2+\delta)$ satisfies

$$\partial_{x_{11}}G = -\frac{G}{1+x_{11}} \leq 0, \tag{40}$$

$$\partial_{x_{22}}G = e^{-\alpha(T_2-t)}\frac{a_2(T_2-t)}{1+x_{11}} \geq 0, \tag{41}$$

therefore the instantaneous covariance between the OIS zero-coupon bond and the Euribor-OIS spread is given by

$$d\langle P(\cdot,T_1), A(\cdot,T_2,T_2+\delta)\rangle_t = \partial_{x_{11}}G\partial_{x_{11}}F d\langle x_{11,\cdot}, x_{11,\cdot}\rangle_t + \partial_{x_{22}}G\partial_{x_{11}}F d\langle x_{11,\cdot}, x_{22,\cdot}\rangle_t. \tag{42}$$

Suppose that $\sigma_{12}=0$, then Eq. (8) implies that the right hand side of Eq. (42) comprises only the leftmost term that is positive thanks to Eq. (3), Eq. (39) and Eq. (40). We conclude that the OIS zero-coupon bond and the Euribor-OIS spread are positively correlated in that particular case. Notice that even if x_{22} is independent of x_{11} the Euribor-OIS spread depends on x_{11} as Eq. (38) clearly shows. Thanks to the second term in Eq. (42), the correlation between the OIS zero-coupon bond and the Euribor-OIS spread of the Wishart multi-curve model proposed here can display any sign. Indeed, Eq. (8), Eq. (39) and Eq. (41) imply that the sign of the

[6] Let us stress the fact that even if the factor driving the discounted spread in Eq. (35) is independent of the factor driving the pricing kernel in Eq. (25), the expected spread given by Eq. (38) does depend on the factor driving the pricing kernel as the presence of $x_{11,t}$ in Eq. (38) clearly shows.

second term is $-\text{sign}(x_{12}\sigma_{12})$. So if x_{12} and σ_{12} have the same signs, the second term in Eq. (42) can lead, if it is large enough in absolute terms, to a negative correlation between the OIS bond price and the Euribor-OIS spread. As such, the Wishart multi-curve model possesses a stochastic basis whose correlation with the OIS term structure is stochastic and can take any sign.

Accounting for the relations (3-8), Equation (42) reads

$$d\langle P(\cdot, T_1), A(\cdot, T_2, T_2+\delta)\rangle_t = 4\partial_{x_{11}} F \left[\partial_{x_{11}} G(\sigma_{11}^2 + \sigma_{12}^2) x_{11,t} + \partial_{x_{22}} G(\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22}) x_{12,t}\right] dt\,, \tag{43}$$

and highlights the fact that our model is actually a three-factor model and not just a two-factor model as might be suggested by the use, so far, of only the two diagonal variables of the matrix x_t. Indeed, the off-diagonal term of the matrix appears as a third factor that drives the instantaneous correlation between the OIS term structure and the Euribor-OIS spread.

Also of interest is the instantaneous covariance of the Euribor-OIS spread term structure. Let $\tau_1 = T_1 - t$ and $\tau_2 = T_2 - t$ two maturities and $A(t, T_1, T_1+\delta)$ and $A(t, T_2, T_2+\delta)$ the Euribor-OIS spreads with time to maturity τ_1 and τ_2, respectively. The instantaneous covariance between those two Euribor-OIS spreads is given by

$$\begin{aligned} \text{cov}(\tau_1, \tau_2) &= \partial_{x_{11}} G(\tau_1)\partial_{x_{11}} G(\tau_2) 4x_{11,t}(\sigma_{11}^2 + \sigma_{12}^2) + \partial_{x_{22}} G(\tau_1)\partial_{x_{22}} G(\tau_2) 4x_{22,t}(\sigma_{12}^2 + \sigma_{22}^2) \\ &\quad + \left(\partial_{x_{11}} G(\tau_1)\partial_{x_{22}} G(\tau_2) + \partial_{x_{22}} G(\tau_1)\partial_{x_{11}} G(\tau_2)\right) 4x_{12,t}\sigma_{12}(\sigma_{11} + \sigma_{22})\,. \end{aligned} \tag{44}$$

From Eq. (39) and Eq. (41) we deduce that the first two terms of the right hand side of Eq. (44) are positive whilst the last term's sign is $-\text{sign}(x_{12,t}\sigma_{12})$. If $\sigma_{12} \neq 0$, the covariance between the Euribor-OIS spreads depends on a factor that does not impact the OIS term structure nor the Euribor-OIS term structure. It is an unspanned stochastic volatility factor (USV). Further to this, the Wishart multicurve model's additional factor x_{12} can take any sign so the last term of Eq. (44) can mitigate the first two terms that are always positive.

3.3 Swaption pricing

The pricing of nonlinear derivatives is important as they are used to calibrate the model on liquid products such as caps/floors and swaptions, often called vanilla products, so that the calibrated model can then be used to price exotic derivatives. It is commonly said that exotic products are priced "consistently" with vanilla products. With exponential affine models, the pricing of caps/floors is often simple from a numerical point of view but, in contrast, the pricing of swaptions is often excessively difficult.

In order to derive the value of a swaption in the Wishart model, let us first compute the time-t value, denoted $C(t, T, T+\delta)$, of a floating coupon fixed at time T and paying $\delta L(T, T+\delta)$ at time $T+\delta$ as

$$C(t, T, T+\delta) = \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_{T+\delta}\delta L(T, T+\delta)\right]\,, \tag{45}$$

$$= \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_T P(T, T+\delta)\delta L(T, T+\delta)\right]\,, \tag{46}$$

$$= P(t, T) - P(t, T+\delta) + A(t, T, T+\delta)\,. \tag{47}$$

Then, let us consider an interest rate swap starting at T_0 and maturing at T_{n_1} where the Euribor based floating leg payment dates are $T_1, \cdots, T_{n_1}$, with $T_j - T_{j-1} = \delta$ for $j = 1, \ldots, n_1$, the fixed leg payment rate K and the fixed leg payment dates are $t_1, \cdots, t_{m_1} = T_{n_1}$, $t_i - t_{i-1} = \Delta$ for $i = 1, \ldots, m_1$ and $t_0 = T_0$. The time $t < T_0$ value of the floating leg of the swap is $\sum_{j=1}^{n_1} C(t, T_{j-1}, T_j) = P(t, T_0) - P(t, T_{n_1}) + \sum_{j=1}^{n_1} A(t, T_{j-1}, T_j)$ while the fixed leg value is $\Delta K \sum_{i=1}^{m_1} P(t, t_i)$. So the fixed-rate payer swap value at time t is

$$\Pi_t^{\text{swap}} = P(t, T_0) - P(t, T_{n_1}) + \sum_{j=1}^{n_1} A(t, T_{j-1}, T_j) - \Delta K \sum_{i=1}^{m_1} P(t, t_i)\,. \tag{48}$$

The time-t forward swap rate, denoted $S_t^{T_0,T_{n_1}}$, is

$$S_t^{T_0,T_{n_1}} = \frac{P(t,T_0) - P(t,T_{n_1}) + \sum_{j=1}^{n_1} A(t,T_{j-1},T_j)}{\Delta \sum_{i=1}^{m_1} P(t,t_i)} \,. \tag{49}$$

Remark 3.3. *The spot swap rate can be obtained from Eq.* (49) *by taking $t = T_0$ and, combined with the zero-coupon bonds extracted from the OIS curve, allows the computation of the current time value of the Euribor-OIS spread, that is the terms $\{A(T_0,T_{i-1},T_i+\delta); i = 1,\dots,n_1\}$.*[7] *These terms can then be used in Eq.* (38) *to estimate the parameters ω_{22}, m_{22} and x_{22,T_0}. This calibration strategy is consistent with the structure of the model that suggests to stage the estimation procedure.*

Given Eq. (48) for the fixed-rate payer swap value at time t, we can derive the value of the corresponding swaption. A striking property of the linear-rational model based on the affine process (whether it be vector or matrix) is the relative simplicity of the swaption pricing formula as the following proposition shows.

Proposition 3.4. *The value at time $t < T_0$ of the European payer swaption with maturity T_0 and swap tenor $T_{n_1} - T_0$ is given by*

$$\begin{aligned}\Pi_t^{swaption} &= \mathbb{E}_t\left[\frac{\zeta_{T_0}}{\zeta_t}(\Pi_{T_0}^{swap})_+\right] , \\ &= \frac{e^{-\alpha(T_0-t)}}{1+x_{11,t}}\mathbb{E}_t\left[(B(T_0,T_{n_1}) + A_1(T_0,T_{n_1})x_{11,T_0} + A_2(T_0,T_{n_1})x_{22,T_0})_+\right] ,\end{aligned} \tag{50}$$

with

$$\begin{aligned}B(T_0,T_{n_1}) :=& b_1(T_0-T_0) - e^{-\alpha(T_{n_1}-T_0)}b_1(T_{n_1}-T_0) + \sum_{j=1}^{n_1} e^{-\alpha(T_{j-1}-T_0)}b_2(T_{j-1}-T_0) \\ &- K\Delta\sum_{i=1}^{m_1} e^{-\alpha(t_i-T_0)}b_1(t_i-T_0)\,,\end{aligned} \tag{51}$$

$$A_1(T_0,T_{n_1}) := a_1(T_0-T_0) - e^{-\alpha(T_{n_1}-T_0)}a_1(T_{n_1}-T_0) - K\Delta\sum_{i=1}^{m_1} e^{-\alpha(t_i-T_0)}a_1(t_i-T_0)\,, \tag{52}$$

$$A_2(T_0,T_{n_1}) := \sum_{j=1}^{n_1} e^{-\alpha(T_{j-1}-T_0)}a_2(T_{j-1}-T_0)\,. \tag{53}$$

Remark 3.5. *The deterministic functions A_1, A_2 and B only involve the diagonal terms of x_0, ω and m which, as already noticed, can be estimated by use of the OIS term structure and the Euribor-OIS spread term structure. Equation* (50) *involves the expectation of a rectified affine combination of the two state variables x_{11} and x_{22}. The non-linear operation introduced by the rectifier is crucial to reveal the dependence between the state variables and its impact on the swaption prices. As a consequence, swaption prices allow the calibration of σ as well as the off-diagonal terms of x_0 and ω (the off-diagonal terms of m are still assumed to be zero).*

As aforementioned, the pricing of the swaption involves a linear functional of the state variable. It sharply contrasts with the classical approach based on the exponential affine framework where the computation of a sum of exponential functions of the state variable is involved for which no simple procedure is available. There are approximation algorithms such as those presented in Singleton and Umantsev (2002) and Schrager and Pelsser (2006) that freeze certain coefficients or the approximation of the density through the cumulant expansion of Collin-Dufresne and Goldstein (2002). In the linear-rational approach, the pricing of a swaption only requires the density of an affine function of the state variables which is known in closed form as we shall see below.

As usual, the caplet pricing formula is obtained by considering a swaption with one fixed payment. More precisely, a caplet with maturity T_0 on the Euribor rate $L(T_0,T_0+\delta)$, pays at time $T_1 = T_0+\delta$ the difference

[7] We remind the reader that T_0 is the current time in that particular case.

$L(T_0, T_0+\delta) - K$, if it's positive, where K is the strike of the caplet. Indeed, standard computations show

$$\Pi_t^{\text{caplet}} = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0+\delta} r_u du}\delta(L(T_0,T_0+\delta)-K)_+\right], \tag{54}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du}P(T_0,T_0+\delta)\delta(L(T_0,T_0+\delta)-K)_+\right], \tag{55}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_0} r_u du}P(T_0,T_0+\delta)\delta\left(\text{Spread}(T_0,T_0+\delta)+\frac{1}{\delta}\left(\frac{1}{P(T_0,T_0+\delta)}-1\right)-K\right)_+\right], \tag{56}$$

$$= \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_{T_0}(1-P(T_0,T_0+\delta)+A(T_0,T_0,T_0+\delta)-K\delta P(T_0,T_0+\delta))_+\right], \tag{57}$$

that is the expression of an option on a swap with one payment, which is a payer swaption.

One striking property of the linear-rational model is that the computational cost of a swaption is at par with the one of the caplet. Indeed, Eq. (50) clearly shows that only the terminal law of an affine function of the marginal of the process is needed, which can be carried out very easily using a Fourier transform (see *e.g.*, Carr and Madan 1999 and Duffie et al. 2000). Define $\Lambda(T_0, T_{n_1})$ the 2×2 matrix with $(A_1(T_0,T_{n_1}), A_2(T_0,T_{n_1}))^\top$ on its diagonal (and 0 elsewhere) and denote the scalar variable $Y = B(T_0,T_{n_1}) + \text{tr}[\Lambda(T_0,T_{n_1})x_{T_0}]$ then the expectation in Eq. (50) rewrites $\mathbb{E}_t[(Y)_+]$. The characteristic function of Y is given by $\Phi_Y(u) = \mathbb{E}_t\left[e^{\mathrm{i}uY}\right] = e^{\mathrm{i}uB(T_0,T_{n_1})}\Phi(T_0-t, \mathrm{i}u\Lambda(T_0,T_{n_1}), 0, x_t)$ with Φ defined by Eq. (9). We have

$$\mathbb{E}_t[(Y)_+] = \frac{1}{\pi}\int_0^{+\infty}\Re\left(\frac{\Phi_Y(u+\mathrm{i}u_i)}{(\mathrm{i}(u+\mathrm{i}u_i))^2}\right)du, \tag{58}$$

with $u_i < 0$. That latter constraint on the integration axis corresponds to a similar constraint in Filipović et al. (2017, Theorem 4).

At that level, the choice of the stochastic process for the state variables is essential. In Crépey et al. (2015b), the authors use exponential martingales based on the Brownian motion and, therefore, need the density of their sum that is not known in closed form and have to rely on a multidimensional integration. In Nguyen and Seifried (2015), a two-factor model is proposed, there it is called the multi-curve rational lognormal model, and leads to a two-dimensional integration of the bivariate Gaussian distribution. As a result, these n-dimensional models imply integrating the n-dimensional Gaussian distribution with the numerical difficulties that come with it when n is larger than two. In the linear-rational based on the Wishart process for the Bru case (*i.e.*, $\omega = \beta\sigma^2$) one could compute the expectation by integrating the distribution Eq. (24) but it will remain numerically tedious. Instead, the formula above shows that in the linear-rational model based on the affine process as presented in Filipović et al. (2017) or the Wishart model as presented here, the pricing of a swaption leads to a one-dimensional integration, irrespective of the size of the model.

3.4 CMS and CMS spread option pricing

The vanilla swaption proved to be surprisingly simple to value in the linear-rational Wishart model and a natural question is whether other exotic products can be also easily priced in that framework. Looking at the interest rate derivatives actively traded on the market, the CMS is certainly the most obvious choice to consider. Following the academic literature (*e.g.*, Brigo and Mercurio 2006, Chapter 13.7) we now recall the characteristics of that product.

Consider a CMS with tenor dates $T_0, \cdots, T_{n_1}$, with $T_j - T_{j-1} = \delta$. The two legs of the CMS have the same payment dates $T_1, \cdots, T_{n_1}$. At a payment date T_{j+1}, with $j = 0, \ldots, n_1 - 1$, one leg pays the Euribor rate resetting at time T_j plus a fixed spread K, while the other leg pays the swap rate $S_{T_j}^{T_{j,0},T_{j,n_s}}$, which is the swap rate with tenor structure and payment dates $T_{j,l} = T_j + l\delta_s$ with $l = 0, \ldots, n_s$ for the floating leg and $t_{j,k} = T_j + k\Delta_s$ for $k = 0, \ldots, m_s$ for the fixed leg and $T_{j,n_s} = t_{j,m_s}$. We suppose that δ and δ_s are equal so that there is no need to introduce another factor (or several factors) to handle the two tenor structures. In practice δ, δ_s and Δ_s are different but we stress the fact that all the computations below can be performed for that more general case without additional significant difficulty.

Proposition 3.6. *The time-t value of the CMS receiving the Euribor (plus a fixed rate K) leg and paying the swap leg is therefore given by*

$$\begin{aligned}\Pi_t^{\text{CMS}} &= P(t,T_0) - P(t,T_{n_1}) + \sum_{j=1}^{n_1} A(t,T_{j-1},T_j) + \delta K \sum_{j=1}^{n_1} P(t,T_j) \\ &\quad - \sum_{j=0}^{n_1-1} \delta \mathbb{E}_t\left[\frac{\zeta_{T_{j+1}}}{\zeta_t} S_{T_j}^{T_j,T_{j,n_s}}\right], \end{aligned} \tag{59}$$

with

$$\mathbb{E}_t\left[\frac{\zeta_{T_{j+1}}}{\zeta_t} S_{T_j}^{T_j,T_{j,n_s}}\right] = \frac{e^{-\alpha(T_{j+1}-t)}}{1+x_{11,t}} \mathbb{E}_t\left[\frac{c_0 + c_1 x_{11,T_j} + c_2 x_{22,T_j} + c_{12} x_{11,T_j} x_{22,T_j} + c_{11} x_{11,T_j}^2}{\mu_0 + \mu_1 x_{11,T_j}}\right], \tag{60}$$

and c_0, c_1, c_2, c_{12}, c_{11}, μ_0 and μ_1 defined in Eqs. (120-126) in the appendix.

To compute the value of the CMS, the expectation Eq. (60) needs to be evaluated but its simple structure, a rational function, combined with the affine property of the Wishart process enable an explicit computation thanks to the following well known remark.

Remark 3.7. *Suppose that we know the moment generating function of the vector (X,Y), that is $G(z_1,z_2) = \mathbb{E}\left[e^{z_1 X + z_2 Y}\right]$. To compute $\mathbb{E}\left[\frac{X}{Y}\right]$, the relation $1/y = \int_0^{+\infty} e^{-sy} ds$ leads to $\mathbb{E}\left[\frac{X}{Y}\right] = \int_0^{+\infty} \mathbb{E}\left[X e^{-sY}\right] ds$, and using the propriety of the moment generating function, we get $\mathbb{E}\left[\frac{X}{Y}\right] = \int_0^{+\infty} \partial_{z_1} \mathbb{E}\left[e^{z_1 X - sY}\right] ds|_{z_1=0}$. As $\mathbb{E}\left[e^{z_1 X - sY}\right]$ is known and can possibly be derived explicitly with respect to z_1, we obtain a quasi closed form for the expectation of the ratio of the two random variables.*

Proposition 3.8. *The integral representation of the ratio of two random variables combined with the moment generating function of the Wishart process Eq. (10) give explicit expressions for the expectations:*

$$\mathbb{E}_t\left[\frac{1}{\mu_0 + \mu_1 x_{11,T_j}}\right], \quad \mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0 + \mu_1 x_{11,T_j}}\right], \tag{61}$$

which are sufficient to compute Eq. (60).[8]

The CMS naturally serves as an underlying for interest rate derivatives but instead of the standard call/put on an CMS rate what is frequently found is the CMS spread single-option which involves two CMS rates as its name suggests. Let us denote by $\Pi_t^{\text{CmsSpSO}}(T_1, n_{s_1}, n_{s_2}, K)$ the t-value of a CMS spread call option with single expiration date T_1 and strike K. It is an option whose value at time T_1 is based on the difference between the spot swap rate $S_{T_1}^{T_{1,0},T_{1,n_{s_1}}}$, starting at time T_1 and ending at time $T_{1,n_{s_1}} > T_1$, and the spot swap rate $S_{T_1}^{T_{1,0},T_{1,n_{s_2}}}$ starting at time T_1 and ending at time $T_{1,n_{s_2}} > T_1$.

The underlying swap $S_{T_1}^{T_{1,0},T_{1,n_{s_1}}}$ floating leg's tenor and payment dates are $T_{1,l} = T_1 + l\delta_s$ with $l = 0,\ldots,n_{s_1}$ whilst the fixed leg's tenor and payment dates are $t_{1,k} = t_1 + k\Delta_s$ with $k = 0,\ldots,m_{s_1}$ and we further have that $T_{1,0} = T_1$, $t_{1,0} = t_1 = T_1$ and $T_{1,n_{s_1}} = t_{1,m_{s_1}}$ which imply that both legs start and end at the same time. The swap $S_{T_1}^{T_{1,0},T_{1,n_{s_2}}}$ is defined similarly.

Those two CMS rates are the underlyings of the CMS spread single-option whose pricing formula is presented in the next proposition.

[8]The first expectation in Eq. (61) admits a closed-form expression in terms of a sum of generalized exponential functions when the matrix m is diagonal. Indeed, in such a case, $(x_{11,t})_{t\geq 0}$ follows a simple square-root process with parameters $\left(\omega_{11}, m_{11}, (\sigma^2)_{11}\right)$. Unfortunately, this result does not extend to the case of a non-diagonal matrix m and does not help evaluate the expectations involved in the valuation of CMS spread options considered below. That is why we do not provide the closed-form expression and rely on a more versatile power series approximation in the sequel (see section 3.5).

Proposition 3.9. *The option time-t value, denoted* Π_t^{CmsSpSO} *for simplicity, is given by:*

$$\Pi_t^{\mathrm{CmsSpSO}} = \mathbb{E}_t\left[\frac{\zeta_{T_1}}{\zeta_t}\left(S_{T_1}^{T_{1,0},T_{1,n_{s_1}}} - S_{T_1}^{T_{1,0},T_{1,n_{s_2}}} - K\right)_+\right], \tag{62}$$

$$= \frac{e^{-\alpha(T_1-t)}}{1+x_{11,t}}\mathbb{E}_t\left[\left(g^1(x_{11,T_1},x_{22,T_1}) - g^2(x_{11,T_1},x_{22,T_1}) - K(1+x_{11,T_1})\right)_+\right], \tag{63}$$

with

$$g^i(x_{11,T_1},x_{22,T_1}) = \frac{c_0^i + c_1^i x_{11,T_1} + c_2^i x_{22,T_1} + c_{12}^i x_{11,T_1}x_{22,T_1} + c_{11}^i x_{11,T_1}^2}{\mu_0^i + \mu_1^i x_{11,T_1}}, \tag{64}$$

and for $i \in \{1,2\}$ *with* c_0^i, c_1^i, c_2^i, c_{12}^i, c_{11}^i μ_0^i *and* μ_1^i *for* $i \in \{1,2\}$ *given in the appendix.*

Similar to caplets (or floorlets) that are not traded individually but as a component of a cap (floor), the CMS spread single-option is traded through a CMS spread multi-option which is just a portfolio of CMS spread single-options and is defined as follows. Let $\Pi_t^{\mathrm{CmsSpMO}}(T_1, T_{n_1}, n_{s_1}, n_{s_2}, K)$ be the t-value of the multi CMS spread call option with exercise dates $T_1, \cdots, T_{n_1}$, with $T_j - T_{j-1} = \delta$ and strike K. It is a sum of CMS spread single call option with maturity dates $T_1, \ldots, T_{n_1}$. All the options' two underlying swaps have the same tenor structures. Using the previous definition, the option time t-value denoted Π_t^{CmsSpMO}, for simplicity and when no confusion is possible, is given by:

$$\Pi_t^{\mathrm{CmsSpMO}} = \sum_{j=1}^{n_1} \Pi_t^{\mathrm{CmsSpSO}}(T_j, n_{s_1}, n_{s_2}, K) \tag{65}$$

Unfortunately, the pricing formula of the CMS spread single-option Eq. (63) is not as simple as the swaption pricing formula. Notice, however, that it only involves $(x_{11,T}, x_{22,T})$, the marginal distribution of the process at time T and not the process path from t to T. In the Bru case, the marginal distribution of the process can be expressed, when the parameter β is an integer, as the square of a matrix Gaussian distribution is therefore computable by Monte Carlo very efficiently. When β is not an integer, Ahdida and Alfonsi (2013) derived an exact and fast simulation algorithm. Still, having accurate price approximations for these products is of interest and the following section shows that such approximations are available thanks to the affine property of the Wishart process.

3.5 Approximation of interest rate derivatives

3.5.1 Swaption price approximation

The pricing of a swaption in the standard exponential affine framework is known to be notoriously tedious as it involves the density of a sum of exponentials of random variables. The swaption price can be computed easily only in some very specific cases, typically when the state variable is one dimensional. In Collin-Dufresne and Goldstein (2002), the authors propose an approximation of the swaption price by approximating the density of a coupon bearing bond. Their result crucially relies on the affine property of the process driving the interest rates. In the approach adopted here, Proposition 3.4 shows that the pricing of a swaption is simple as it only requires a one-dimensional integration. The affine property of the Wishart process enables us to derive an approximation in the spirit of Collin-Dufresne and Goldstein (2002) and derive an even faster option pricing formula. Notice that the affine property is used in two different ways. In Collin-Dufresne and Goldstein (2002), the authors use the fact that the expected value of the exponential of an affine variable is exponential affine whereas here we use the fact that the expected value of a polynomial function of a given order of an affine process can be expressed as a polynomial function of the process, in order words the set of polynomials is stable for the infinitesimal generator of the Wishart process.

To establish the approximation, we need the two following lemmas.

Lemma 3.10. *Let $y(t)$ a function solution of the ordinary differential equation*

$$\frac{dy(t)}{dt} = \kappa y(t) + \sum_{i=1}^{l} \bar{a}_i + \bar{b}_i e^{\kappa_i t}\,, \tag{66}$$

with $\kappa \neq \kappa_i$ $\forall i \in \{1,\ldots,l\}$ and $\bar{a}_i, \bar{b}_i$ $\forall i \in \{1,\ldots,l\}$ some constants. Then it can be integrated to

$$y(t) = \bar{c} + \sum_{i=1}^{l+1} \bar{d}_i e^{\kappa_i t}\,, \tag{67}$$

with $\kappa_{l+1} = \kappa$ and

$$\bar{c} = -\sum_{i=1}^{l} \frac{\bar{a}_i}{\kappa}\,, \tag{68}$$

$$\bar{d}_i = \frac{\bar{b}_i}{\kappa_i - \kappa}\,, \quad i = 1,\ldots,l\,, \tag{69}$$

$$\bar{d}_{l+1} = y(0) + \sum_{i=1}^{l} \frac{\bar{a}_i}{\kappa} - \sum_{i=1}^{l} \frac{\bar{b}_i}{\kappa_i - \kappa}\,. \tag{70}$$

For notational convenience, it is useful to introduce $\bar{\sigma} = \sigma^2$ and notice that in Eqs. (3–8), $(\sigma_{11}^2 + \sigma_{12}^2) = (\sigma^2)_{11} = \bar{\sigma}_{11}$, $(\sigma_{12}^2 + \sigma_{22}^2) = (\sigma^2)_{22} = \bar{\sigma}_{22}$ and $(\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22}) = (\sigma^2)_{12} = \bar{\sigma}_{12}$. It can be seen that the infinitesimal generator $\mathcal{G}$ given by Eq. (2) only involves σ^2. The following lemma shows that the affine property of the Wishart process implies a simple expression for the expected value of a polynomial function of the process.

Lemma 3.11. *Let us denote $g(t,i,k,j) = \mathbb{E}[x_{11,t}^i x_{12,t}^k x_{22,t}^j]$ where $(x_{11,t}, x_{12,t}, x_{22,t})$ are the components of a 2×2 Wishart process. Then using the Eqs. (3-8) and Itô's Lemma we get*

$$\frac{dg(t,i,k,j)}{dt} = (i2m_{11} + k(m_{11}+m_{22}) + 2jm_{22})\, g(t,i,k,j) \tag{71}$$

$$+ (i\omega_{11} + 2ik\bar{\sigma}_{11} + 2i(i-1)\bar{\sigma}_{11})\, g(t,i-1,k,j) \tag{72}$$

$$+ (k\omega_{12} + k(k-1)\bar{\sigma}_{12} + 2ik\bar{\sigma}_{12} + 2jk\bar{\sigma}_{12})\, g(t,i,k-1,j) \tag{73}$$

$$+ (j\omega_{22} + 2j(j-1)\bar{\sigma}_{22} + 2jk\bar{\sigma}_{22})\, g(t,i,k,j-1) \tag{74}$$

$$+ \frac{k(k-1)}{2}\bar{\sigma}_{22} g(t,i+1,k-2,j) \tag{75}$$

$$+ \frac{k(k-1)}{2}\bar{\sigma}_{11} g(t,i,k-2,j+1) \tag{76}$$

$$+ 4ij\bar{\sigma}_{12} g(t,i-1,k+1,j-1)\,. \tag{77}$$

Notice that Eqs. (72–77) involve polynomials with degree lower or equal to $i+k+j-1$ whilst Eq. (71) involves a polynomial of degree $i+k+j$, it is a consequence of the affine property of the Wishart process. As $g(t,1,0,0)$, $g(t,0,1,0)$ and $g(t,0,0,1)$ can be written in the form $\bar{a}_0 + \bar{b}_0 e^{\kappa t}$ with suitable $\bar{a}_0, \bar{b}_0$ and κ coefficients then we deduce by recurrence that $g(t,i,k,j)$ solves an ODE of the form Eq. (66) and therefore Lemma 3.10 applies. Notice that the condition $\kappa \neq \kappa_i$ $\forall i \in \{1,\ldots,l\}$ of Lemma 3.10 is satisfied as $m_{11} < 0$ and $m_{22} < 0$.

Starting from Eq. (50), define $Y_{T_0} = B(T_0,T_{n_1}) + A_1(T_0,T_{n_1})x_{11,T_0} + A_2(T_0,T_{n_1})x_{22,T_0}$. To apply Collin-Dufresne and Goldstein (2002)'s swaption price approximation one needs to compute the q^{th} moment of Y_{T_0} that is simply given by

$$\mathbb{E}[Y_{T_0}^q] = \sum_{l_0+l_1+l_2=q} \binom{q}{l_0,l_1,l_2} B(T_0,T_{n_1})^{l_0} A_1(T_0,T_{n_1})^{l_1} A_2(T_0,T_{n_1})^{l_2} \mathbb{E}[x_{11,T_0}^{l_1} x_{22,T_0}^{l_2}]\,. \tag{78}$$

As $\mathbb{E}[x_{11,T_0}^{l_1} x_{22,T_0}^{l_2}] = g(T_0,l_1,0,l_2)$ is known thanks to Lemma 3.11, the moments are also known.

Remark 3.12. *Notice that although only terms of the form* $\mathbb{E}[x_{11,T_0}^{l_1} x_{22,T_0}^{l_2}] = g(T_0, l_1, 0, l_2)$ *are needed to determine the moment* $\mathbb{E}[Y_{T_0}^q]$*, Lemma 3.11 shows that these terms depend on moments involving* x_{12,T_0} *(through Eq. 77).*

Starting from Collin-Dufresne and Goldstein (2002, Eq. 17), which presents an expansion of the density of Y_{T_0}, given by

$$\frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(y-c_1)^2}{2c_2}} \left(\sum_{j \geq 0} \gamma_j (y - c_1)^j \right) , \tag{79}$$

the expectation in Eq. (50) can be expressed as

$$\mathbb{E}_t \left[(Y_{T_0})_+ \right] \sim \sum_{j \geq 0} \gamma_j \int_0^{+\infty} \frac{1}{\sqrt{2\pi c_2}} y (y - c_1)^j e^{-\frac{(y-c_1)^2}{2c_2}} dy \, , \tag{80}$$

$$= \sum_{j \geq 0} \gamma_j \sqrt{c_2} c_2^{j/2} \int_{\frac{-c_1}{\sqrt{c_2}}}^{+\infty} z^{j+1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \, , \tag{81}$$

$$+ \sum_{j \geq 0} \gamma_j c_1 c_2^{j/2} \int_{\frac{-c_1}{\sqrt{c_2}}}^{+\infty} z^j \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \, , \tag{82}$$

$$= \sum_{j \geq 0} \gamma_j \lambda_{j+1} + c_1 \sum_{j \geq 0} \gamma_j \lambda_j \, , \tag{83}$$

where $\{\gamma_j; j \in \mathbb{N}\}$ are related to the cumulants through Eqs. (B.18–B.25) in Collin-Dufresne and Goldstein (2002), $\{c_j; j \in \mathbb{N}\}$ are the cumulants of the variable Y_{T_0} that are related to the moments of that variable through Eqs. (A.1–A.7) in Collin-Dufresne and Goldstein (2002) whilst $\{\lambda_j; j \in \mathbb{N}\}$ are related to the normal density/cumulative distribution (see Collin-Dufresne and Goldstein, 2002, Eqs. B.10–B.17).

3.5.2 CMS and CMS derivative approximations

To evaluate a CMS, one needs to compute the expectation given in Eq. (63), it can be done exactly thanks to Proposition 3.8 but it requires a one or two dimensional integration depending on whether the Bru condition (*i.e.*, $\omega = \beta\sigma^2$) is satisfied or not. For standard interest rate models, such as exponential affine models, there is no closed form solution for that expectation and one needs to rely on some approximations, see Brigo and Mercurio (2006) and Hanton and Henrard (2012). It is useful to notice that Eq. (63) is the expectation of a ratio of two polynomials of the Wishart process and a series expansion of the denominator enables us to rewrite the problem as an expectation of a series of the Wishart process that when truncated leads to an expectation of a polynomial function of the Wishart process. The moments of the Wishart process being known, thanks to Lemma 3.11, we obtain an approximation of the expectation as the following proposition shows.

Proposition 3.13. *The time-t value expectation*

$$I = \mathbb{E}_t \left[\frac{c_0 + c_1 x_{11,T_j} + c_2 x_{22,T_j} + c_{12} x_{11,T_j} x_{22,T_j} + c_{11} x_{11,T_j}^2}{\mu_0 + \mu_1 x_{11,T_j}} \right] , \tag{84}$$

can be approximated by

$$\begin{aligned} I(M) = \; & \frac{c_0}{\mu_0} \sum_{m=0}^{M} (-1)^m \left(\frac{\mu_1}{\mu_0} \right)^m \mathbb{E}_t \left[x_{11,T_j}^m \right] + \frac{c_1}{\mu_0} \sum_{m=0}^{M} (-1)^m \left(\frac{\mu_1}{\mu_0} \right)^m \mathbb{E}_t \left[x_{11,T_j}^{m+1} \right] \\ & + \frac{c_2}{\mu_0} \sum_{m=0}^{M} (-1)^m \left(\frac{\mu_1}{\mu_0} \right)^m \mathbb{E}_t \left[x_{11,T_j}^m x_{22,T_j} \right] + \frac{c_{12}}{\mu_0} \sum_{m=0}^{M} (-1)^m \left(\frac{\mu_1}{\mu_0} \right)^m \mathbb{E}_t \left[x_{11,T_j}^{m+1} x_{22,T_j} \right] \\ & + \frac{c_{11}}{\mu_0} \sum_{m=0}^{M} (-1)^m \left(\frac{\mu_1}{\mu_0} \right)^m \mathbb{E}_t \left[x_{11,T_j}^{m+2} \right] , \end{aligned} \tag{85}$$

where M is the truncation order of the series $1/(1+(\mu_1/\mu_0)x)$ and the constants c_0, c_1, c_2, c_{12}, c_{11}, μ_0 and μ_1 are those of Proposition 3.6 while the expectations in Eq. (85) *are given by Lemma 3.11. Notice that $0 < \mu_1/\mu_0 < 1$ by construction.*

In the Collin-Dufresne and Goldstein (2002) swaption price approximation, the key ingredient is the set of moments of the random variable whose law, which is unknown, is needed to compute the expectation associated option price. In the CMS option case, the underlying random variable is the discounted swap rate that is given by a ratio of polynomial functions of the Wishart process. The ratio can be expanded as a series, by performing a series expansion of the denominator, and after a truncation it gives an approximation of the variable of interest by a polynomial function of the Wishart process. The moments of the Wishart process being known, the different moments of the CMS option underlying variable are known and therefore Collin-Dufresne and Goldstein (2002) can be readily applied. To illustrate the method, the CMS option spread is used with the following proposition providing the details.

Proposition 3.14. *Consider the CMS spread option of Proposition 3.9, and define the expectation in Eq.* (63) *by $\mathbb{E}_t\left[(Y_T)_+\right]$ where $Y_T = g^1(x_{11,T_1}, x_{22,T_1}) - g^2(x_{11,T_1}, x_{22,T_1}) - K(1 + x_{11,T_1})$ with $g^1(.,.)$ and $g^2(.,.)$ defined by Eq.* (64)*. Define the approximation Y_T^M of order M of Y_T by*

$$Y_T^M = \sum_{i=0}^{2M+4} \zeta_i y_i\,, \tag{86}$$

with $\{\zeta_i; i = 0, \dots, 2M+4\}$ constants given in the proof while $y_i = x^i_{11,T}$ for $i = 0, \dots M+2$ and $y_i = x_{22,T}x^{i-M-2}_{11,T}$ for $i = M+3, \dots, 2M+4$. The q^{th} moment of Y_T can be approximated by the q^{th} moment of Y_T^M given by

$$\mathbb{E}\left[(Y_T^M)^q\right] = \sum_{k_0+\ldots+k_{2M+4}=q} \binom{q}{k_0, \dots, k_{2M+4}} \prod_{j=0}^{2M+4} \zeta_j^{k_j} \mathbb{E}\left[\prod_{l=0}^{2M+4} y_l^{k_l}\right]. \tag{87}$$

The expectation $\mathbb{E}\left[\prod_{l=0}^{2M+4} y_l^{k_l}\right]$ is known thanks to Lemmas 3.10 and 3.11, so the q^{th} moment of Y_T^M is known.

Notice that to compute the moment of order q of Y_T^M, we need the moments of order $q(M+2)$ of x_T.

4 Model implementation

4.1 The data

This study considers the Euro market and the data comprise the OIS term structure, the Euribor term structure and the ATM swaption prices for the period 4 October 2011 to 12 March 2012. For the term structures, either OIS or Euribor, we restrict to a maturity smaller than 15 years.[9] For the OIS, we use Eonia rates that have floating and fixed legs that pay annually (when the swap's maturity is larger than 1 year). For the Euribor rates, the floating leg pays semi-annually while the fixed leg pays annually. Table I reports the basic descriptive statistics, mean and standard deviation, for each term structure of interest rates. Both term structures are increasing and as expected the Euribor curve is above the OIS curve reflecting its credit risk component. For both curves, long term rates display lower standard deviations.

[Insert Table I here]

The swaption data is usually quoted in terms of normal or log-normal volatility. In our data set, the normal volatility quotes are converted into prices using the Bachelier formula for at-the-money call options, it is the

[9] The current framework is designed to generate positive interest rates, indeed Rogers (1997) generalizes the work of Constantinides (1992) that focuses on nominal interest rates, whilst they have been close to zero and even negative during a recent period. This forces us to consider the period at the beginning of 2012.

market practice and the approach used in Filipović et al. (2017, online appendix). Section A.1 of the appendix presents the basic formulas. By definition the at-the-money swaption is the option with a strike equal to the forward swap rate that can be synthesized using two spot swap rates which are quoted (see A.2 of the appendix for the details). Furthermore, for the swaption strike we follow Filipović et al. (2017) and set it to the model-implied forward swap rate. We consider the swaption maturities 1Y, 2Y, 3Y, 4Y and 5Y whilst for the swap tenor we restrict to 1Y, 2Y, 3Y, 4Y and 5Y. Table II reports mean and standard deviation of the normal implied volatility for each option. For a given swap tenor, the implied volatility is increasing with the swaption maturity while for a given swaption maturity, the implied volatility is increasing with the swap tenor for swaption maturities less than or equal to two years and decreasing for swaption maturities greater than or equal to three years. Regarding the standard deviations, for a given swaption maturity, the standard deviation decreases as the swap tenor increases while for a given swap tenor the standard deviation decreases as the swaption maturity increases.

[Insert Table II here]

4.2 Calibration results and analysis

For the implementation, we follow the common market practice of performing a daily calibration and rolling it but we take into account the specifics of the model by staging the estimation procedure. More precisely, we proceed as follows. First, relying on Eq. (31), α is estimated as the long-term zero-coupon bond yield. Then the parameters $x_{11,t}$, ω_{11} and m_{11} are estimated by solving the optimization problem

$$\min \frac{1}{N} \sum_{i=1}^{N} \left(P_{\text{model}}(t, T_i) - P_{\text{market}}(t, T_i) \right)^2 , \tag{88}$$

where $P_{\text{market}}(t, T_i)$ is the market price at time t of a zero-coupon with maturity T_i, obtained by bootstrapping the OIS term structure, whilst $P_{\text{model}}(t, T_i)$ stands for the corresponding model price given by Eq. (28) and N is the number of zero-coupon prices available for that day. Using the Euribor swap rates along with the OIS zero-coupon bond market prices, we extract the market spreads given by Eq. (38) and then calibrate for each day the parameters $x_{22,t}$, ω_{22} and m_{22} by solving the optimization problem

$$\min \frac{1}{N} \sum_{j=1}^{N} \left(A_{\text{model}}(t, T_{j-1}, T_j) - A_{\text{market}}(t, T_{j-1}, T_j) \right)^2 . \tag{89}$$

Lastly, using the swaptions we calibrate $\sigma_{11}, \sigma_{12}, \sigma_{22}$ and $x_{12,t}$, ω_{12} by solving

$$\min \frac{1}{N} \sum_{i=1}^{N} \frac{1}{M_i} \sum_{j=1}^{M_i} \left(\sigma_{\text{model}}(t, T_i, T_{i,j}) - \sigma_{\text{market}}(t, T_i, T_{i,j}) \right)^2 , \tag{90}$$

with $\sigma_{\text{model}}(t, T_i, T_{i,j})$ the swaption model (normal) implied volatility for day t, swaption maturity T_i and swap tenor $T_{i,j} - T_i$ given by Eq. (50), $\sigma_{\text{market}}(t, T_i, T_{i,j})$ stands for the corresponding market (normal) implied volatility while N is the number of swaption maturities and M_i is the number of tenors for the i^{th} maturity available for that day.

The mean value as well as the standard deviation of the estimated parameters are reported in Table III while Table IV reports the eigenvalues of x, ω and σ in order to provide a sanity check of the estimates. Table V reports the correlations associated with x, ω and σ as well as the long term mean value $\bar{x}_\infty$ given by Eq. (19). Table VI contains the average as well as the standard deviation of the root mean square errors of the calibrations Eqs. (88-90).

[Insert Table III here]

[Insert Table IV here]

[Insert Table V here]

[Insert Table VI here]

As per Eq. (31), the value of α corresponds to the long term yield and the mean value is equal to 2.4% with a small standard deviation. Combined with Eq. (30) we find that the model short term rate is not positive. Let us point out that Filipović et al. (2017, Table IA.III, online appendix) proceed the other way around; α is such that the short term rate is positive and therefore the natural question is whether the long term yield is accurately fitted.[10] The mean values of x_{11} and x_{22} are positive with small standard deviations. The value of x_{12} is on average negative and when combined with x_{11} and x_{22} leads to matrix (*i.e.,* the matrix with x_{11} and x_{22} on the diagonal and x_{12} on the off-diagonal) that has positive eigenvalues according to Table IV whilst the correlation associated with the matrix x is on average equal to -0.423 as shown in Table V. The mean values of ω_{11} and ω_{22} are positive with a small standard deviation for the ω_{11} but a rather large (compared to the mean) one for ω_{22}. The value of ω_{12} is on average positive and leads to a matrix ω which has positive eigenvalues according to Table IV, the correlation associated with the matrix ω is on average equal to 0.223 as shown in Table V. We find that for each day m_{11} and m_{22} are negative with the mean estimated values reported in Table III along with the standard deviations that are small. All the elements of σ are positive with small standard deviations, the eigenvalues of σ reported in Table IV are positive and the correlation associated with the matrix σ is 0.497, that is rather strong. As expected all the matrices belong to $\mathbb{S}_2^{++}$ and the correlations associated with these matrices give an indication of the dependency between the factors and therefore the curves in the model. Notice that $\bar{x}_\infty$ given by Eq. (19) is a positive definite matrix whose correlation associated with the off-diagonal term is 0.208 according to Table V, there is a change in the correlation sign between the long term mean value of the Wishart process and its initial value.

Tables III, IV and V also report min and max values for the estimates. They confirm that the model properties deduced from the mean and the standard deviation values are not only valid on average over the sample but also punctually in time for each day.

The calibration errors are reported in Table VI, they are overall very reasonable if we take into account the parsimony of the model. Regarding the standard deviation of the errors, compared to the mean it is small for the OIS curve but rather large for the spread and translates the large standard deviation observed for ω_{22}. The rather large (compared to the mean) standard deviation of the spread calibration error is due to the calibration procedure that is sequential. Any variation in the OIS calibration error will impact the spread calibration error that builds upon it.[11] For the swaptions, the error is 36.24 with a small standard deviation showing the ability of the model to capture the daily variation of the data.

Although the calibration errors reported in Table VI are quite reasonable, they could be improved in two directions. First, the fit of the initial yield curve at the first stage (see Eq. 88) can be improved following Filipović et al. (2017) who suggest to add a time dependent function for $T \geq t$ as follows

$$\tilde{P}(t,T) = e^{-\int_t^T \nu_1(u)du} P(t,T)\,, \tag{91}$$

and it implies to redefine the pricing kernel as

$$\tilde{\zeta}_T = e^{-\int_t^T \nu_1(u)du} \zeta_T\,. \tag{92}$$

Second, a similar approach applies to improve the fit of the Euribor-OIS spread term structure. Indeed, suppose we observe the market spreads $\{\tilde{A}(t,T_i,T_i+\delta); i = 1,\ldots,n_1\}$ (extracted from the Euribor-OIS spread curve) and that the model implied spread A is given by the formula Eq. (38) above with α already estimated (from the OIS curve) then the fit can be improved by adding a function $(\nu_2(u); u \in [t,\, T])$ such that

$$\tilde{\zeta}_T \tilde{P}(T,T+\delta)\delta \mathrm{Spread}(T,T+\delta) = e^{-\int_t^T \nu_2(u)du} e^{-\alpha T} x_{22,T}\,, \tag{93}$$

which leads to

$$\tilde{A}(t,T,T+\delta) = e^{-\int_t^T \nu_2(u)du} A(t,T,T+\delta)\,. \tag{94}$$

[10] In Crépey et al. (2015b), the authors develop a linear-rational model based on a log-normal process, which is a non affine process, and their results show that when calibrated the model generates negative interest rates.

[11] Notice that it seems to affect more ω_{22} than x_{22}.

Improving the fit by making some of the parameters time dependent is common in the interest rate literature and is central to the Heath-Jarrow-Morton approach, see Brigo and Mercurio (2006) for standard models and to Jin and Glasserman (2001) or Crépey et al. (2015b) for models based on the potential approach.[12] Notice that the improvements mentioned above barely change the swaption calibration error. This latter can be reduced by making the volatility σ time dependent; such a time dependent parameter strategy is used in Nguyen and Seifried (2015), but for the model presented here this has far reaching numerical consequences that are beyond the scope of that work.

Once the model is calibrated, it is relevant to analyze the distribution of the variable $Y_{T_0} = B(T_0, T_{n_1}) + A_1(T_0, T_{n_1})x_{11,T_0} + A_2(T_0, T_{n_1})x_{22,T_0}$ that is involved in the swaption pricing in Eq. (50) and whose moments are known and given by Eq. (78). Using the characteristic function of that variable, we report in Figures 1-2 its density for two pairs of maturity/tenor: (1 year, 1 year) and (5 years, 5 years). These pairs are the extremes of the swaption data reported in Table II. All the other pairs look similar to those reported. Figures show two distributions that are uni-modal and slightly asymmetric. It suggests that the first three moments of the distribution could be used in the swaption price approximation developed in Collin-Dufresne and Goldstein (2002) and produce reasonably accurate results. The approximate density given by Eq. (79) is reported for the maturity/tenor pair (1 year, 1 year) in Figure 1 and in Figure 2 for maturity/tenor pair (5 years, 5 years). For the first pair, the approximate density is very close to the true one, suggesting that a fairly accurate swaption price can be obtained using the first three moments, while for the second pair the difference is more pronounced but remains reasonable, the swaption price approximation should also be close to the exact price. To assess that latter point, we follow the details of section 3.5.1 and price the options using the calibrated parameters and the approximation formula Eq. (83). We restrict to the first three cumulants, as we found that taking higher cumulants deteriorates the results, then the average root mean square swaption pricing error between the normal volatility computed using the exact swaption pricing formula and the normal volatility using the approximate swaption pricing formula is 9.410.[13] That error reduces to 7.301 when restricted to swaptions with maturity equal to 1 year and increases to 9.724 when restricted to swaptions with a maturity greater than or equal to 4 years.

[Insert Figure 1 here]

[Insert Figure 2 here]

To the matrices x, ω and σ correspond certain correlation matrices with off-diagonal terms reported in Table V confirming that the model does not have a diagonal structure. In particular, $\sigma_{12} \neq 0$ implies that the last term in Eq. (42) (or Eq. 43) does not vanish (*i.e.*, $\langle x_{11,.}, x_{22,.} \rangle_t$ depends linearly on σ_{12} according to Eq. 8). Furthermore, x_{12} is on average negative, as the calibrated value or the correlation associated with x shows, combined with σ_{12} that is positive, we deduce that the last term in Eq. (42) is positive and contributes to increase the covariance between the two curves. Whether that covariance is mainly driven by the first term or the second term of Eq. (42) (or Eq. 43) determines the importance of the off-diagonal terms x_{12} and σ_{12} of the Wishart process and therefore the degree of dependency that exists between the two diagonal terms x_{11} and x_{22} of the Wishart process or factors and, by extension, the OIS and Euribor-OIS curves.

The importance of off-diagonal terms illustrates the result of Benabid et al. (2009) according to which the law of the diagonal terms of the Wishart process for a given time t (*i.e.*, $(x_{t,11}, x_{t,22})$), which are the only terms involved in the argument of the characteristic function $\Phi_Y(.)$ in Eq. (58) as $\Lambda(T_0, T_{n_1})$ is diagonal, is not the product of two noncentral chi-squared distributions. As a consequence, the off-diagonal terms x_{12}, ω_{12} and σ_{12} of the Wishart process are essential for the model to capture the dependency between the OIS and the Euribor-OIS curves and Table V, which reports the correlations associated with these matrices, clearly show that they are significant.

[12] The corresponding results are available upon request.

[13] The fact that higher order cumulants do not improve the results as in Collin-Dufresne and Goldstein (2002) is likely to be related to distribution of the variable that is close to a noncentral chi-squared distribution while in Collin-Dufresne and Goldstein (2002) the variable is a sum of exponentials of normal variables/noncentral chi-squared variables as they use an exponential affine model.

To further illustrate the importance of the correlation between the two curves, and the relevance of the Wishart process to capture that dependency, we compare the market correlation with the model correlation. Following Eq. (49), let us denote $\bar{A}(t, T_{n_1}) = \sum_{j=1}^{n_1} A(t, T_{j-1}, T_j)$ the sum of spreads up to T_{n_1} involved in a swap contract with maturity T_{n_1} and $P(t, T_{n_1})$ the OIS zero-coupon bond with maturity T_{n_1}. We are interested in

$$\mathrm{Corr}(dP(t, T_{n_1}), d\bar{A}(t, T_{n_1}))\,, \tag{95}$$

the correlation between $P(t, T_{n_1})$ increments and $\bar{A}(t, T_{n_1})$ increments. Thanks to Eq. (42), it is known that it can take any sign but also that it is driven by two terms, the first one depending only on the first factor x_{11} of the model while the second one depends on off-diagonal term x_{12}. To assess the quality of the linear-rational Wishart model, we compare the market correlation with the model correlation (*i.e.*, the correlation given by the calibrated model, that is the right hand side of Eq. 42) and report the results in Table VII. The market correlation, reported in the line "Market", is positive and declines with the zero-coupon bond/spread maturity as the table shows. The model correlation is given by the right hand side of Eq. (42), calibrated parameters lead to correlations reported in "Model" in Table VII, and check whether the model correlation is close to the market correlation.[14] The values reported in line "Model" are consistent with those of reported in the line "Market" and confirm the model's ability to handle the non trivial dependency that exists between the two curves.

[Insert Table VII here]

To clarify further the analysis of the model, we decompose the correlation Eq. (95) into two terms thanks to the relation Eq. (42) (or Eq. 43), the first one depending on $\langle x_{11,.}, x_{11,.}\rangle_t$ named the "diagonal term", and the second one depending on $\langle x_{11,.}, x_{22,.}\rangle_t$ that is linear in x_{12} named the "off-diagonal term". It allows us to quantify the contribution of these two terms to the correlation between the OIS zero-coupon bond with a given maturity and sum of spreads up to that maturity. Obviously, summing them leads to the correlations of Table VII. According to Eq. (8), $\langle x_{11,.}, x_{22,.}\rangle_t$ depends linearly on x_{12} and σ_{12} and therefore in a diagonal model (*i.e.*, $\sigma_{12} = 0$), such as the standard vector affine model of Duffie and Kan (1996) used in Filipović et al. (2017), the correlation is only controlled by the factor x_{11} and diagonal parameters. Table VIII contains the values and shows that for all the maturities, the main contributor to the correlation is by far the off diagonal term. Notice that $\partial_{x_{11}} G$ given by Eq. (40), which contributes to the diagonal term, is comparable to the spread whereas $\partial_{x_{22}} G$ given by Eq. (41), which contributes to the off-diagonal term, is comparable $a_2(t) = e^{2m_{22}t}$ according to Proposition 3.2. The first term can only be small and any significant correlation necessarily comes from the off-diagonal term. As a result, using a diagonal model, one cannot capture the correlation between the OIS curve and Euribor-OIS curve. In conclusion, the non trivial dependency between the two curves can be handled by the linear-rational Wishart model that provides, compared to the standard vector affine process, an additional factor that is crucial.

[Insert Table VIII here]

4.3 Pricing exotic derivatives

Once the model is calibrated on liquid products such as swaptions, it can be used to price exotic derivatives. We focus on the CMS and CMS spread options as these are important products for which section 3.5.2 provides price approximations that we now evaluate. For the model parameters, we consider those of Table III while for the product parameters in Eq. (60) we take $T_j = 1Y$ and $5Y$, the swap tenor is either $1Y$ or $5Y$ and $\delta_s = \Delta_s = 0.5$. Regarding the truncation level M in Proposition 3.13, we consider $M = 3$ and $M = 5$. We benchmark the approximation given by Eq. (85) with a Monte-Carlo method with 50000 paths and a time discretisation of 250 days per year. The results reported in Table IX confirm the quality of the approximation as evidenced by the small discrepancy between the two methods. Not surprisingly, the error decreases with the truncation level M in Eq. (85) and deteriorates with the maturity of the CMS (everything else being equal).

[Insert Table IX here]

[14] When analyzing the right hand side of Eq. (42) ω_{12} is not needed.

Following these encouraging results, we consider the CMS spread option of Proposition 3.9 using the approximation of the underlying variable given by Proposition 3.14 and the Collin-Dufresne and Goldstein (2002) approximation formula Eq. (83) where we restrict to the first three moments/cumulants. The two underlying swaps have a tenor of 1Y and 5Y, respectively, while $\delta_s = \Delta_s = 0.5$. For the CMS spread option maturities, we take $1Y$ and $5Y$. For the CMS spread option approximation, we consider $M = 3$ and $M = 5$. As for the CMS, we compare the price approximation with a Monte-Carlo method with 50000 paths and a time discretisation of 250 days per year and report in Table X the absolute error between these two prices expressed in percent. The table confirms the accuracy of the approximation for all the parameters selected. Taking into account the importance of the CMS and CMS spread option, it shows an interesting property of the linear-rational Wishart model as it enables a better integration between the swaption market, which is used to calibrate the model, and the exotic interest rate derivatives market, in this case the CMS and CMS spread option market, as these products can be priced easily using a polynomial approximation that is accurate.

[Insert Table X here]

5 Conclusion

We propose a linear-rational multi-curve term structure model based on the Wishart process. Following Filipović et al. (2017)'s modeling strategy that is based on the potential approach presented in Rogers (1997), we use the Wishart process to build a multi-curve model that allows for a stochastic correlation between the curves. We develop the pricing formulas for interest rate products commonly traded on the market such as interest swaps, swaptions, constant maturity swap (CMS) and CMS spread options. One striking property of the model is that the swaptions have the same computational cost as caps/floors, a property very interesting as these products are commonly used to calibrate interest rate models. Thus, being able to efficiently price these derivatives is essential. Pricing formulas for more complex interest rate derivatives such as CMS and CMS spread options are also derived but, unfortunately, they do not lead to simple mathematical expressions. Thanks to the affine property of Wishart process, we develop a swaption price approximation in the spirit of Collin-Dufresne and Goldstein (2002) that is accurate and simple to implement. Fortunately, the technique is rather generic and also applies to the CMS and CMS spread options with excellent results. To illustrate the framework, we analyze the model empirical properties, that is we perform a daily calibration of the model using a three-month sample of OIS term structure, Euribor-OIS term structure and ATM swaption prices. The calibration errors are stable and show the model's ability to handle the data fluctuations. The estimated parameters lead to a model that possesses the right statistical properties. The estimated parameters have small standard deviations, the model is therefore robust. What is more, the estimated parameters illustrate the ability of the model to capture the non null relationship that exists between the OIS curve and the Euribor-OIS spread curve that critically relies on the Wishart process properties. Further to this, the calibrated model is then used to price exotic derivatives such as CMS and CMS spread options using the approximation formulas that prove to be very accurate. Overall, the results clearly underline the linear-rational model based on the Wishart process ability to encompass interest rate dependencies, calibration of liquid derivatives and pricing of exotic derivatives in an efficient way.

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A Appendix

A.1 Black formula for swaption pricing

Let us consider a swap starting at T_0 and ending at T_{n_1}, with floating leg payment dates $(T_j)_{j=1,\cdots,n_1}$ (and reset dates $(T_j)_{j=0,\cdots,n_1-1}$) and fixed rate leg payment dates given by $(t_i)_{i=1,\cdots,m_1}$ with $T_{j+1}-T_j=\delta$, $t_{i+1}-t_i=\Delta$, $t_{m_1}=T_{n_1}$ and $t_0=T_0$. At time t, the floating leg value is given by $P(t,T_0)-P(t,T_{n_1})+\sum_{j=1}^{n_1}A(t,T_{j-1},T_j)$ and the fixed leg value is $K\sum_{i=1}^{m_1}\Delta P(t,t_i)$, where K is the fixed rate.

We can therefore derive the time-t forward swap rate, $S_t^{T_0,T_{n_1}}$ as:

$$S_t^{T_0,T_{n_1}}=\frac{P(t,T_0)-P(t,T_{n_1})+\sum_{j=1}^{n_1}A(t,T_{j-1},T_j)}{\Delta\sum_{i=1}^{m_1}P(t,t_i)}\,. \tag{96}$$

Let us denote $An_t^{T_0,T_{n_1}}=\sum_{i=1}^{m_1}\Delta P(t,t_i)$ the annuity. The swap rate is a martingale under the swap numeraire (also called annuity numeraire) and if we assume that the swap rate follows a normal process, its dynamic under the swap numeraire can then be written as:

$$dS_t^{T_0,T_{n_1}}=\sigma dW_t. \tag{97}$$

This leads to $S_T^{T_0,T_{n_1}}\sim\mathcal{N}\left(S_t^{T_0,T_{n_1}},\,\sigma\sqrt{T-t}\right)$. The value V_t^{swaption} at time t of a swaption associated with the swap described above, with option expiry date T_0, is given by (below we simplify the notation by replacing $An_t^{T_0,T_{n_1}}$ with An_t):

$$\begin{aligned}
V_t^{\text{swaption}} &= An_t\mathbb{E}_t\left[\frac{1}{An_{T_0}}\left(P(T_0,T_0)-P(T_0,T_{n_1})+\sum_{j=1}^{n_1}A(T_0,T_{j-1},T_j)-K\sum_{i=1}^{m_1}\Delta P(T_0,t_i)\right)_+\right], && (98)\\
&= An_t\mathbb{E}_t\left[\left(S_{T_0}^{T_0,T_{n_1}}-K\right)_+\right], && (99)\\
&= \sum_{i=1}^{m_1}\Delta P(t,t_i)\left(\left(S_t^{T_0,T_{n_1}}-K\right)\Phi\left(\frac{S_t^{T_0,T_{n_1}}-K}{\sigma\sqrt{T_0-t}}\right)+\sigma\sqrt{T_0-t}\Phi'\left(\frac{S_t^{T_0,T_{n_1}}-K}{\sigma\sqrt{T_0-t}}\right)\right), && (100)
\end{aligned}$$

where Φ is the cumulative distribution function of the standard normal variable and Φ' its derivative. When the swaption is at the money then its price simplifies to

$$V_t^{\text{swaption}}=\sum_{i=1}^{m_1}\Delta P(t,t_i)\sqrt{\frac{T_0-t}{2\pi}}\sigma\,. \tag{101}$$

It is a market practice to quote the swaption price through its normal volatility σ.

A.2 Synthesizing a forward swap with two spot swaps

Following the notation of the swap above, let us consider the swap $S_0^{T_0=0,T_{n_1}}$ (the swap starting at time $T_0=0$ and ending at T_{n_1}, $T_0=0<T_{n_1}$) with the floating leg reset and payment dates $T_0,T_1,\cdots,T_{n_1}$, with $T_j-T_{j-1}=\delta$, and the fixed leg payment dates $t_1,\cdots,t_{m_1}=T_{n_1}$ and $t_i-t_{i-1}=\Delta$, $(T_0=t_0=0)$.

Similarly, we consider a second swap $S_0^{T_0=0,T_{n_2}}$ with floating leg reset and payment dates $T_0,T_1,\cdots,T_{n_2}$, with $T_j-T_{j-1}=\delta$, and fixed leg payment dates $t_1,\cdots,t_{m_2}=T_{n_2}$ and $t_i-t_{i-1}=\Delta$, (with $t_{m_2}=T_{n_2}$, and $T_0=0$).

The par swap rates are given by:

$$\begin{aligned}
S_0^{0,T_{n_1}} &= \frac{1-P(0,T_{n_1})+\sum_{j=1}^{n_1}A(0,T_{j-1},T_j)}{\Delta\sum_{i=1}^{m_1}P(0,t_i)}, && (102)\\
S_0^{0,T_{n_2}} &= \frac{1-P(0,T_{n_2})+\sum_{j=1}^{n_2}A(0,T_{j-1},T_j)}{\Delta\sum_{i=1}^{m_2}P(0,t_i)}. && (103)
\end{aligned}$$

Suppose that $T_{n_1}<T_{n_2}$, then the forward starting swap rate $S_0^{T_{n_1},T_{n_2}}$ with floating leg reset and payment dates $T_{n_1},T_{n_1+1},\cdots,T_{n_2}$ and fixed leg payment dates $t_{m_1+1},\cdots,t_{m_2}=T_{n_2}$ can be expressed as a function of $S_0^{T_0=0,T_{n_1}}$ and $S_0^{T_0=0,T_{n_2}}$ as follows

$$\begin{aligned}
S_0^{T_{n_1},T_{n_2}} &= \frac{P(0,T_{n_1})-P(0,T_{n_2})+\sum_{j=n_1+1}^{n_2}A(0,T_{j-1},T_j)}{\Delta\sum_{i=m_1+1}^{m_2}P(0,t_i)}, && (104)\\
&= \frac{S_0^{T_0,T_{n_2}}\left(\Delta\sum_{i=1}^{m_2}P(0,t_i)\right)-S_0^{T_0,T_{n_1}}\left(\Delta\sum_{i=1}^{m_1}P(0,t_i)\right)}{\Delta\sum_{i=m_1+1}^{m_2}P(0,t_i)}. && (105)
\end{aligned}$$

For the model calibration purpose, we will consider spot swap rates $(t=T_0=0)$. Further, in order to apply the formula (105) above, the start date T_{n_1} of the underlying swap of the swaption $S_{t=0}^{T_{n_1},T_{n_2}}$ should be one of the payment dates of the spot swap $S_{t=0}^{T_0=0,T_{n_2}}$.

A.3 Tables

Table I: Descriptive statistics

Maturity	0.5	1	3	5	7	10	12	15
					OIS			
Mean	0.466	0.460	0.713	1.189	1.598	1.995	2.187	2.365
Std. dev.	0.150	0.138	0.170	0.187	0.174	0.149	0.142	0.140
					Euribor			
Mean	1.575	1.423	1.423	1.806	2.143	2.464	2.618	2.756
Std. dev.	0.183	0.176	0.189	0.200	0.182	0.159	0.152	0.151

Note: Mean value and standard deviation of the OIS and Euribor term structures (with the maturity expressed in years). Rates are expressed in percentage and the data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table II: Swaption volatilities

Swap tenor	1	2	3	4	5
			1Y		
Mean	70.72	75.06	79.24	84.12	88.06
Std. dev.	16.34	15.49	13.79	11.31	9.80
			2Y		
Mean	85.51	85.83	87.87	90.34	91.49
Std. dev.	12.39	10.43	8.71	7.12	6.36
			3Y		
Mean	94.74	92.02	91.29	91.41	92.07
Std. dev.	8.27	6.52	5.55	5.07	4.41
			4Y		
Mean	96.58	92.52	91.44	90.91	90.63
Std. dev.	5.74	4.71	4.11	3.49	3.14
			5Y		
Mean	95.04	91.28	89.70	88.97	88.49
Std. dev.	3.71	3.33	2.98	2.72	2.60

Note: Mean value and standard deviation of the normal implied swaption volatilities (expressed in basis points) for the swaption maturities 1Y, 2Y, 3Y, 4Y and 5Y (in years) and swap tenors (in years). The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table III: Calibrated parameters

Param.	Mean	Std. dev.	Min	Max
α	0.024	1.440×10^{-3}	0.021	0.027
x_{11}	0.125	0.037	0.045	0.218
x_{12}	-1.121×10^{-2}	4.038×10^{-3}	-2.713×10^{-2}	-6.675×10^{-3}
x_{22}	5.745×10^{-3}	6.590×10^{-4}	0.004	0.006
ω_{11}	0.130	0.029	0.068	0.203
ω_{12}	1.797×10^{-3}	2.011×10^{-3}	1.950×10^{-4}	0.011
ω_{22}	4.660×10^{-4}	2.320×10^{-4}	5.026×10^{-5}	9.610×10^{-4}
m_{11}	-0.375	0.016	-0.416	-0.352
m_{22}	-0.181	0.053	-0.284	-0.069
σ_{11}	0.050	0.013	0.019	0.121
σ_{12}	0.024	6.904×10^{-3}	0.009	0.038
σ_{22}	0.047	0.085	0.020	0.067

Note: Mean value, standard deviation, min value and max value of the calibrated parameters obtained by rolling the daily calibration. The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table IV: Eigenvalues of the matrices

	First				Second			
Param.	Mean	Std. dev.	Min	Max	Mean	Std. dev.	Min	Max
x	0.126	0.037	0.046	0.219	4.601×10^{-3}	9.890×10^{-4}	0.002	0.006
ω	0.130	0.029	0.068	0.203	4.130×10^{-4}	2.150×10^{-4}	5.000×10^{-5}	9.050×10^{-4}
σ	0.056	0.028	0.014	0.124	0.041	0.024	0.010	0.092

Note: Mean value, standard deviation, min value and max value of the eigenvalues of the estimated parameters. The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table V: Correlation associated with parameters

Param.	Mean	Std. dev.	Min	Max
x	-0.423	0.124	-0.837	-0.258
ω	0.223	0.190	0.063	0.885
σ	0.497	0.118	0.292	0.754
$\bar{x}_\infty$	0.208	0.183	0.058	0.867

Note: Mean value, standard deviation, min value and max value of the correlation associated with the estimated parameters with $\bar{x}_\infty$ defined in Eq. (19). The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table VI: Calibration errors

	OIS error	Spread error	Swaption error
Mean	115.04	3.58	36.24
Std. dev.	8.63	2.11	4.27

Note: Mean value and standard deviation of the daily root mean square errors of the calibrations. OIS error stands for the square root of the error Eq. (88) expressed in basis points, Spread error stands for the square root of the error Eq. (89) expressed in basis points and Swaption error stands for the square root of the error Eq. (90) expressed in basis points. The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table VII: Market vs. model correlations

Maturity	1	3	5	7	10	12	15
Market	0.514	0.270	0.241	0.174	0.184	0.181	0.128
Model	0.353	0.342	0.322	0.301	0.272	0.256	0.235

Note: Market and model correlations, given by Eq. (95), between the OIS zero-coupon bond with maturity T and the sum of spreads up to maturity T for different values for T (in years). "Model" stands for the computation of Eq. (95) using the right hand side of Eq. (42) and the calibrated parameters solving Eq. (90) (along with x_{11} and x_{22} obtained from Eq. (88) and Eq. (89), respectively). The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table VIII: Correlation decomposition

Maturity	1	3	5	7	10	12	15
Diagonal term	0.025	0.025	0.025	0.025	0.024	0.024	0.024
Off-diagonal term	0.328	0.316	0.296	0.275	0.247	0.231	0.211

Note: Decomposition of the correlation Eq. (95) into two terms using Eq. (42), the first one depending on $\langle x_{11,.}, x_{11,.}\rangle_t$ named the "diagonal term" and the second one depending on $\langle x_{11,.}, x_{22,.}\rangle_t$, named the "off-diagonal term". The correlation decomposition is performed for different values for T (in years). The calibrated parameters are obtained by solving Eq. (90) (along with x_{11} and x_{22} obtained from Eq. (88) and Eq. (89), respectively). The data sample period is 4/10/2011 to 12/03/2012 at daily frequency.

Table IX: CMS approximation

	Tenor 1Y		Tenor 5Y	
	$M = 3$	$M = 5$	$M = 3$	$M = 5$
Maturity 1Y	0.129	0.009	0.064	0.002
Maturity 5Y	0.130	0.059	0.093	0.061

Note: Absolute error expressed in % between the approximation $I(M)$ given by Eq. (85) and I given by the expectation on the right hand side of Eq. (60) or Eq. (84) computed using a Monte-Carlo method. The model parameters are those of Table III, the CMS maturity is $T_j = 1Y$ or $5Y$, the swap tenor is equal to $1Y$ or $5Y$ and $\delta_s = \Delta_s = 0.5$. The Monte-Carlo method is based on 50000 paths and a daily discretisation of the time interval.

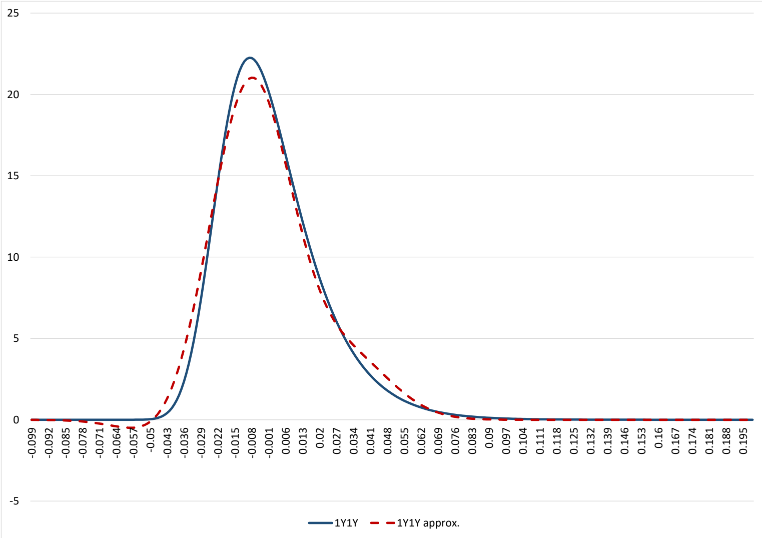
Table X: CMS spread option approximation

	$M = 3$	$M = 5$
Maturity 1Y	0.640	0.632
Maturity 5Y	1.004	0.613

Note: Absolute error expressed in % between the CMS spread option price approximation given by Proposition 3.14 and the Monte-Carlo price given by Proposition 3.9. The two underlying rates are the swaps with tenors 1Y and 5Y with fixed and floating legs such that $\delta_s = \Delta_s = 0.5$. The strikes of the options are such that they are at the money. The Monte-Carlo method is based on 50000 paths and a daily discretisation of the time interval.

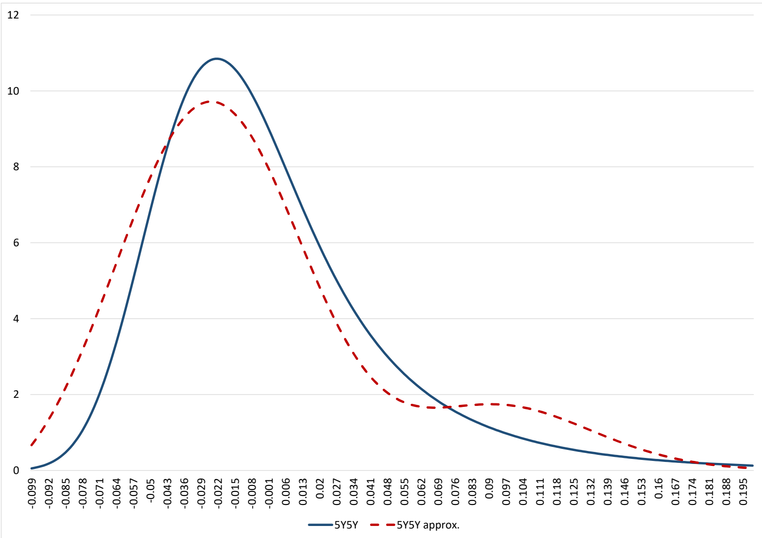
A.4 Figures

Figure 1: Density of the variable Y with maturity 1Y and tenor 1Y



Note: Density of the variable Y defined as $Y_{T_0} = B(T_0, T_{n_1}) + A_1(T_0, T_{n_1})x_{11,T_0} + A_2(T_0, T_{n_1})x_{22,T_0}$ involved in the pricing formula Eq. (50) for a swaption with maturity $1Y$ and tenor $1Y$ given by the solid blue line (1Y1Y) and in red dash line (1Y1Y approx.) its approximation using the first three cumulants and Eq. (79). The parameters used to compute the density are those of 4/10/2011.

Figure 2: Density of the variable Y with maturity 5Y and tenor 5Y



Note: Density of the variable Y defined as $Y_{T_0} = B(T_0, T_{n_1}) + A_1(T_0, T_{n_1})x_{11,T_0} + A_2(T_0, T_{n_1})x_{22,T_0}$ involved in the pricing formula Eq. (50) for a swaption with maturity $5Y$ and tenor $5Y$ given by the solid blue line (5Y5Y) and in red dash line (5Y5Y approx.) its approximation using the first three cumulants and Eq. (79). The parameters used to compute the density are those of 4/10/2011.

A.5 Proofs

Proof of Proposition 2.1. Thanks to the property of the exponential function, when $\theta_2 = 0_n$ (with 0_n the $n \times n$ null matrix) Eq. (14) leads to the system of matrix ODEs (*e.g.*, see Van Loan 1978)

$$A'_{11} = mA_{11} - 2\sigma^2 A_{21}\,, \tag{106}$$

$$A'_{12} = mA_{12} - 2\sigma^2 A_{22}\,, \tag{107}$$

$$A'_{21} = -m^\top A_{21}\,, \tag{108}$$

$$A'_{22} = -m^\top A_{22}\,, \tag{109}$$

and the initial conditions $A_{11}(0) = I_n$, $A_{12}(0) = 0_n$, $A_{21}(0) = 0_n$ and $A_{22}(0) = I_n$ (with I_n the $n \times n$ identity matrix). Solving these ODEs leads to: $A_{21}(t) = 0$, $A_{11}(t) = e^{mt}$, $A_{22}(t) = e^{-m^\top t}$ and

$$A_{12}(t) = \int_0^t e^{(t-s)m}(-2\sigma^2)e^{-sm^\top}\,ds\,. \tag{110}$$

As a result, $e^{\mathrm{tr}(a(t)x_0)}$ in Eq. (10) after some transformations is given by

$$\mathrm{etr}\left(e^{m^\top t}\left(\theta_1 \int_0^t e^{(t-s)m}(-2\sigma^2)e^{(t-s)m^\top}\,ds + I\right)^{-1}\theta_1 e^{mt}x_0\right)\,, \tag{111}$$

where $\mathrm{etr(A)} := e^{\mathrm{tr}(A)}$.

In the Bru case the term $e^{b(t)}$ in Eq. (10) rewrites as

$$\mathrm{etr}\left(-\frac{\beta}{2}m^\top t\right)\left(\mathrm{etr}\left(\log(\theta_1 A_{12} + A_{22})\right)\right)^{-\beta/2}\,, \tag{112}$$

and thanks to the relation $\det(e^A) = e^{\mathrm{tr}(A)}$ we get

$$e^{b(t)} = \det\left(I + \theta_1\int_0^t e^{(t-s)m}(-2\sigma^2)e^{(t-s)m^\top}\,ds\right)^{-\beta/2}\,. \tag{113}$$

Combining Eq. (111) and Eq. (113) gives the moment generating function of x_t.

Consider Eq. (9) with $\theta_2 = 0_n$ and θ_1 replaced with $-\theta_1$ with $\theta_1 \in \mathbb{S}_n^{++}$, it is the Laplace transform of x_t that is given by

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{etr}(-\theta_1 x_t)\right] &= \det\left(I - \theta_1\int_0^t e^{(t-s)m}(-2\sigma^2)e^{(t-s)m^\top}\,ds\right)^{-\beta/2} \\ &\times \mathrm{etr}\left(-e^{mt}x_0 e^{m^\top t}\theta_1\left(I - \int_0^t e^{(t-s)m}(-2\sigma^2)e^{(t-s)m^\top}\,ds\theta_1\right)^{-1}\right)\,,\end{aligned} \tag{114}$$

and defining Ξ_t as in Eq. (21) and Λ_t as in Eq. (22) leads to the result after reorganizing the terms. $\square$

Proof of Proposition 3.6. It is known from Eq. (49) that the swap rate is given by

$$S_{T_j}^{T_{j,0},T_{j,n_s}} = \frac{P(T_j,T_j) - P(T_j,T_{j,n_s}) + \sum_{l=1}^{n_s} A(T_j,T_{j,l-1},T_{j,l})}{\Delta_s \sum_{k=1}^{m_s} P(T_j,t_{j,k})}\,. \tag{115}$$

The time t-value (with $t \le T_0$) of the leg that pays the Euribor rate plus a fixed rate K is given by

$$P(t,T_0) - P(t,T_{n_1}) + \sum_{j=1}^{n_1} A(t,T_{j-1},T_j) + \delta K\sum_{j=1}^{n_1} P(t,T_j)\,, \tag{116}$$

while the time t-value (with $t \le T_0$) of the leg that pays the swap rate is given by

$$\mathbb{E}_t\left[\sum_{j=0}^{n_1-1}\frac{\zeta_{T_{j+1}}}{\zeta_t}\delta S_{T_j}^{T_j,T_{j,n_s}}\right]\,, \tag{117}$$

and taking into account Eq. (115), it leads to evaluate

$$\mathbb{E}_t\left[\frac{\zeta_{T_{j+1}}}{\zeta_t}\frac{P(T_j,T_j) - P(T_j,T_{j,n_s}) + \sum_{l=1}^{n_s} A(T_j,T_{j,l-1},T_{j,l})}{\Delta_s\sum_{k=1}^{m_s} P(t_i,t_{i,k})}\right]\,. \tag{118}$$

Taking into account Eq. (28), Eq. (38) and $\mathbb{E}_{T_j}\left[e^{-\alpha T_{j+1}}\zeta_{T_{j+1}}\right] = e^{-\alpha T_{j+1}}(b_1(\delta_s) + a_1(\delta_s)x_{11,T_j})$, the above expectation is equal to

$$\frac{e^{-\alpha(T_{j+1}-t)}}{1 + x_{11,t}}\mathbb{E}_t\left[\frac{c_0 + c_1 x_{11,T_j} + c_2 x_{22,T_j} + c_{12}x_{11,T_j}x_{22,T_j} + c_{11}x_{11,T_j}^2}{\mu_0 + \mu_1 x_{11,T_j}}\right]\,, \tag{119}$$

with

$$c_0 = b_1(\delta_s)\left(b_1(T_j - T_j) - e^{-\alpha(T_{j,n_s}-T_j)}b_1(T_{j,n_s} - T_j) + \sum_{l=1}^{n_s} e^{-\alpha(T_{j,l-1}-T_j)}b_2(T_{j,l-1} - T_j)\right), \tag{120}$$

$$c_1 = b_1(\delta_s)\left(a_1(T_j - T_j) - e^{-\alpha(T_{j,n_s}-T_j)}a_1(T_{j,n_s} - T_j)\right) + c_0\frac{a_1(\delta_s)}{b_1(\delta_s)}, \tag{121}$$

$$c_2 = b_1(\delta_s)\sum_{l=1}^{n_s} e^{-\alpha(T_{j,l-1}-T_j)}a_2(T_{j,l-1} - T_j), \tag{122}$$

$$c_{12} = a_1(\delta_s)\sum_{l=1}^{n_s} e^{-\alpha(T_{j,l-1}-T_j)}a_2(T_{j,l-1} - T_j), \tag{123}$$

$$c_{11} = a_1(\delta_s)\left(a_1(T_j - T_j) - e^{-\alpha(T_{j,n_s}-T_j)}a_1(T_{j,n_s} - T_j)\right), \tag{124}$$

$$\mu_0 = \Delta_s\sum_{k=1}^{m_s} e^{-\alpha(t_{j,k}-t_i)}b_1(t_{i,k} - t_i), \tag{125}$$

$$\mu_1 = \Delta_s\sum_{k=1}^{m_s} e^{-\alpha(t_{j,k}-t_i)}a_1(t_{i,k} - t_i), \tag{126}$$

which is the announced result. □

Proof of Proposition 3.8. The expectation Eq. (60) can be expressed

$$\begin{aligned}
I = &\left(\frac{c_1}{\mu_1} - \frac{c_{11}}{\mu_1}\frac{\mu_0}{\mu_1}\right) + \frac{c_{11}}{\mu_1}\cdot\mathbb{E}_t\left[x_{11,T_j}\right] + \frac{c_{12}}{\mu_1}\cdot\mathbb{E}_t\left[x_{22,T_j}\right] &(127)\\
&+\left(c_0 - c_1\frac{\mu_0}{\mu_1} + c_{11}\left(\frac{\mu_0}{\mu_1}\right)^2\right)\cdot\mathbb{E}_t\left[\frac{1}{\mu_0 + \mu_1 x_{11,T_j}}\right] &(128)\\
&+\left(c_2 - c_{12}\frac{\mu_0}{\mu_1}\right)\cdot\mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0 + \mu_1 x_{11,T_j}}\right], &(129)\\
= &\left(\frac{c_1 - c_{11}}{\mu_1} - \frac{c_{11}}{\mu_1}\frac{\mu_0}{\mu_1}\right) + \frac{c_{11}}{\mu_1}\cdot\left[b_1\left(T_j - t\right) + a_1\left(T_j - t\right)x_{11,t}\right] &(130)\\
&+\frac{c_{12}}{\mu_1}\cdot\left[b_2\left(T_j - t\right) + a_1\left(T_j - t\right)x_{22,t}\right] &(131)\\
&+\left(c_0 - c_1\frac{\mu_0}{\mu_1} + c_{11}\left(\frac{\mu_0}{\mu_1}\right)^2\right)\cdot\mathbb{E}_t\left[\frac{1}{\mu_0 + \mu_1 x_{11,T_j}}\right] &(132)\\
&+\left(c_2 - c_{12}\frac{\mu_0}{\mu_1}\right)\cdot\mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0 + \mu_1 x_{11,T_j}}\right], &(133)
\end{aligned}$$

which leaves us with only two expectations to evaluate. Remark 3.7 allows the computation of the expectations

$$\mathbb{E}_t\left[\frac{1}{\mu_0 + \mu_1 x_{11,T_j}}\right] = \int_0^{+\infty} e^{-s\mu_0}\Phi(\tau,\theta_1,0,x_t)ds, \tag{134}$$

$$\mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0 + \mu_1 x_{11,T_j}}\right] = \int_0^{+\infty} e^{-s\mu_0}\partial_z\Phi(\tau,\theta_2,0,x_t)ds|_{z=0}, \tag{135}$$

with $\tau = T_j - t$ and

$$\theta_1 = \mu_1 s e_{11}, \tag{136}$$

$$\theta_2 = z e_{22} - \mu_1 s e_{11}. \tag{137}$$

For Eq. (134), integrating the moment generating function leads to the result. For Eq. (135), one needs to differentiate the moment generating function of the Wishart process which can be explicitly carried out. From Eq. (13) (if we drop the dependency of the matrices A_{ij} on t), if θ_2 is given by Eq. (137) then the computation of Eq. (135) leads to

$$\frac{d}{dz}a(t,\theta_2,0) = -(\theta_2 A_{12} + A_{22})^{-1}e_{22}A_{12}a(t) + (\theta_2 A_{12} + A_{22})^{-1}e_{22}A_{11}. \tag{138}$$

We might denote the above derivative as $d_z a(t,\theta_2,0)$. Under the hypothesis that $\omega = \beta\sigma^2$, consider $z \to \mathrm{tr}\left[\log(\theta_2 A_{12} + A_{22})\right]$, denote $c = \theta_2 A_{12} + A_{22}$ and $l = c - I_n$ then using the Taylor expansions for $\ln(I_n + X)$ and $(I_n + X)^{-1}$ we get $\frac{d}{dz}\mathrm{tr}[\ln c] = \frac{d}{dz}\mathrm{tr}[\ln(I_n + l)] = \mathrm{tr}[\frac{d}{dz}\{l - l^2/2 + \ldots\}] = \mathrm{tr}[\frac{dl}{dz}\{I_n - l + l^2 - \ldots\}] = \mathrm{tr}[\frac{dc}{dz}c^{-1}]$ (thanks to $\mathrm{tr}[\frac{dl}{dz}l] = \mathrm{tr}[l\frac{dl}{dz}]$) and as a result

$$\frac{d}{dz}b(t,\theta_2,0) = -\frac{\beta}{2}\mathrm{tr}\left[e_{22}A_{12}(\theta_2 A_{12} + A_{22})^{-1}\right]. \tag{139}$$

We might denote the above derivative as $d_z b(t,\theta_2,0)$. Combining these two derivatives, we get the expression for $\partial_z \Phi(T-t,\theta_2,0,x_t) = (\mathrm{tr}[d_z a(t,\theta_2,0)x_t] + d_z b(t,\theta_2,0))\,\Phi(T-t,\theta_2,0,x_t)$ that is involved in Eq. (135) and when evaluated at $z=0$ then $\theta_2 = -\mu_1 s e_{11}$ and it leads to:

$$\mathbb{E}_t\left[\frac{x_{22,T_j}}{\mu_0+\mu_1 x_{11,T_j}}\right] = \int_0^{+\infty} e^{-s\mu_0}\left(\mathrm{tr}[d_z a(\tau,\theta_2,0)x_t] + d_z b(\tau,\theta_2,0)\right)\Phi(\tau,\theta_2,0,x_t)ds|_{z=0}\,, \tag{140}$$

with $\tau = T_j - t$. As a result, the expectations Eq. (61) are known up to an integration of dimension one. In the non Bru case (*i.e.*, $\omega \neq \beta\sigma^2$) then $b(T_j-t) = \int_t^{T_j} \mathrm{tr}[\omega a(u)]du$, the derivative of $b(T_j-t)$ can also be computed but it involves another integration. □

Proof of Proposition 3.9. Starting from Eq. (62) and replacing $S_{T_1}^{T_{1,0},T_{1,n_{s_1}}}$ and $S_{T_1}^{T_{1,0},T_{1,n_{s_2}}}$ with Eq. (115) leads to the announced results after performing computations similar to those of Proposition 3.6 but with

$$c_0^i = \left(b_1(T_1-T_1) - e^{-\alpha(T_{1,n_{s_i}}-T_1)}b_1(T_{1,n_{s_i}}-T_1) + \sum_{l=1}^{n_{s_i}} e^{-\alpha(T_{1,l-1}-T_1)}b_2(T_{1,l-1}-T_1)\right)\,, \tag{141}$$

$$c_1^i = \left(a_1(T_1-T_1) - e^{-\alpha(T_{1,n_{s_i}}-T_1)}a_1(T_{1,n_{s_i}}-T_1)\right) + c_0^i\,, \tag{142}$$

$$c_2^i = \sum_{l=1}^{n_{s_i}} e^{-\alpha(T_{1,l-1}-T_1)}a_2(T_{1,l-1}-T_1)\,, \tag{143}$$

$$c_{12}^i = \sum_{l=1}^{n_{s_i}} e^{-\alpha(T_{1,l-1}-T_1)}a_2(T_{1,l-1}-T_1)\,, \tag{144}$$

$$c_{11}^i = \left(a_1(T_1-T_1) - e^{-\alpha(T_{1,n_{s_i}}-T_1)}a_1(T_{1,n_{s_i}}-T_1)\right)\,, \tag{145}$$

$$\mu_0^i = \Delta_s \sum_{k=1}^{m_{s_i}} e^{-\alpha(t_{1,k}-t_1)}b_1(t_{1,k}-t_1)\,, \tag{146}$$

$$\mu_1^i = \Delta_s \sum_{k=1}^{m_{s_i}} e^{-\alpha(t_{1,k}-t_1)}a_1(t_{1,k}-t_1)\,. \tag{147}$$

□

Proof of Proposition 3.14. Using the series expansion $1/(1+(\mu_1/\mu_0)x) = \sum_{i=0}^{+\infty}(-1)^i(\frac{\mu_1}{\mu_0})^i x^i$ and truncating it at the order M gives an approximation of Y_T by Y_T^M defined by

$$Y_T^M = \sum_{m=0}^{M+2} v_m x_{11,T}^m + \sum_{m=0}^{M+1} u_m x_{22,T} x_{11,T}^m\,, \tag{148}$$

where $v_0 = v_0^1 - v_0^2 - K$, $v_1 = v_1^1 - v_1^2 - K$, $\{v_m = v_m^1 - v_m^2\,,\ m = 2,\dots,M\}$, $v_{M+1} = v_{M+1}^1 - v_{M+1}^2$, $v_{M+2} = v_{M+1}^1 - v_{M+1}^2$, $u_0 = u_0^1 - u_0^2$, $u_1 = u_1^1 - u_1^2$, $\{u_m = u_m^1 - u_m^2\,,\ m=2,\dots,M\}$, $u_{M+1} = u_{M+1}^1 - u_{M+1}^2$ with for $i \in \{1,2\}$

$$v_0^i = \frac{c_0^i}{\mu_0^i} - K\,, \tag{149}$$

$$v_1^i = -\frac{\mu_1^i}{\mu_0^i}\frac{c_0^i}{\mu_0^i} + \frac{c_1^i}{\mu_0^i} - K\,, \tag{150}$$

$$v_m^i = (-1)^m\left(\frac{\mu_1^i}{\mu_0^i}\right)^m \frac{c_0^i}{\mu_0^i} + (-1)^{m-1}\left(\frac{\mu_1^i}{\mu_0^i}\right)^{m-1}\frac{c_1^i}{\mu_0^i} + (-1)^{m-2}\left(\frac{\mu_1^i}{\mu_0^i}\right)^{m-2}\frac{c_{11}^i}{\mu_0^i}\,,\ m=2,\dots,M \tag{151}$$

$$v_{M+1}^i = (-1)^M\left(\frac{\mu_1^i}{\mu_0^i}\right)^M \frac{c_1^i}{\mu_0^i} + (-1)^{M-1}\left(\frac{\mu_1^i}{\mu_0^i}\right)^{M-1}\frac{c_{11}^i}{\mu_0^i}\,, \tag{152}$$

$$v_{M+2}^i = (-1)^M\left(\frac{\mu_1^i}{\mu_0^i}\right)^M \frac{c_{11}^i}{\mu_0^i}\,, \tag{153}$$

$$u_0^i = \frac{c_2^i}{\mu_0^i}\,, \tag{154}$$

$$u_1^i = (-1)\left(\frac{\mu_1^i}{\mu_0^i}\right)\frac{c_2^i}{\mu_0^i} + \frac{c_{12}^i}{\mu_0^i}\,, \tag{155}$$

$$u_m^i = (-1)^m\left(\frac{\mu_1^i}{\mu_0^i}\right)^m \frac{c_2^i}{\mu_0^i} + (-1)^{m-1}\left(\frac{\mu_1^i}{\mu_0^i}\right)^{m-1}\frac{c_{12}^i}{\mu_0^i}\,,\ m=2,\dots,M \tag{156}$$

$$u_{M+1}^i = (-1)^M\left(\frac{\mu_1^i}{\mu_0^i}\right)^M \frac{c_{12}^i}{\mu_0^i}\,. \tag{157}$$

Rewrite Eq. (148) as

$$Y_T^M = \sum_{i=0}^{2M+4} \zeta_i y_i \,, \tag{158}$$

with the first $M+2$ terms given by $v_m x_{11,T}^m$ and the last $M+2$ terms given by $u_m x_{22,T} x_{11,T}^m$. Using the standard multinomial expansion we get

$$(Y_T^M)^q = \left(\sum_{i=0}^{2M+4} \zeta_i y_i \right)^q = \sum_{k_0+\ldots+k_{2M+4}=n} \binom{q}{k_0, \ldots, k_{2M+4}} \prod_{j=0}^{2M+4} (\zeta_j y_i)^{k_j} \ , \tag{159}$$

and therefore taking the expectation of Eq. (159) leads to the announced results. $\square$